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LÉVY PROCESSES, SUBORDINATORS AND CRIME MODELLING

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Abstract. We investigate some properties of Lévy processes in the context of subordinators. Lévy walks can be represented as subordinators of random walks and Lévy flights are random walks with trajectories composed of self-similar jumps. Lévy processes provide a framework for modelling many physical phenomena. In this paper we consider, as an illustration, crime models based on Brownian motion and Lévy flights. We propose an efficient implementation of the models by using high performance computing techniques. Numerical simulations on different scenarios allows us to analyze some properties of the processes, particularly regimes of aggregation and first passage time.

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1. Introduction

Applications of probability theory and stochastic processes arise in science and engineering, e.g. quantum mechanics, finance, biomathematics, etc. For example, in financial mathematics modelling fluctuations rely on Lévy processes. Lévy processes are stochastic processes with stationary and independent increments properties which satisfy a mild sample path regularity condition [1]. They are examples of random motion whose sample paths are right-continuous and have at most countable number of random jump discontinuities occurring at random times, on each finite time interval. Semimartingales and Markov processes are special subclasses of Lévy processes, which include a number of very important processes: Brownian motion, Poisson process, stable and self-decomposable processes and subordinators [18]. Precisely, Brownian motion and homogeneous Poisson process appear in models of finance and insurance. The key difference between these two processes lies in the sample path behavior, Brownian motion has continuous sample paths whereas Poisson process is a counting process, a jump process. Lévy flight is a class of random walks that is characterized by a heavy tailed step length distribution [13].

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Lévy processes have an important role in modelling social behavior, urban evolution, phenomena and processes appearing in biology and natural sciences. They appear, for example in predator-pray models, foraging patterns of animals [23], plasma physics and cell science [7], solute transport in heterogeneous media [24], social aspects, financial models [10], crime models [6, 22] etc.

Simple random walks models, also known as gambler ruin models, are used to approximate one dimensional diffusions, such as Brownian motion, where physical particle is exposed to a great number of molecular collisions which produce random motion. The variance of diffusion is scaled linearly with time. Moreover, the Central limit theorem can be applied to random walks without long-term correlations [8]. In this paper we focus on Lévy flights, stable pure jump Lévy processes, which model anomalous diffusions, i.e. the diffusion phenomena whose variance is not linear. The nonlinearity of variance is due to non Gaussian power-law tails of the probability distribution function, i.e. they are of the form t^γ, $\gamma \neq 1$. Fractional diffusion equations are widely used to describe anomalous diffusion processes [20, 25]. Especially, Lévy walks are Lévy flights with finite velocity. The first found super diffusion phenomenon is a fractional Brownian motion. The variance of fractional Brownian motion with the Hurst parameter $H \in (0, 1)$ can be related to the power scaling exponent, i.e. t^{2H}. For $H = \frac{1}{2}$ we retain a standard Brownian motion. In [11] the authors showed how a power law truncated Lévy stable distribution evolves in time to a distribution with power law asymptotics. Subordination of a process by another one, also called random time-change, is a technique from stochastic analysis widely used for constructing new Markov processes or strongly continuous semigroups [18]. It was first introduced by Bochner. We study simple random walks, continuous-time random walks and Lévy flights, where in the subordination a new operational time scale is used to measure the time between two steps of the random walk.

In the class of Lévy processes that can be written as Brownian motion time changed by independent Lévy subordinator, a question concerning the precise relation between the standard first passage time arise [12]. The first passage time, also called the hitting time, is the time at which a given process reaches a given subset of the state space. The first passage time is also referred as the inverse subordinator [26]. Exit times and return times are also examples of hitting times [8]. First passage problems arise in financial mathematics, e.g. credit risk modelling, pricing barrier options etc. They also arise in the study of fractional kinetics and the scaling limits of continuous random walks [26].

In this paper, we investigate the properties of different classes of Lévy processes that are applied for modelling crime. Following [6, 22], we propose an efficient implementation of the models by using high performance computing techniques, in particular we speed up simulations of the Lévy flight model [6] by using MEX functions. This allows us to perform simulations of the Lévy model at the computational cost of the Brownian motion based model, making feasible to simulate crime modelling and other real life applications. We also discuss several regimes of aggregation, like hotspots of high criminal activity and we provide a study of the fist passage time. In addition we generalize the one dimensional Lévy flight model proposed in [6] to a two dimensional one allowing a detailed comparison with the model based on Brownian motion. The latter together with the efficient implementation are the main contributions of this paper. We point out that the same ideas can be applied straightforward to other applications in financial mathematics or biology.

A strategy for the police to adapt dynamically to changing crime patterns has been introduced in [29]. The authors use optimization techniques to solve the problem numerically. In case when the resulting model is linear (or can be linearized), a linear quadratic control problem approach can be used making the computation with real data feasible as efficient numerical solvers have been proposed, e.g. [2, 3, 4, 16]. The latter we intend to investigate in our future work.

The paper is organized as follows: in Section 2 we present a brief overview of Lévy processes, subordinators and random walks. Then, in Section 3 we describe the crime models for burglar locomotion. Afterwards, in Section 4 we perform numerical simulations for different scenarios and we study the first passage time. Finally, in Section 5 we present some conclusions.

2. Theoretical background

2.1. Lévy processes

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$, $T > 0$. The σ-algebra $\mathcal{F}_T$ represents the information available in the model up to time T. A real valued stochastic process $X = X_t(\omega) = (X_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called an one dimensional *Lévy process* if $X_0 = 0$ *a.s.*, X has independent and stationary increments and X is stochastically continuous. Both Brownian motion and Poisson process are temporary homogeneous Lévy processes, i.e. the probability distribution of the increment $X_{t+h} - X_t$, for $h > 0$ is independent of t. Almost all trajectories of Brownian motion are continuous, while those of Poisson process are discontinuous, and they increase only by jumps of unit magnitude. Other important examples of Lévy processes are compound Poisson process, Gamma process, α-stable processes, self decomposable processes.

A real valued random variable Y has an *infinitely divisible distribution* if for each $n \in \mathbb{N}$ there exist a sequence of independent identically distributed random variables $Y_{1,n}, ..., Y_{n,n}$ such that Y and $Y_{1,n} + ... + Y_{n,n}$ have equal distribution. Hence, the probability law μ of a real valued random variable is infinitely divisible if it can be decomposed to n-fold convolution $\mu = \mu_n^{*n}$, $n \in \mathbb{N}$, for some probability law μ_n. The *characteristic exponent* of a random variable Y is defined by

$$\Psi(u) = -\log \mathbb{E}(e^{iuY}), \qquad u \in \mathbb{R}.$$

Therefore, Y has an infinitely divisible distribution if for all $n \in \mathbb{N}$ there exists a characteristic exponent Ψ_n of a probability distribution such that

$\Psi(u) = n\Psi_n(u)$, $u \in \mathbb{R}$. Complete characterization of an infinitely divisible distribution in terms of its characteristic exponent is given by the Lévy-Khintchine formula [1, 21]

Theorem 2.1 (The Lévy-Khintchine formula)**.** *The probability law μ of a real valued random variable is infinitely divisible with the characteristic exponent Ψ*

$$\int_{\mathbb{R}} e^{i\theta x}\, \mu(dx) \;=\; e^{-\Psi(\theta)}, \qquad \theta \in \mathbb{R} \tag{2.1}$$

if and only if there exists the triplet (a, σ, Π), where $a \in \mathbb{R}$, $\sigma \geq 0$ and Π is the Lévy measure, i.e. the measure concentrated on $\mathbb{R} \setminus \{0\}$ which satisfies

$$\int_{\mathbb{R}} (1 \wedge x^2)\, \Pi(dx) \;<\; \infty, \tag{2.2}$$

such that

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x 1_{|x|<1})\, \Pi(dx), \quad \theta \in \mathbb{R}. \tag{2.3}$$

The proof of the Lévy-Khintchine formula can be found, for example in [1, 21].

