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FACTORIZATION OF OPERATORS WITH $G_\alpha^\alpha(\mathbb{R}_+^d)$ AND $g_\alpha^\alpha(\mathbb{R}_+^d)$ KERNELS

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Abstract. The aim of this paper is to prove that any linear operator with kernel in the spaces $G_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha \geq 1$ and $g_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha > 1$ is a composition of two operators in the same space.

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1. Introduction

The test space $\mathcal{S}(\mathbb{R}_+)$ for the space of tempered distributions supported by $[0,\infty)$ is studied in [3], [9] and [12]; recently, the space $\mathcal{S}(\mathbb{R}_+^d)$ is examined in [5]. In [6] G-type spaces, $G_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha \geq 1$ and their dual spaces, i.e. the spaces of ultradistributions of Roumier type over $[0,\infty)^d$, are characterized in terms of their Fourier-Laguerre coefficients; cf. Duran [4] for the one-dimensional case. Actually, the result of [4] is extended and the full topological characterization is given in all dimensions, as well as applications to pseudo-differential operators with radial symbols.

In this paper we introduce g-type spaces, $g_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha > 1$ and their dual spaces i.e. the spaces of ultradistributions of Beurling type over $[0,\infty)^d$. We give the kernel theorem for $g_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha > 1$.

For any topological vector space $\mathcal{B}$, the set of continuous linear functionals on $\mathcal{B}$, denoted by $\mathcal{M} = \mathfrak{L}(\mathcal{B})$, is a factorization algebra (also the term decomposition algebra can be used). This means that any operator T in $\mathcal{M}$ is a composition of two operators T_1, T_2 in $\mathcal{M}$ since we can choose T_1 as the identity operator and $T_2 = T$. If $\mathcal{B}$ is a Hilbert space, then it follows from spectral decomposition that the set of compact operators on $\mathcal{B}$ is a factorization algebra, where the factorization properties are obtained by straightforward applications of the spectral theorem.

An interesting subclass of linear and continuous operators on an L^2 space concerns the set of all linear operators whose kernels belong to the Schwartz space (see e.g. [1], [7],[11]). Similar facts hold true for the set of operators with kernels in Gelfand-Shilov spaces (cf. [10]) and Pilipović spaces (cf. [2]).

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In this paper we consider the case when $\mathcal{M}$ is the set of all linear operators with kernels in $G_\alpha^\alpha(\mathbb{R}^d_+)$, $\alpha \geq 1$ and $g_\alpha^\alpha(\mathbb{R}^d_+)$, $\alpha > 1$ spaces. We prove that any such $\mathcal{M}$ is a factorization algebra. Note that the identity operator does not belong to these operator classes.

2. Preliminaries

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of positive integers, integers, real and complex numbers, respectively; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0,\infty)$, $\mathbb{R}^d_+ = (0,\infty)^d$ and $\overline{\mathbb{R}^d_+} = [0,\infty)^d$. We use the standard multi-index notation. Let $x = (x_1,\dots,x_d) \in \mathbb{R}^d$, $k = (k_1,\dots,k_d) \in \mathbb{N}_0^d$. Then $|x| = \sqrt{x_1^2+\ldots+x_d^2}$, $|k| = k_1+\ldots+k_d$, $k! = k_1!\cdots k_d!$, $x^k = \prod_{i=1}^d x_i^{k_i}$, $D^k = \prod_{i=1}^d \partial^{k_i}/\partial x_i^{k_i}$. Furthermore, if $x,\gamma \in \overline{\mathbb{R}^d_+}$ we also use $x^\gamma = \prod_{j=1}^d x_j^{\gamma_j}$. In this case, if $x_j = 0$ and $\gamma_j = 0$, we use the convention $0^0 = 1$.

For $j \in \mathbb{N}_0$ and $\gamma > -1$, the j-th Laguerre polynomial of order γ is defined by

$$L_j^\gamma(x) = \frac{x^{-\gamma}e^x}{j!}\frac{d^j}{dx^j}(e^{-x}x^{\gamma+j}), \quad x \geq 0.$$

For $\gamma = (\gamma_1,\dots,\gamma_d) \in \mathbb{R}^d$ such that $\gamma_j > -1$, $j = 1,\dots,d$ and $n \in \mathbb{N}_0^d$, the d-dimensional n-th Laguerre polynomial of order γ is defined by $L_n^\gamma(x) = L_{n_1}^{\gamma_1}(x_1)\dots L_{n_d}^{\gamma_d}(x_d)$. For $\gamma = 0$, we write $L_n(x)$ instead of $L_n^0(x)$.