It is clear that every Lévy process can be related to some infinite divisible distribution. If X is a Lévy process, then X_t has an infinitely divisible distribution for each $t \geq 0$. Moreover, for every infinite divisible process, it is possible to construct a Lévy process X_t so that X_1 has the given infinite divisible distribution. This is the statement of the following theorem.

Theorem 2.2. *Given a triplet (a, σ, Π), where $a \in \mathbb{R}$, $\sigma \geq 0$ and Π is the Lévy measure such that* (2.2) *is satisfied. Then there exists a unique Lévy process X such that (2.1) and (2.3) hold.*

Note that every Lévy process can be decomposed into the sum of three terms. The first term, seen as a continuous part of a Lévy process is represented by a Brownian motion with drift, the second term represents a compensated sum of small jumps, while the third term describes the large jumps and is represented by compound Poisson process.

2.1.1. Subordinators

Lévy processes with almost sure nondecreasing paths are called *subordinators.* Such processes can be thought of as random models of time evolution. Their characteristic exponent (2.3) is of the form

$$\Psi(\theta) = ia\theta + \int_0^{+\infty} (1 - e^{ix\theta})\, \Pi(dx)\,.$$

A wide class of Lévy processes appearing in applications is obtained by subordination of Brownian motion with drift [14, 18].

Theorem 2.3. *Let $X = (X_t)_{t\geq 0}$ be a Lévy process with the characteristic exponent Ψ and $S = (S_\tau)_{\tau\geq 0}$ an independent subordinator defined on the same probability space as X with the characteristic exponent Φ. Then, the process $Y_\tau = (X_{S_\tau})_{\tau\geq 0}$ is a Lévy process with the characteristic exponent $\Phi \circ \Psi$.*

The proof of the previous theorem can be found for example in [1, 15]. In [9] the authors suggested that the value of a risky asset can evolve as the process on an abstract time scale suitable to the rate of business translations called the *business* (operational) *time.* The subordinator S represents the link between business time and real time. Particularly, in the models of crime activities we are considering in Section 3, the position Y of the burglar at the certain time is distributed by S_t and follows the process $Y_t = X \circ S_t = X_{S_t}$. This means that at real time $t > 0$, S_t units of business ("robbing") time pass and the value of a burglar activity is positioned at X_{S_t}.

A *stable* Lévy process is a Lévy process X for which X_t is a stable random variable for all $t \geq 0$. These processes are important in applications because they display the self-similarity property.

A random time τ taking values in $[0, +\infty)$ is a *stopping time* with respect to the process X if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. The σ-algebra associated with τ is defined by $\mathcal{F}_\tau = \{B \in \mathcal{F} : B \cap \{\tau \leq t\} \in \mathcal{F}_t, \text{ for all } t \geq 0\}$. Stopping time can be seen as a random time that does not require knowledge about the future. From the available information we can only tell whether or not $\tau \leq t$ holds [18].

Example 2.4. *Let B_t, $t \geq 0$ be a standard Brownian motion, $\sigma > 0$, $\theta > 0$ and S an independent subordinator. Then, $W_t = \sigma B_t + \theta t$, $t \geq 0$ is a Brownian motion with drift θ and scaling (dispersion) coefficient σ [14]. A subordinated process B_{S_t} performs jumps in random time, where time is passing according to S_t, $t \geq 0$. For a linear transformation, when $S_t = b^2 t$ the process $B_{S_t} = B_{b^2 t}$, $b \neq 0$ is a Brownian motion with drift b^2 and dispersion coefficient b.*
We subordinate Brownian motion and by time changing $X = W \circ S$ construct a Lévy process $X_t = W_{S_t} = \sigma B_{S_t} + \theta S_t$. Let τ be a stopping time with respect to the process W. Define a process $Y_t = W_{t+\tau} - W_\tau$, $t \geq 0$. Then $Y = (Y_t)_{t\geq 0}$ is also a Brownian motion with the same drift and scale parameter.

The *first hitting time*, also called *the first passage time* and *the inverse subordinator*, is the time at which a given process reaches a given subset of the state space. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space and let M be a measurable state space. Let $X : \Omega \times [0, +\infty) \to M$ be a stochastic process and A a measurable subset of the state space M. Then, the first hitting time $\tau_A : \Omega \to [0, +\infty)$ is the random variable defined by

$$\tau_A(\omega) = \inf\{t \geq 0 : X_t(\omega) \in A\}.$$

Any stopping time is a hitting time for a properly chosen process and target set. Precisely, the hitting time of a measurable set, for a progressively measurable process, is a stopping time. Progressively measurable processes include, in particular, all right and left-continuous adapted processes. The converse also holds, every stopping time defined with respect to a filtration over a real valued time index set can be represented by a hitting time [1]. The mean first passage time is called the *renewal function*. All the moments of the first passage time can be computed if the renewal function is first computed [26].

Example 2.5. *For $\alpha \in (0,1)$ and $u \geq 0$ it holds*

$$u^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} (1 - e^{-ux}) \frac{dx}{x^{1+\alpha}}.$$

Thus, there exists α-stable subordinator S *with the characteristic exponent* $\Psi(u) = u^\alpha$ *and the characteristics of* S *are* $(0, \lambda)$, *where* $\lambda(dx) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{dx}{x^{1+\alpha}}$. *Moreover, the renewal function for the* α*-stable subordinator is given by* $\frac{t^\alpha}{\Gamma(\alpha+1)}$. *The* $\frac{1}{2}$*-stable subordinator is called the Lévy subordinator and its probabilistic interpretation is the first hitting time for one dimensional Brownian motion* B_t, *i.e.* $S_t = \inf\{s > 0 : B_s = \frac{t}{\sqrt{2}}\}$. *Recall, the probability density function of a random variable* $S_{b\sqrt{2}} = \inf\{s > 0 : B_s = b\}$ *obtained by the reflection principle is given in the close form* $\frac{|b|}{\sqrt{2\pi t^3}} e^{-\frac{b^2}{2t}}$. *Moreover, for an* α*-stable subordinator* S *independent on a Brownian motion* B, *the process* $Z = B_{S_\tau}$ *is rotationally invariant* 2α*-stable process. For more details we refer to [1, 14, 18, 26].*

2.2. Fractional Laplacian

We recall briefly some facts on fractional discrete Laplace operator. Let $\mathbb{Z}_h = \{h_j, j \in \mathbb{Z}\}$ be a mesh on $\mathbb{R}$ of size $h > 0$. The *discrete Laplacian* Δ_h is given by

$$-\Delta_h u_j = -\frac{1}{h^2} (u_{j+1} - 2u_j + u_{j-1}),$$

where u_j, $j \in \mathbb{Z}$ ia a function on $\mathbb{Z}_h$. The *fractional discrete Laplacian* $(-\Delta_h)^s$ is defined by

$$(-\Delta_h)^s u_j = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\Delta_h} u_j - u_j) \frac{dt}{t^{1+s}},$$

where $v_j(t) = e^{t\Delta)h} u_j$ is the solution of the semidiscrete heat equation

$$\begin{aligned} \partial_t v_j &= \Delta_h v_j, && \text{in } \mathbb{Z}_h \times (0, \infty), \\ v_j(0) &= u_j, && \text{on } \mathbb{Z}_h. \end{aligned}$$

2.3. Random walk

Random walk is a process by which randomly moving objects wander away from where they started. The simplest random walk is an one dimensional simple random walk. Two or three dimensional random walks are commonly found in nature. For example, when gas particles bounce around in a room, changing direction every time they collide with another particle, it is random walk that determines how long it will take them to travel from one location to another. The particles in a drop of ink added to water will spread out partially due to a random walk [13]. For all random walks it holds that the total distance traveled from where it started is approximately $\sqrt{N}$, where N is the number of steps taken. This is called the *universal scaling property* [8]. Random walks have various interesting mathematical properties that vary depending on the dimension in which the walk occurs [5, 7, 13, 20].

Random walks generated by steps taken at regular intervals can be generalized by introducing a probability density function for pausing times between successive steps in the walk, i.e. the *waiting time distribution* [13].

2.3.1. Simple random walk

We recall briefly the concept of an one dimensional simple random walk [8, 13]. The walker starts at the zero on a number line. It can move one step, in each moment of time, either forward or backward, with same probability. In this case only the current location of the walker determines the random motion, the past is not relevant. The position of the walk after N steps can be represented as a sum of consecutive displacements ΔX_n, i.e.

$$X_N = \sum_{i=1}^{N} \Delta X_i, \tag{2.4}$$

where ΔX_i are independent identically distributed random variables with the variance σ^2. Each displacement has the same probability density function $p(\Delta x)$. In the case of symmetric single step, from the strong law of large numbers it follows that the average velocity vanishes, i.e. $\lim\limits_{N\to\infty} \frac{X_N}{N} = \mathbb{E}(\Delta X_1) = 0$, where $\mathbb{E}$ is the expectation with respect to the measure $\mathbb{P}$. Since the walker makes large excursions both forward and backward, so that $\limsup\limits_{N\to\infty} X_N = +\infty$, $\liminf\limits_{N\to\infty} X_N = -\infty$ $\mathbb{P}$-a.s. a random walker visits the origin, and any other integer point on the line, infinitely often. Fluctuations of the random walk can be characterized by the Central limit theorem. The distribution of X_N is asymptotically normal with zero mean and variance $\sigma^2 N$. Clearly, the probability density function $f_Y(y, N)$ for the scaled position $Y_N = \frac{X_N}{\sqrt{N}} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \Delta X_i$ is Gaussian and independent of N when $N \to \infty$. Thus the universal scaling relation for ordinary random walks $X_N \sim \sqrt{N}$ follows and the probability density function $f_X(x, N)$ for the position X_N is asymptotically Gaussian.

Example 2.6. *Brownian motion is a process that could be obtain as a limit of a simple random walk. Moreover, one dimensional Brownian motion can be represented as a random Fourier series*

$$B_t = \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \frac{\sin\left(\pi t(n + \frac{1}{2})\right)}{n + \frac{1}{2}} Z_n,$$

for each $t \geq 0$, where Z_n, $n \in \mathbb{N}_0$ is a sequence of independent identically distributed standardized Gaussian random variables [1].

2.3.2. Lévy flights

Lévy flights are a class of non Gaussian random processes whose stationary increments are distributed according to a Lévy stable distribution. The probability density functions of Lévy stable laws decay in asymptotic power law form with diverging variance. Lévy flights are stochastic processes which satisfy the Markov property, whose individual jumps have lengths that are distributed with the probability density function decaying at large x as $|x|^{-1-\beta}$ with $0 \leq \beta \leq 2$. Due to the divergence of their variance, they can have extremely long jumps, typical trajectories are self-similar. Similar to the emergence of the Gaussian as limit distribution of independent identically distributed random variables with finite variance due to the Central limit theorem, Lévy stable distributions represent the limit distributions of independent identically distributed random variables with diverging variance [13].

Lévy flights belong to a class of random walks for which the Central limit theorem does not apply [13, 17]. They can be defined in a similar way as simple random walks, by a sum of independent identically distributed random increments (2.4). If the single step probability density functions have tails, i.e. when the second moment of the single step is divergent

$$p(\Delta x) \sim \frac{1}{\Delta x^{1+\beta}}, \quad 0 \leq \beta \leq 2,$$

one can apply the Lévy-Khinchin theorem. It is a generalization of the Central limit theorem and it states if the position of a Lévy flight is scaled by $Y_N = \frac{X_N}{N^{\frac{1}{\beta}}}$, and the scaled variable has a probability density function independent of N when $N \to \infty$. The limiting density $f_{Y,\beta}$ is refered as a stable Lévy law of index β which is not Gaussian. Asymptotically, the limiting density has the same power law behavior as the single step distribution $f_{Y,\beta} \sim \frac{1}{|y|^{1+\beta}}$. Hence, the position of a Lévy flight scales superdiffusively with step number $X_N \sim N^{\frac{1}{\beta}}$.

(a) (b) (c)

Figure 1: Paths of a Brownian motion (a), a 2D Lévy flight (b) and a 3D Lévy flight

Trajectories of a Lévy flight and a Brownian motion differ. For an illustration, in Figure 1 are ploted sample paths of a Brownian motion (a), of a Lévy flight of 1000 steps in two dimensions (b) and in three dimensions (c). Paths of Brownian motion are continuous and almost nowhere differential. Each part of the path is again a trajectory of a Brownian motion [14]. On the other hand, Lévy flight is characterized by many small moves combined with a few longer trajectories, i.e. longer rutes are taken on occasion. The characteristic size of the Lévy flight is the size of the largest step and the flight is self- similar at higher magnifications [13].

2.3.3. Continuous time random walks

Temporally continuous random walks can be constructed from time discrete simple random walks by identifying the step number N with the time elapsed t and associated time increment $\Delta t = \frac{t}{N}$ between successive steps. A generalization of this concept leads to the continuous time random walk [13]. Its simple version is defined by two probability density functions, one for spatial displacements $g_1(\Delta x)$ and one for random temporal increments $g_2(\Delta t)$. Thus, continuous time random walk consists of pairwise random independent events, spacial displacement Δx and temporal increment Δt and the probability density functions

$$p(\Delta x, \Delta t) = g_1(\Delta x)\, g_2(\Delta t).$$

After N steps, the position of the walker is given by $X_N = \sum_{i=1}^{N} \Delta x_i$ and the time elapsed is $T_N = \sum_{i=1}^{N} \Delta t_i$. The probability density function $f(x,t)$ of the position X_t after time t is calculated in [17].

Particularly, the *ordinary diffusion* occurs when the variance of the spatial steps and the expectation of the temporal increments exist. Then, the continuous time random walks are equivalent to Brownian motion on large spatio-temporal scales. When the spatial displacements are drawn from a power law probability density function and the temporal increment have finite expectation, the continuous time random walk is equivalent to ordinary *Lévy flights* with a superdiffusive scaling with time $X_t \sim t^{\frac{1}{\beta}}$. When the ordinary spatial steps are combined with a power law in the probability density function, the time between successive spatial increments can be very long, effectively slowing down the random walk. In this case one obtains the scaling relation $X_t \sim t^{\frac{\alpha}{2}}$. Since $\alpha < 1$ these processes are *subdiffusive* and are refered to *fractional Browniam motion.*

Random walks in random environments are studied in [5].