The j-th Laguerre function (of order 0) is defined by $l_j(x) = L_j(x)e^{-x/2}$, $x \geq 0$, $j \in \mathbb{N}_0$ and in a d-dimensional case we have $l_n(x) = l_{n_1}(x_1)\dots l_{n_d}(x_d)$, $x \in \overline{\mathbb{R}^d_+}$, $n \in \mathbb{N}_0^d$. The Laguerre functions form an orthonormal basis for $L^2(\mathbb{R}^d_+)$. Also, they have a special role for the characterisation of the spaces $G_\alpha^\alpha(\mathbb{R}^d_+)$, $\alpha \geq 1$ and $g_\alpha^\alpha(\mathbb{R}^d_+)$, $\alpha > 1$ considered below.

2.1. Basic spaces

We denote by $\mathbb{R}^d_+$ the set $(0,\infty)^d$ and by $\overline{\mathbb{R}^d_+}$ its closure, i.e. $[0,\infty)^d$. The space $\mathcal{S}(\mathbb{R}^d_+)$ consists of all $f \in \mathcal{C}^\infty(\mathbb{R}^d_+)$ such that all derivatives $D^p f$, $p \in \mathbb{N}_0^d$, extend to continuous functions on $\overline{\mathbb{R}^d_+}$ and

$$\sup_{x\in\mathbb{R}^d_+} x^k|D^p f(x)| < \infty \ , \forall k,p \in \mathbb{N}_0^d.$$

Let $A > 0$. We denote by $G_{\alpha,A}^{\alpha,A}(\mathbb{R}^d_+)$ the space of all $f \in \mathcal{S}(\mathbb{R}^d_+)$ for which

$$\sup_{p,k\in\mathbb{N}_0^d} \frac{\|t^{(p+k)/2}D^p f(t)\|_2}{A^{|p+k|}k^{(\alpha/2)k}p^{(\alpha/2)p}} < \infty.$$

With the following seminorms

$$\sigma_{A,j}(f) = \sup_{p,k\in\mathbb{N}_0^d} \frac{\|t^{(p+k)/2}D^p f(t)\|_{L^2(\mathbb{R}^d_+)}}{A^{|p+k|}k^{(\alpha/2)k}p^{(\alpha/2)p}} + \sup_{\substack{|p|\leq j\\ |k|\leq j}} \sup_{t\in\mathbb{R}^d_+} |t^k D^p f(t)|, \ j \in \mathbb{N}_0,$$

one easily verifies that it becomes an (F)-space.

Define $G^{\alpha}_{\alpha}(\mathbb{R}^d_+) = \varinjlim_{A\to\infty} G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ and $g^{\alpha}_{\alpha}(\mathbb{R}^d_+) = \varprojlim_{A\to 0} G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$. Clearly, $G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$, $\alpha\geq 1$ and $g^{\alpha}_{\alpha}(\mathbb{R}^d_+)$, $\alpha>1$ are continuously injected into $\mathcal{S}(\mathbb{R}^d_+)$ i.e.

$$g^{\alpha}_{\alpha}(\mathbb{R}^d_+) \hookrightarrow G^{\alpha}_{\alpha}(\mathbb{R}^d_+) \hookrightarrow \mathcal{S}(\mathbb{R}^d_+).$$

We immediately have

$$(G^{\alpha}_{\alpha}(\mathbb{R}^d_+))' = \bigcup_{A\in\mathbb{N}} (G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+))' \tag{2.1}$$

in the set theoretical sense (see [8, (1.2), p.34]). Since the sequence $G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$, when $A\to\infty$, is reduced, we have

$$(g^{\alpha}_{\alpha}(\mathbb{R}^d_+))' = \bigcap_{A\in\mathbb{N}} (G^{\alpha,1/A}_{\alpha,1/A}(\mathbb{R}^d_+))' \tag{2.2}$$

in the set theoretical sense (see [8, (1.1), p.33]).

Let $\alpha\geq 1$ and $a>1$. We define $s^{\alpha,a}$ as the space of all complex sequences $\{a_n\}_{n\in\mathbb{N}^d_0}$ for which $\|\{a_n\}_{n\in\mathbb{N}^d_0}\|_{s^{\alpha,a}} = \sup_{n\in\mathbb{N}^d_0} |a_n| a^{|n|^{1/\alpha}} < \infty$. With this norm $s^{\alpha,a}$ becomes a (B)-space. We define $s^{\alpha} = \varinjlim_{a\to 1^+} s^{\alpha,a}$ (resp. $\sigma^{\alpha} = \varprojlim_{a\to\infty} \sigma^{\alpha,a}$). In particular, s^{α} is a (DFN)-space (resp. σ^{α} is a (FN)-space). The strong dual $(s^{\alpha})'$ of s^{α} is an (FN)-space of all complex valued sequences $\{b_n\}_{n\in\mathbb{N}^d_0}$ such that, for each $a>1$, $\|\{b_n\}_{n\in\mathbb{N}^d_0}\|_{(s^{\alpha,a})'} = \sum_{n\in\mathbb{N}^d_0} |b_n| a^{-|n|^{1/\alpha}} < \infty$ (resp. the strong dual $(\sigma^{\alpha})'$ of σ^{α} is a (DFN)-space of all complex valued sequences $\{b_n\}_{n\in\mathbb{N}^d_0}$ such that there exists $a>1$ such that $\|\{b_n\}_{n\in\mathbb{N}^d_0}\|_{(s^{\alpha,a})'} < \infty$).

Theorem 2.1. *([6, Theorem 5.7.]) Let $\alpha\geq 1$. For $f\in L^2(\mathbb{R}^d_+)$ let $a_n = \int_{\mathbb{R}^d_+} f(t) l_n(t)dt$, $n\in\mathbb{N}^d_0$. Then $f\in G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ if and only if there exist $c>0$ and $a>1$ such that $|a_n|\leq c a^{-|n|^{1/\alpha}}$.*

With small modifications of the arguments (using the closed graph theorem for an F-space), we have the next theorem

Theorem 2.2. *Let $\alpha>1$. For $f\in L^2(\mathbb{R}^d_+)$ let $a_n = \int_{\mathbb{R}^d_+} f(t) l_n(t)dt$, $n\in\mathbb{N}^d_0$. Then $f\in g^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ if and only if for every $a>1$ there exists $c>0$ such that $|a_n|\leq c a^{-|n|^{1/\alpha}}$.*

Theorem 2.3. *([6, Theorem 6.1]) Let $\alpha\geq 1$. The mapping $\iota : G^{\alpha}_{\alpha}(\mathbb{R}^d_+)\to s^{\alpha}$, $\iota(f) = \{\langle f, l_n\rangle\}_{n\in\mathbb{N}^d_0}$, is a topological isomorphism between $G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ and s^{α}.*

For each $f\in G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$, $\sum_{n\in\mathbb{N}^d_0}\langle f, l_n\rangle l_n$ is summable to f in $G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$.

In a similar way, we prove the next theorem.

Theorem 2.4. *Let $\alpha>1$. The mapping $\iota : g^{\alpha}_{\alpha}(\mathbb{R}^d_+)\to \sigma^{\alpha}$, $\iota(f) = \{\langle f, l_n\rangle\}_{n\in\mathbb{N}^d_0}$, is a topological isomorphism between $g^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ and σ^{α}.*

For each $f\in g^{\alpha}_{\alpha}(\mathbb{R}^d_+)$, $\sum_{n\in\mathbb{N}^d_0}\langle f, l_n\rangle l_n$ is summable to f in $g^{\alpha}_{\alpha}(\mathbb{R}^d_+)$.

The last four results are crucial and we will often tacitly apply them throughout the rest of this article.