3. Crime modelling as an application

In this section as an application of Lévy proceses and Lévy flights we discuss different models of criminal activity (residential burglaries where the targets are stationary). In particular, we discuss models proposed in [6, 22]. The first model, also called the UCLA model, describes the locomotion of criminal agents by Brownian motion [22] and the second one is based on Lévy flights motion [6]. The second model relies on the advantages of Lévy distributed excursion lengths, which optimize the search compared to the Brownian search. The considered models may be also applied for modelling processes appearing in nature such as the foraging behavior of bacteria and animals or the spreading of diseases [7, 23].

Due to social, economic, geographic structure the criminal activity is not distributed uniformly. The regions with elevated criminal activities are called hotspots. Different spatial-temporal methods for crime analysis were studied in details in [19, 28]. In [22] the authors proposed a discrete model of the formation of hotspots of criminal activity based on a random walk biased toward the attractive burglary sites, such that the criminals can move only to adjacent cites in each time step. On the basis of the discrete system, a continuum model is obtained as the limit of the discrete one. The continuous model is thus based on biased Brownian motion and described by a system of coupled partial differential equations (PDEs) for criminal density and the attractiveness field. In [6] the authors assumed that criminals can make movements not only to the neighborhood sites, but also can exhibit a long range of jumps. The model is nonlocal and involves occasional long jumps spread with a local random walks, i.e. in its continuous version it involves a Lévy flight motion. Criminals are able to examine the potential robbery spots and have knowledge, beside their local environment, also of far away surroundings. In this case the continuous model is governed by PDEs involving a fractional Laplace operator, which allows the superdiffusion of criminal density. Note that different mobility patterns are due to different types of criminals [27].

Following [6, 22] we perform numerical simulations for different parameter choices. This allows us to visualize several regimes of aggregation, like hotspots of high criminal activity. In addition, we study the fist passage time. One of the contributions of this paper is to generalize the one dimensional Lévy flight model proposed in [6] to a two dimensional one allowing a detailed comparison with the model based on Brownian motion.

Parameter	Meaning
l	grid spacing
Δt	time step
ω	attractiveness decay rate
η	measurement of the neighborhood effects
θ	increase of attractiveness due to one burglarisation
A_s^0	static attractiveness of site s
n_s	number of criminals at site s
Γ	rate of burglars generation

Table 1: Parameters for the UCLA and Lévy flight models

3.1. UCLA model

We briefly describe the UCLA burglary hotspot model introduced in [22]. The model contains a two dimensional lattice with spacing l, where a house is located at each grid point s and the burglars are imagined to walk on this lattice. Each house is assigned with an attractiveness $A_s(t)$, which displays the burglars thinking of the value of the house as a burglary target. The model is based upon the assumption, that the attractiveness is not modeled by properties like value, security or location but rather with the broken windows effect and near-repeated victimization, where the broken windows effect explains that the inhibition threshold recedes due to previous burglaries. Therefore, the attractiveness is divided into two parts, a static part A_s^0 and the dynamic part $B_s(t)$. Thus,

$$A_s(t) = A_s^0 + B_s(t).$$

The static parts A_s^0 measures values like e.g. location and accessibility, whereas the dynamic part $B_s(t)$ changes through interactions with the burglars.
In one time step a criminal agent can either decide to burglarize the house where he is located or move on to an adjacent site. The probability for the burglar to commit burglary in a time step of length Δt is given by a standard Poisson process

$$p_s(t) = 1 - e^{-A_s(t)\Delta t}, \tag{3.1}$$

where the expected value of events is $A_s(t)\Delta t$. It is assumed that a criminal agent committed burglary, gets removed from the grid. Furthermore, burglars are generated at each grid point with rate Γ. In the case that the burglar does not burglarize, it moves to an adjacent site into direction of areas with high attractiveness. The locomotion of the criminal agents follows a Brownian motion. Hence, the transition probability to move from a site s to an adjacent site n is given by

$$q_{s,n}(t) = \frac{A_n(t)}{\sum\limits_{s' \sim s} A_{s'}(t)}, \tag{3.2}$$

with $s' \sim s$ being all neighboring sites of s.

The dynamic part of the attractiveness $B_s(t)$ depends on former burglaries at site s. Thus, $B_s(t)$ is increased every time the house is getting burglarized by a value θ. For this increment affecting the attractiveness only for a finite time period the dynamic part is modeled by

$$B_s(t + \Delta t) = B_s(t)(1 - \omega\Delta t) + \theta E_s(t), \tag{3.3}$$

where $E_s(t)$ is the number of burglaries, Δt is the timescale and ω represents the decay rate of the dynamic attractiveness field. For letting the attractiveness spread over adjacent houses the dynamic part becomes

$$B_s(t + \Delta t) = \left[(1 - \eta)B_s(t) + \frac{\eta}{z}\sum_{s' \sim s} B_{s'}(t)\right](1 - \omega\Delta t) + \theta E_s(t), \tag{3.4}$$

where z is the number of adjacent sites (four in two dimensions) and $\eta \in [0, 1]$ is a parameter to measure near repeated victimization, higher value of η leads to higher attractiveness generated by any burglary event, i.e. to more robberies. Rewriting (3.4) using the discrete Laplacian operator (see Section 2.2),

$$\Delta B_s(t) = \frac{1}{l^2} \cdot \left(\sum_{s' \sim s} B_{s'}(t) - zB_s(t), \right)$$

with l being the grid spacing leads to

$$B_s(t+\Delta t) = \left[B_s(t) + \frac{\eta l^2}{z} \Delta B_s(t) \right] (1-\omega \Delta t) + \theta E_s(t). \tag{3.5}$$

Further we consider the simplest version of the discrete system (3.5), i.e. we will obtain an homogeneous equilibrium solution. We assume that all sites have the same attractiveness $\bar{A}$ and same number of criminals $\bar{n}$ on average. More details are given in Section 4.

A continuous version of the discrete UCLA model is obtained as a limit of the discrete one. It corresponds to the reaction diffusion model of the form

$$\begin{aligned} \frac{\partial B}{\partial t} &= \frac{\eta D}{z} \nabla^2 B - \omega B + \varepsilon D \rho A \\ \frac{\partial \rho}{\partial t} &= \frac{D}{z} \vec{\nabla} \cdot \left(\vec{\nabla} \rho - \frac{2\rho}{A} \vec{\nabla} A \right) - \rho A + \gamma \end{aligned} \tag{3.6}$$

where $\rho = \frac{n_s(t)}{l^2}$, $\gamma = \frac{\Gamma}{l^2}$ and fixed values $D = \frac{l^2}{\Delta t}$ and $\varepsilon = \theta \Delta t$. In (3.6), the first equation gives the dynamics of the attractiveness and the second one the criminal activity. The attractiveness diffuses throughout the environment and simultaneously decays in time and reacts with criminals to create more attractiveness [22].

3.2. Crime models with Lévy flights

In [6] the authors presented an one dimensional approach for the locomotion of the criminal agents and suggested a model involving Lévy flights. Allowing the burglars to move via Lévy flights, the burglars can search more efficiently for houses with high attractiveness by doing larger jumps. Thereby the distribution of step lengths obeys a power law. In this paper, we expand the model described in [6] to a two dimensional model. Moreover we compare the results of simulations of two dimensional UCLA and Lévy flight models.