3. G- and g- kernels

Firstly, we state the kernel theorems:

Theorem 3.1. *([6, Theorem 6.4.]) Let $\alpha \geq 1$. We have the following canonical isomorphism:*

$$(G_\alpha^\alpha(\mathbb{R}_+^{d_1}))'\hat{\otimes}(G_\alpha^\alpha(\mathbb{R}_+^{d_2}))' \cong (G_\alpha^\alpha(\mathbb{R}_+^{d_1+d_2}))' \cong \mathcal{L}(G_\alpha^\alpha(\mathbb{R}_+^{d_1}), (G_\alpha^\alpha(\mathbb{R}_+^{d_2}))'). \tag{3.1}$$

Theorem 3.2. *Let $\alpha > 1$. We have the following canonical isomorphism:*

$$(g_\alpha^\alpha(\mathbb{R}_+^{d_1}))'\hat{\otimes}(g_\alpha^\alpha(\mathbb{R}_+^{d_2}))' \cong (g_\alpha^\alpha(\mathbb{R}_+^{d_1+d_2}))' \cong \mathcal{L}(g_\alpha^\alpha(\mathbb{R}_+^{d_1}), (g_\alpha^\alpha(\mathbb{R}_+^{d_2}))'). \tag{3.2}$$

Proof. The proof for $(g_\alpha^\alpha(\mathbb{R}_+^d))'$ can be obtained it the same way as for $(G_\alpha^\alpha(\mathbb{R}_+^d))'$ in Theorem 3.1. □

The isomorphisms (3.1) and (3.2) require some comment. In what follows we use the convention that if T is a linear and continuous operator from $G_1^1(\mathbb{R}_+^{d_1})$ to $(G_1^1(\mathbb{R}_+^{d_2}))'$, and $g \in (G_1^1(\mathbb{R}_+^{d_0}))'$, then $T \otimes g$ is the linear and continuous operator from $G_1^1(\mathbb{R}_+^{d_1})$ to $(G_1^1(\mathbb{R}_+^{d_2+d_0}))'$, given by

$$(T \otimes g) : f \mapsto (Tf) \otimes g.$$

The following theorem is the main result of this paper.

Theorem 3.3. *Let T be a linear and continuous operator from $G_1^1(\mathbb{R}_+^{d_1})$ to $(G_1^1(\mathbb{R}_+^{d_2}))'$ with the kernel K, and let $d_0 \geq \min(d_1, d_2)$. Then the following is true:*

- *(R) If $\alpha \geq 1$ and $K \in G_\alpha^\alpha(\mathbb{R}_+^{d_2+d_1})$, then there are operators T_1 and T_2 with kernels $K_1 \in G_\alpha^\alpha(\mathbb{R}_+^{d_0+d_1})$ and $K_2 \in G_\alpha^\alpha(\mathbb{R}_+^{d_2+d_0})$, respectively, such that $T = T_2 \circ T_1$.*
- *(B) If $\alpha > 1$ and $K \in g_\alpha^\alpha(\mathbb{R}_+^{d_2+d_1})$, then there are operators T_1 and T_2 with kernels $K_1 \in g_\alpha^\alpha(\mathbb{R}_+^{d_0+d_1})$ and $K_2 \in g_\alpha^\alpha(\mathbb{R}_+^{d_2+d_0})$, respectively, such that $T = T_2 \circ T_1$.*

Remark 3.4. Let $\alpha \geq 1$. An operator with kernel in $G_\alpha^\alpha(\mathbb{R}_+^{2d})$ is sometimes called a regularizing operator with respect to $G_\alpha^\alpha(\mathbb{R}_+^d)$, because it extends uniquely to a continuous mapping from $(G_\alpha^\alpha(\mathbb{R}_+^d))'$, into $G_\alpha^\alpha(\mathbb{R}_+^d)$.

Let $\alpha > 1$. An operator with kernel in $g_\alpha^\alpha(\mathbb{R}_+^{2d})$ is sometimes called a regularizing operator with respect to $g_\alpha^\alpha(\mathbb{R}_+^d)$.

Proof. First we assume that $d_0 = d_1$, and start to prove (R). Let $l_{d,n}(x)$ be the Laguerre function on $\mathbb{R}_+^d$ of order $n \in \mathbb{N}^d$. Then K possesses the expansion

$$K(x, y) = \sum_{n\in\mathbb{N}^{d_2}} \sum_{k\in\mathbb{N}^{d_1}} a_{n,k} l_{d_2,n}(x) l_{d_1,k}(y), \tag{3.3}$$

where the coefficients $a_{n,k}$ satisfy

$$\sup_{n,k} |a_{n,k} e^{A(|n|^{1/\alpha}+|k|^{1/\alpha})}| < \infty, \tag{3.4}$$

for some $A > 0$. Let $z \in \mathbb{R}_+^{d_1}$ and

$$\begin{aligned} K_{0,1}(z,y) &= \sum_{n,k\in\mathbb{N}^{d_1}} c_{n,k} l_{d_1,n}(z) l_{d_1,k}(y) \\ K_{0,2}(x,z) &= \sum_{n\in\mathbb{N}^{d_2}} \sum_{k\in\mathbb{N}^{d_1}} b_{n,k} l_{d_2,n}(x) l_{d_1,k}(z), \end{aligned} \tag{3.5}$$

where

$$c_{n,k} = \chi_{n,k} e^{-\frac{A}{2}|n|^{1/\alpha}} \quad \text{and} \quad b_{n,k} = a_{n,k} e^{\frac{A}{2}|k|^{1/\alpha}}$$

and $\chi_{n,k}$ is the Kronecker delta. Then we have

$$\int K_{0,2}(x,z) K_{0,1}(z,y)\,dz = \sum_{n\in\mathbb{N}^{d_2}} \sum_{k\in\mathbb{N}^{d_1}} a_{n,k} l_{d_2,n}(x) l_{d_1,k}(y) = K(x,y).$$

Hence, if T_j is the operator with kernel $K_{0,j}$, $j = 1,2$, then $T = T_2 \circ T_1$. Furthermore,

$$\sup_{n,k} |b_{n,k} e^{\frac{A}{2}(|n|^{1/\alpha}+|k|^{1/\alpha})}| \le \sup_{n,k} |a_{n,k} e^{A(|n|^{1/\alpha}+|k|^{1/\alpha})}| < \infty$$

and

$$\sup_{n,k} |c_{n,k} e^{\frac{A}{4}(|n|^{1/\alpha}+|k|^{1/\alpha})}| = \sup_{n} |e^{-\frac{A}{2}|n|^{1/\alpha}} e^{\frac{A}{2}|n|^{1/\alpha}}| < \infty.$$

This implies that $K_{0,1} \in G_\alpha^\alpha(\mathbb{R}_+^{d_1+d_1})$ and $K_{0,2} \in G_\alpha^\alpha(\mathbb{R}_+^{d_2+d_1})$ (see Theorem 2.1). If we put $K_1 = K_{0,1}$ and $K_2 = K_{0,2}$, we proved (R) in the case when $d_0 = d_1$.

In order to prove (B) assume that $K \in g_\alpha^\alpha(\mathbb{R}_+^{d_2+d_1})$ and let $a_{n,k}$ be the same as the above. Then (3.4) holds for any $A > 0$, which implies that if $N \ge 0$ is an integer, then

$$\Sigma_N = \sup\{|k| : |a_{n,k}| \ge e^{-2(N+1)(|n|^{1/\alpha}+|k|^{1/\alpha})} \text{ for some } n \in \mathbb{N}^{d_2}\} \tag{3.6}$$

is finite. Let $I_1 = \{k \in \mathbb{N}^{d_1} : |k| \le \Sigma_1 + 1\}$ and define inductively

$$I_j = \{k \in \mathbb{N}^{d_1} \setminus I_{j-1} : |k| \le \Sigma_j + j\},\ j \ge 2.$$

Then

$$I_j \cap I_k = \emptyset \ \text{ when } j \ne k, \ \text{ and } \ \mathbb{N}^{d_1} = \bigcup_{j\ge 1} I_j.$$

Let $K_{0,1}$ and $K_{0,2}$ be as in (3.5) where

$$c_{n_1,k} = \chi_{n_1,k} e^{-j|k|^{1/\alpha}} \quad \text{and} \quad b_{n_2,k} = a_{n_2,k} e^{j|k|^{1/\alpha}},$$

$n_1 \in \mathbb{N}^{d_1}$, $n_2 \in \mathbb{N}^{d_2}$ and $k \in I_j$. If T_j is the operator with kernel $K_{0,j}$, $j = 1, 2$, then it follows that $T = T_2 \circ T_1$. Moreover, if $A > 0$ we have

$$\sup_{n,k} |b_{n,k} e^{A(|n|^{1/\alpha}+|k|^{1/\alpha})}| \leq J_1 + J_2,$$

where

$$J_1 = \sup_{j \leq A+1} \sup_n \sup_{k \in I_j} |b_{n,k} e^{A(|n|^{1/\alpha}+|k|^{1/\alpha})}| \tag{3.7}$$

and

$$J_2 = \sup_{j > A+1} \sup_n \sup_{k \in I_j} |b_{n,k} e^{A(|n|^{1/\alpha}+|k|^{1/\alpha})}|. \tag{3.8}$$

We will prove that J_1 and J_2 are finite. Since in (3.7) we have the finite numbers of k, from (3.4) and the definition of $b_{n,k}$ we obtain that J_1 is finite.