Each time step, every burglar in the system either choose to move from his location to a new site or to commit a crime. Burglars are appearing randomly with the probability (3.1). Let $E_s(t)$ denote the number of crimes at each site s during the time interval $(t, t+\Delta t)$ and $N_s(t)$ the average number of criminals at the site s in the time interval $(t, t+\Delta t)$. Then, the dynamic part of the attractiveness is given by (3.3).

The relative weight of a criminal moving from a site $s = (s_1, s_2)$ to a different site $s' = (s'_1, s'_2)$ is given by

$$w_{s,s'} = \frac{A_{s'}}{l^\mu \|s - s'\|^\mu} = \frac{A_{s'}}{l^\mu \sqrt{(s_1 - s'_1)^2 + (s_2 - s'_2)^2}^{\,\mu}}, \tag{3.7}$$

where μ is the exponent of the underlying power law for the Lévy flight and l is the grid spacing. The transition probability $q_{s,s'}$ for a burglar to move from site s to a different site s' is then

$$q_{s,s'} = \frac{w_{s,s'}}{\sum_{r\in\mathbb{Z}^2, r\neq s} w_{s,r}}. \tag{3.8}$$

As in the UCLA model, here is also assumed that during the time interval Δt the burglar either commits a crime or else moves on according to a biased flight. New criminals appear with the rate Γ. The modelling of the attractiveness follows the same way as in the UCLA model.

Since the criminals can appear at the site s by moving there from some site $\hat{s}$, as is governed by $q_{\hat{s},s}$, or by the birth with $\Gamma\Delta t$, it follows

$$N_s(t+\Delta t) = \sum_{y\in\mathbb{Z},\, i\neq s} N_i\,(1 - A_i\Delta t)\cdot q_{i,s} \;+\Gamma\Delta t.$$

The corresponding continuum version of the model is again reaction diffusion system that involves in this case the fractional Laplace operator, i.e.

$$\begin{aligned} \frac{\partial A}{\partial t} &= \eta A_{xx} - A + \alpha + A\rho \\ \frac{\partial \rho}{\partial t} &= D\cdot\Big(A(-\Delta)^s\,(\frac{\rho}{A}) - \frac{\rho}{A}\,(-\Delta)^s\,(A)\Big) - A\rho + \beta, \end{aligned} \tag{3.9}$$

for a suitable choice of the parameters $\eta = \frac{l^2\hat{\eta}}{2\omega\Delta t}$, $D = \frac{l^{2s}}{2^{2s}\Delta t}\frac{\sqrt{\pi}|\Gamma(-s)|}{z\omega\Gamma(2s+1)}$, $\alpha = \frac{A^0}{\omega}$ and $\beta = \frac{\Gamma\theta}{\omega^2}$. For more details on the derivation of the system (3.9) and its stability properties we refer to [6].

In Algorithm 1 we sketch the UCLA and Lévy flight models.

Algorithm 1 UCLA and Leváy flight model

```
t ← 0                                                        ▷ Time
A = A^0 + B                          ▷ A = Field of attractiveness
while t <= t_max do
    for all Burglars do                              ▷ Criminal loop
        if RandomNumber <= ProbabilityForBurglary then
            Burglar commit burglary and gets removed from field
            Save position of burglary
        else
            Compute probability of moving to adjacent sites
            Move burglar to one site according to probability
    Compute dynamic part of field of attractiveness B and set A ← A^0 + B
    Place new burglars on the field with rate Γ
    t ← t + Δt
```

As UCLA and Lévy flight models differ only in the way the burglar locomotion is computed, we sketch these procedures in Algorithm 2 and Algorithm 3, respectively.

Algorithm 2 Burglar locomotion for the UCLA model

```
for all Burglars do
    Get position of a burglar
    Compute probability to move to one of the four adjacent sites by
    the formula (3.2)
    Decide where burglar moves
```

Algorithm 3 Burglar locomotion for the Lévy flight model

```
for all Burglars do
    Get position of a burglar
    Compute the weights (3.7)
    Compute the probability to move to one site on grid by using
    the formula (3.8)
    Decide where the burglar moves
```

3.3. High performance computing implementation

The numerical treatment of the UCLA and Lévy flight models is a challenging task, specially in the Lévy flight model. Note that although these models differ only in the way the burglar locomotion is computed, the computational cost of the Lévy flight model grows exponentially comparing to the UCLA model, see Figure 2. There, we plot the computing time vs grid size for different computing blocks for a given set of parameters. On the other hand, although storage is in principle not a problem, a careful implementation is required in both models. We point out that a Matlab implementation of the UCLA model seems to provide accurate results even for simulations involving *real data*, i.e. larger domains. Thus, in this case the use of high performance computing (HPC) implementation techniques is not needed. The latter does not apply to the Lévy flight model as it can be visualized in Figure 2.

Therefore, following Algorithm 3, we develop a HPC implementation based on MEX functions, i.e. we use subroutines implemented on C that take advantage of the structure and the potential parallelization of the problem. Our approach reduces the computing time in approximately one order. This can be verified for all computing blocks: complete simulation (Full), the dynamic part (Dynamic_part), the criminal locomotion (Criminal_loop), the burglary and movement, see Figure 2. Note that by using our HPC implementation we are able to simulate the Lévy flight model at essentially the same computational cost of the ULCA model, this was verified for different set of parameters.

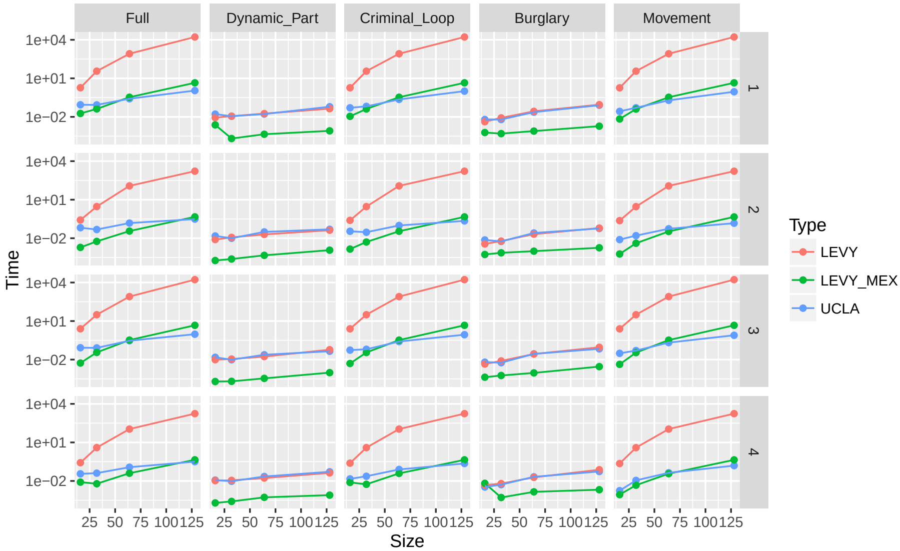

Figure 2: Computing time vs grid size for different parameter choices in the models.

4. Numerical simulations

We perform numerical simulations for different choices of the parameters given in Table 1. In Subsection 4.2 we generalize the one dimensional Lévy flight model proposed in [6] to a two dimensional one allowing a detailed comparison with the UCLA model from Subsection 4.1. A comparison of the burglar locomotion is provided in Subsection 4.3. Finally, in Subsection 4.4 we present a study of the first passage time.

All the experiments in this section were performed in Matlab. The value t_{max} in Algorithm 1, represents the number of days that the simulation ran. Double-precision floating-point arithmetic was employed in all cases. An efficient implementation on C of the main routines was developed.

4.1. UCLA model

All simulations were performed with $l = 1$, $\Delta t = 1/100$, $\omega = 1/15$ and $A^0 = 1/30$. The variation of the parameters η, θ and Γ can be seen in each figure description.

While in Figure 3 one can observe no hotspot forming, the other simulations form hotspots in different ways. Figure 4 exhibit dynamic hotspots, whilst in contrary Figure 5 leads to static hotspots. Finally, Figure 6 form spatially static hotspots with more deformations over time.

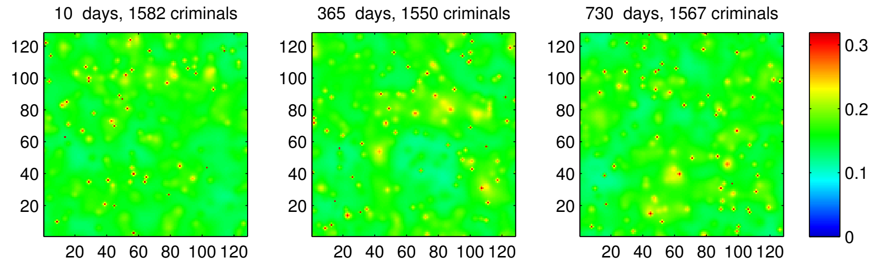

Figure 3: UCLA-$\eta = 0.2, \theta = 0.56, \Gamma = 0.019$

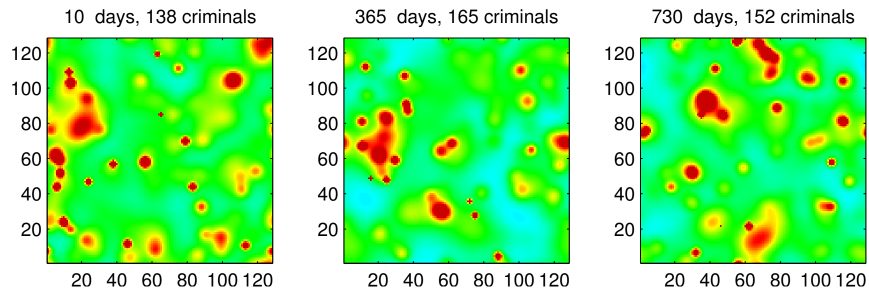

Figure 4: UCLA-$\eta = 0.2, \theta = 5.6, \Gamma = 0.002$

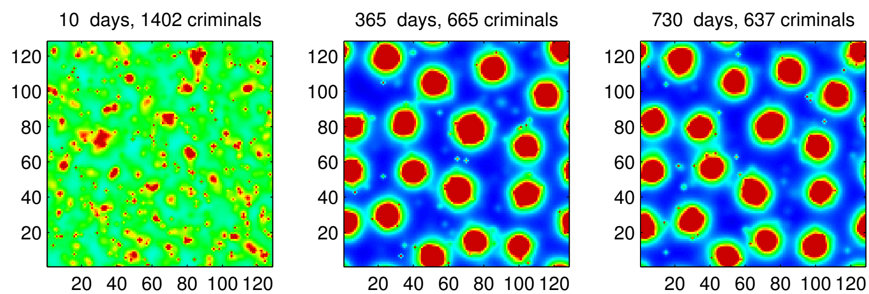

Figure 5: UCLA-$\eta = 0.03, \theta = 0.56, \Gamma = 0.019$

4.2. Lévy flight model

The numerical simulations of the Lévy flight model were run with the same parameters as for the UCLA models. We observe that for all the parameter setting the systems exhibit less number of hotspots. The simulations of the one dimensional model exhibit also only one or two hotspots, which is shown also in [22]. This is due to the different types of burglar locomotion. In the Lévy flight model, a criminal agent can make larger jumps and therefore all burglars move to the area of highest attractiveness.