For J_2 we have

$$\begin{aligned} J_2 &= \sup_{j > A+1} \sup_n \sup_{k \in I_j} |a_{n,k} e^{A|n|^{1/\alpha}+(A+j)|k|^{1/\alpha}}| \\ &\leq \sup_{j > A+1} \sup_n \sup_{k \in I_j} |e^{-2j(|n|^{1/\alpha}+|k|^{1/\alpha})} e^{A|n|^{1/\alpha}+(A+j)|k|^{1/\alpha}}| < \infty, \end{aligned}$$

where the first inequality follows from (3.6). Hence,

$$\sup_{n,k} |b_{n,k} e^{A(|n|^{1/\alpha}+|k|^{1/\alpha})}| < \infty,$$

which implies that $K_{0,2} \in g^{\alpha}_{\alpha}(\mathbb{R}^{d_2+d_1}_+)$.

In a similar way if we replace $b_{n,k}$ with $c_{n,k}$ in the definition of J_1 and J_2) we obtain

$$\sup_{n,k} |c_{n,k} e^{A(|n|^{1/\alpha}+|k|^{1/\alpha})}| < \infty,$$

which implies $K_{0,1} \in g^{\alpha}_{\alpha}(\mathbb{R}^{d_1+d_1}_+)$ and (B) follows in the case $d_0 = d_1$.

Next, assume that $d_0 > d_1$ and let $d = d_0 - d_1 \geq 1$. Then we set

$$K_1(z, y) = K_{0,1}(z_1, y) l_{d,0}(z_2) \quad \text{and} \quad K_2(x, z) = K_{0,2}(x, z_1) l_{d,0}(z_2),$$

where $z_1 \in \mathbb{R}^{d_1}_+$, $z_2 \in \mathbb{R}^{d}_+$ and hence, $z = (z_1, z_2) \in \mathbb{R}^{d_0}_+$. Next, we obtain

$$\int_{\mathbb{R}^{d_0}_+} K_2(x, z) K_1(z, y) dz = \int_{\mathbb{R}^{d_1}_+} K_{0,2}(x, z_1) K_{0,1}(z_1, y) dz_1 = K(x, y).$$

In this case the assertion (R) and (B) follows from the equivalences

$$K_1 \in G^{\alpha}_{\alpha}(\mathbb{R}^{d_0+d_1}_+) \ (g^{\alpha}_{\alpha}(\mathbb{R}^{d_0+d_1}_+)) \iff K_{0,1} \in G^{\alpha}_{\alpha}(\mathbb{R}^{d_1+d_1}_+) \ (g^{\alpha}_{\alpha}(\mathbb{R}^{d_1+d_1}_+))$$

and

$$K_2 \in G^{\alpha}_{\alpha}(\mathbb{R}^{d_2+d_0}_+) \ (g^{\alpha}_{\alpha}(\mathbb{R}^{d_2+d_0}_+)) \iff K_{0,1} \in G^{\alpha}_{\alpha}(\mathbb{R}^{d_2+d_1}_+) \ (g^{\alpha}_{\alpha}(\mathbb{R}^{d_2+d_1}_+)).$$

It remains to prove the result in the case $d_0 \geq d_2$. The rules of d_1 and d_2 are interchanged when taking the adjoint opeartors. Hence, the result follows from the first part of the proof in combination with the facts that G^{α}_{α} and g^{α}_{α} are invariant under pullbacks of bijective linear transformations i.e. $(x, y) \mapsto F(x, y)$ belongs to $G^{\alpha}_{\alpha}(\mathbb{R}^{d_1}_+ \times \mathbb{R}^{d_2}_+)$ if and only if $(y, x) \mapsto \overline{F(x, y)}$ belongs to $G^{\alpha}_{\alpha}(\mathbb{R}^{d_2}_+ \times \mathbb{R}^{d_1}_+)$. The proof is complete. $\square$

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