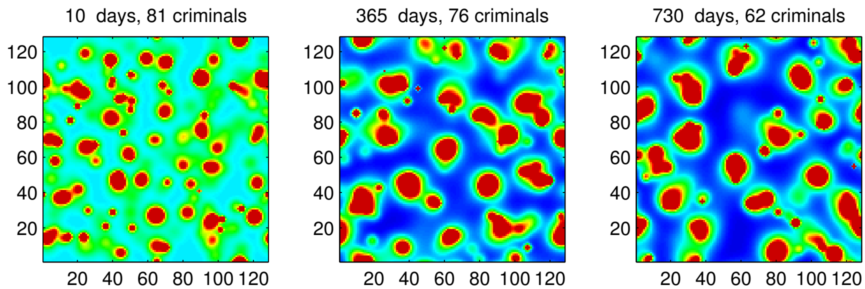

Figure 6: UCLA-$\eta = 0.03, \theta = 5.6, \Gamma = 0.002$

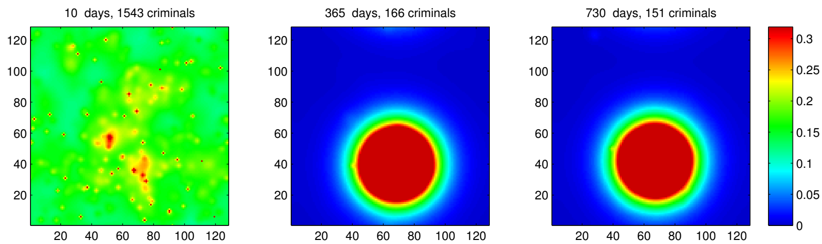

Figure 7: Lévy flight-$\eta = 0.2, \theta = 0.56, \Gamma = 0.019$

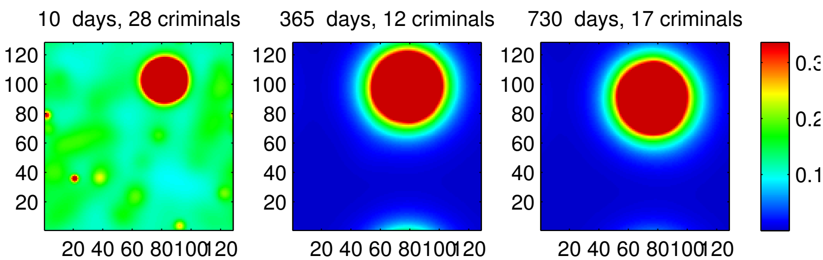

Figure 8: Lévy flight-$\eta = 0.2, \theta = 5.6, \Gamma = 0.002$

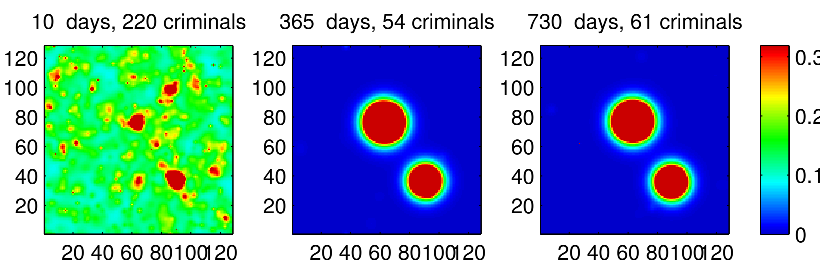

Figure 9: Lévy flight-$\eta = 0.03, \theta = 0.56, \Gamma = 0.019$

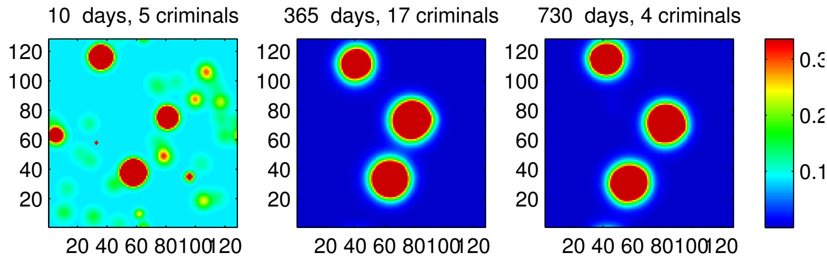

Figure 10: Lévy flight-$\eta = 0.03, \theta = 5.6, \Gamma = 0.002$

4.3. Comparison of burglar locomotion

In order to compare the locomotion in the two models, we displayed the motion of two burglars for three different sets of parameters. One is moving according to the UCLA model while the other is moving via the Lévy flight model. The output of these simulations are presented in Figure 11, Figure 12 and Figure 13. We observe that for the burglar moving via the Lévy flight model (blue) the areas with dense motion activity fits to the areas of the hotspots. Furthermore the burglar moving via the UCLA model, i.e. via Brownian motion (red) stays in the area of the same hotspot the whole time, for all parameter choices.

4.4. First passage time

We have computed the first passage times as the mean value of the time differences between the times where the burglars enter and where they leave (cross the border) the system. Our main conclusion is that all first passage times have essentially the same value, which verifies the theoretical results from [12, 14, 26]. In Figure 14, we plot the field of attractiveness for different times. Subfigures at time $t = 10$, $t = 100$ and $t = 200$ look very similar. This it due to the fact that as soon as the time is long enough (in subfigures $t = 0$ and $t = 1$ the dynamics has not evolved completely) the behavior of the first passage time is the same. The high peaks, represent that the burglar is approaching to the border.

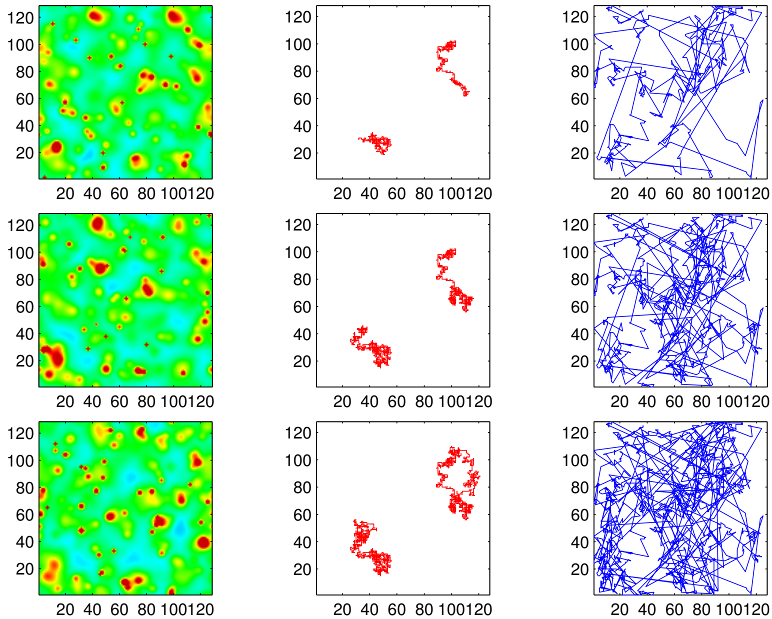

Figure 11: Comparison of burglar locomotion-$\eta = 0.1$, $\theta = 8$, $\Gamma = 0.004$

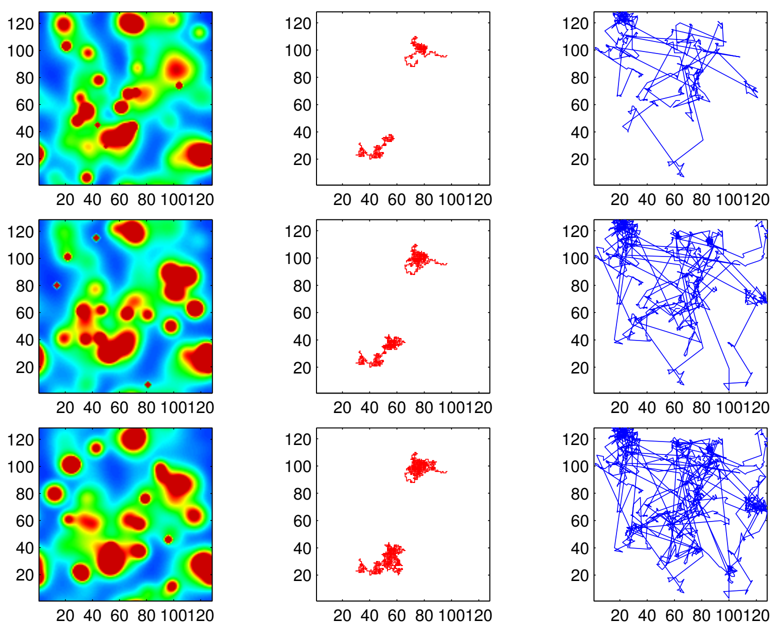

Figure 12: Comparison of burglar locomotion-$\eta = 0.1$, $\theta = 10$, $\Gamma = 0.0005$

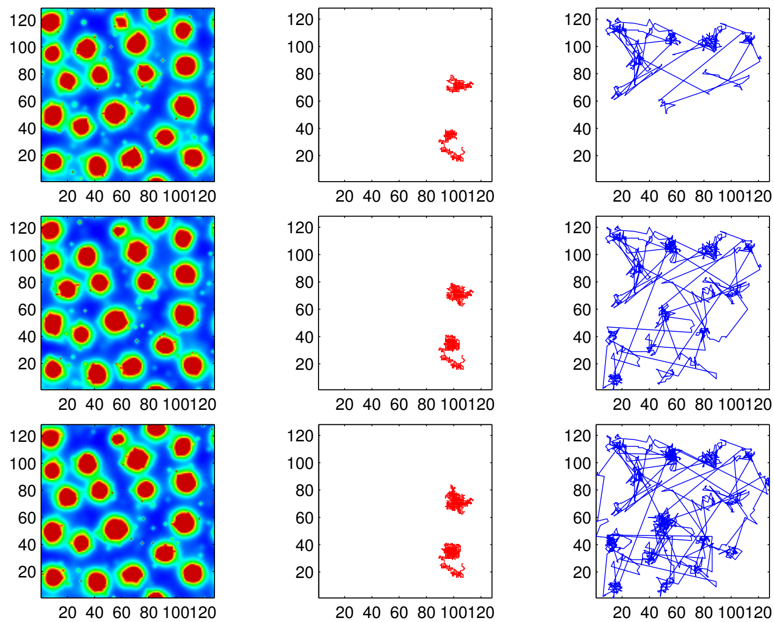

Figure 13: Comparison of burglar locomotion-$\eta = 0.03$, $\theta = 0.56$, $\Gamma = 0.019$

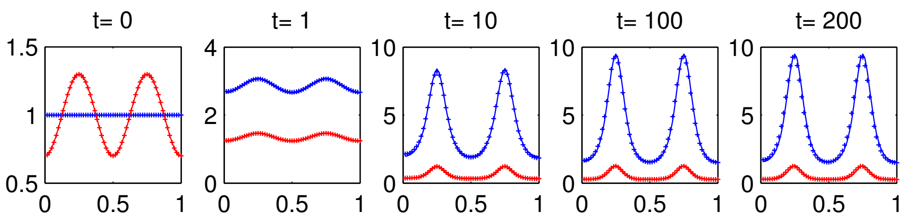

Figure 14: Field of attractiveness for different number of days.

5. Conclusion

In this paper, we studied random walks, subordinated Lévy processes and Lévy flights. We considered models based on Brownian motion and Lévy flights. Following [6, 22] we performed numerical simulations of the models with different parameter choices. This allowed us to visualize several regimes of aggregation, like hotspots of high criminal activity. In addition, we studied the first passage time and we generalize the one dimensional Lévy flight model proposed in [6] to a two dimensional one. We provided an efficient implementation of the models, which allowed us to compare the models. The model based on Brownian motion is suitable for small environments where only few hotspots arise. In the presence of many hotspots, the model based on Lévy flight motion is more appropriate. In addition, we proposed an efficient implementation of the Lévy model by using high performance computing techniques. This allowed us to perform numerical simulations with the Lévy model at the computational cost of the Brownian motion based model, making feasible for real life applications.

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References

[1] Applebaum, D., Lévy processes and stochastic calculus. 2nd edition, Cambridge studies in advanced mathematics 116, Cambridge University Press, 2009.

[2] Benner, P., Ezzatti, P., Mena, H., Quintana-Ort, E.S., Remon, A., Solving matrix equations on multi-core and many-core architectures. Algorithms 6 (4) (2013), 857–870.

[3] Benner, P., Mena H., Numerical solution of the infinite-dimensional LQR-problem and the associated Riccati differential equations. J. Numer. Math., De Gruyter just accepted, (2016), DOI: 10.1515/jnma-2016-1039.

[4] Benner, P., Mena H., Rosenbrock methods for solving differential Riccati equations. IEEE Trans. Automat. Control 58 (11) (2013), 2950–2957.

[5] Bogachev, L. V., Random walks in random environments. In: Encyclopedia o Mathematical Physics 4, (J.-P. Francoise et al. eds.), pp. 353–371, Oxford: Elsevier, 2006.

[6] Chaturapruek, S., Breslau, J., Yazdi, D., Kolokolnikov, T., McCalla, S. G., Crime modelling with Lévy flights. SIAM J. Appl. Math. 73 (4) (2013), 1703–1720.

[7] Chechkin, A. V., Metzler, R., Klafter, J., Gonchar, V. Yu., Introduction to the theory of Lévy flights. In: Anomalous Transport: Foundations and Applications, (R. Klages et al. eds.), pp. 129–162. Wiley-VCH Verlag GmbH & Co. KGaA, 2008.

[8] Feller, W., An introduction to probability theory and its applications. Vol. I, 3rd edition, New York - London - Sydney: John Wiley & Sons, Inc., 1968.

[9] Geman, H., Madan, D., Yor, M., Asset prices are Brownian motion: only in business time. Quantitative Analysis in Financial Markets, River Edge, NY: World Sci. Publ., (2001), 103–146.

[10] Hilber, N., Reichmann, O., Schwab, C., Winter, C., Computational methods for quantitative finance: Finite element methods for derivative pricing. Heidelberg: Springer, 2013.

[11] Huang, Y., Oberman, A., Numerical methods for fractional Laplacian: a finite difference-quadrature approach. SIAM J. Numer. Anal. 52 (2014), 3056–3084.

[12] Hurd, T. R., Kuznetsov, A., On the first passage time for Brownian motion subordinated by a Lévy process. J. Appl. Probab. 46 (1) (2009), 181–198.

[13] Hughes, B. D., Random walks and random environments. Volume 1: Random walks, Oxford Science Publications. New York: The Clarendon Press, Oxford University Press, 1995.

[14] Karatzas, I., Shreve, S. E., Brownian motion and stochastic calculus. Graduate Texts in Mathematics 113, 2nd edition, New York: Springer-Verlag, 1991.

[15] Kyprianou, A. E., Introductory lectures on fluctuations of Lévy processes with applications. Universitext. Berlin: Springer-Verlag, 2006.

[16] Lang, N., Mena, H., Saak, J., On the benefits of the LDL factorization for large-scale differential matrix equation solvers. Linear Algebra Appl. 480 (2015), 44–71.

[17] Metzler, R., Klafter, J., The random walks guide to anomalous diffusion: A fractional dynamic approach. Phys. Rep. 339 (1) (2000), 1–77.

[18] Revuz, D., Yor, M., Continuous martingales and Brownian motion. Fundamental Principles of Mathematical Sciences 293, 3rd edition, Berlin: Springer-Verlag, 1999.

[19] Rey, S. J., Mack, E. A., Koschinsky, J., Exploratory space-time analysis of burglary patterns. J. Quant. Criminol. 28 (3) (2012), 509–531.

[20] Samorodnitsky, G., Taqqu, M., Stable non-Gaussian random processes. Stochastic models with infinite variance. Stochastic Modeling. New York: Chapman & Hall, 1994.

[21] Sato, K., Lévy processes and infinitely divisible distributions. Cambridge Studies in Advanced Mathematics 68, Cambridge: Cambridge University Press, 2013.

[22] Short, M. B., D'Orsogna, M. R., Pasour, V. B., Tita, G. E., Brantingham, P. J., Bertozzi, A. L., Chayes, L. B., A statistical model of criminal behavior. Math. Models Methods Appl. Sci. 18 (2008), 1249–1267.

[23] Sims, D. W., Humphries, N. E., Bradford, R. W., Bruce, B. D., Lévy flight and Brownian search patterns of a free-ranging predator reflect different prey field. J Anim Ecol. 81 (2) (2012), 432–442.

[24] Sokolov, I. M., Lévy flights from continuous-time process. Phys. Rev. E 63 (011104) (2000), 1–10.

[25] Song, R., Vondraček, Z., Potential theory of special subordinators and subordinate killed stable processes. J. Theoret. Probab. 19 (4) (2006), 817–847.

[26] Taqqu, M., Veillette, M., Numerical computation of first-passage times of increasing Levy processes. Methodol. Comput. Appl. Probab. 12 (4) (2010), 695–729.

[27] Van Koppen, P. J., Jansen, R. W. J., The road to the robbery: Travel patterns in commercial robberies. Br. J. Criminol. 38 (2) (1998), 230–246.

[28] Warren, J., Yor, M., The Brownian burglar: conditioning Brownian motion by its local time process. Séminare de Probabilités, XXXII, pp. 328–342, Lecture Notes in Math. 1686, Berlin: Springer, 1998.

[29] Zipkin, J. R., Short, M. B., Bertozzi A. L., Cops on the dots in a mathematical model of urban crime and police response. Discrete Contin. Dyn. Syst. Ser. B 19 (5) (2014), 1479–1506.

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