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VALUE DISTRIBUTION AND UNIQUENESS OF q-SHIFT DIFFERENCE POLYNOMIALS[1]

Pulak Sahoo[2] and Gurudas Biswas[3]

Abstract

In this paper, we deal with the distribution of zeros of q-shift difference polynomials of transcendental entire functions of zero order. At the same time we also investigate the uniqueness problems when two difference products of entire functions share one value with finite weight. The results of the paper improve and generalize some recent results due to Xu, Liu and Cao [Math. Commun. 20 (2015), 97-112].

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1 Introduction, Definitions and Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations of the Nevanlinna's theory of meromorphic functions as explained in [7], [10] and [21]. For a nonconstant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic function of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ for all r outside a possible exceptional set of the finite logarithmic measure, $S(f)$ denotes the family of all meromorphic functions α such that $T(r, \alpha) = S(r, f)$, where $r \to \infty$ outside a possible exceptional set of the finite logarithmic measure. For convenience, we assume that $S(f)$ includes all constant functions and $\hat{S} = S(f) \cup \{\infty\}$.

Let f and g be two nonconstant meromorphic functions and a be a value in the extended complex plane. If the zeros of $f - a$ and $g - a$ coincide in locations and multiplicity, then we say that f and g share the value a CM (counting multiplicities). On the other hand, if the zeros of $f - a$ and $g - a$ coincide in their locations only, then we say that f and g share the value a IM (ignoring multiplicities). For a positive integer p, $N_p(r, a; f)$ denotes the counting function of the zeros of $f - a$, where an m-fold zero is counted m times if $m \leq p$ and p times if $m > p$. The difference operators for a meromorphic function f are defined as

$$\triangle_c f(z) = f(z + c) - f(z), \ (c \neq 0),$$

[1]The first author is thankful to DST-PURSE programme for financial assistance.

[2]Department of Mathematics, University of Kalyani, West Bengal-741235, India, e-mail: sahoopulak1@gmail.com

[3]Department of Mathematics, University of Kalyani, West Bengal-741235, India, e-mail: gurudasbiswas@yahoo.com

$$\triangle_q f(z) = f(qz) - f(z),\ (q \neq 0, 1).$$

Many research works on entire and meromorphic functions whose differential polynomials share certain value or fixed point have been done by many mathematicians in the world (see [4], [12], [16], [17], [20]). Recently, there has been an increasing interest in studying difference equations, the difference product and the q-difference in the complex plane $\mathbb{C}$, a number of papers (see [3], [5], [6], [11], [13], [14]) have focused on the uniqueness of difference analogue of Nevanlinna theory. The difference logarithmic derivative lemma, given by R.G. Halburd and R.J. Korhonen [5] in 2006 plays an important role in considering the difference analogues of Nevanlinna theory. Afterwards, Barnett, Halburd, Korhonen and Morgan [2] also established an analogue of the logarithmic derivative lemma on q-difference operators.

In 2007, Laine and Yang [11] studied zero distributions of difference polynomials of entire functions and obtained the following result.

Theorem A. *Let f be a transcendental entire function of finite order and η be a nonzero complex constant. Then for $n \geq 2$, $f^n(z)f(z+\eta)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

In 2010, Qi, Yang and Liu [15] proved the following uniqueness result corresponding to Theorem A.

Theorem B. *Let f and g be two transcendental entire functions of finite order, and η be a nonzero complex constant, and let $n \geq 6$ be an integer. If $f^n(z)f(z+\eta)$ and $g^n(z)g(z+\eta)$ share 1 CM, then either $fg = t_1$ or $f = t_2 g$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = t_2^{n+1} = 1$.*

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0$ be a nonzero polynomial, where $a_n (\neq 0)$, a_{n-1}, ... , a_0 are complex constants. We denote Γ_1, Γ_2 by $\Gamma_1 = m_1 + m_2$, $\Gamma_2 = m_1 + 2m_2$ respectively, where m_1 is the number of simple zeros of $P(z)$ and m_2 is the number of multiple zeros of $P(z)$. Throughout the paper we denote $d = \gcd(\lambda_0, \lambda_1, ..., \lambda_n)$, where $\lambda_i = n + 1$ if $a_i = 0$, $\lambda_i = i + 1$ if $a_i \neq 0$.

In 2011, Xudan and Lin [19] considered the zeros of one certain type of difference polynomial and obtained the following result.

Theorem C. *Let f be a transcendental entire function of finite order and η be a fixed nonzero complex constant. Then for $n > \Gamma_1$, $P(f(z))f(z+\eta) - \alpha(z) = 0$ has infinitely many solutions, where $\alpha(z) \in S(f) \setminus \{0\}$.*

In that paper the authors also established the following uniqueness result which corresponded to Theorem C.

Theorem D. *Let f and g be two transcendental entire functions of finite order, η be a nonzero complex constant, and $n > 2\Gamma_2 + 1$ be an integer. If $P(f)f(z+\eta)$ and $P(g)g(z+\eta)$ share 1 CM, then one of the following results hold:*
(i) $f = tg$, where $t^d = 1$;
(ii) f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(w_1, w_2) = P(w_1)w_1(z+\eta) - P(w_2)w_2(z+\eta)$;

(iii) $f = e^{\alpha}$, $g = e^{\beta}$, *where* α *and* β *are two polynomials and* $\alpha + \beta = c$, c *is a constant satisfying* $a_n^2 e^{(n+1)c} = 1$.

In 2010, Zhang and Korhonen [22] studied the value distribution of q-difference polynomials of meromorphic functions and obtained the following result.

Theorem E. *Let* f *be a transcendental meromorphic (resp. entire) function of zero order and* q *nonzero complex constant. Then for* $n \geq 6$ *(resp.* $n \geq 2$*),* $f^n(z)f(qz)$ *assume every nonzero value* $a \in \mathbb{C}$ *infinitely often.*

In the same paper the authors also proved the following uniqueness result for the q- difference of entire functions corresponding to Theorem E.

Theorem F. *Let* f *and* g *be two transcendental entire functions of zero order. Suppose that* q *is a nonzero complex constant and* $n \geq 6$ *is an integer. If* $f^n(z)(f(z)-1)f(qz)$ *and* $g^n(z)(g(z)-1)g(qz)$ *share* 1 *CM, then* $f \equiv g$.

Recently Xu, Liu and Cao [18] investigated value distributions for a q-shift of meromorphic functions and obtained the following result.

Theorem G. *Let* f *be a zero-order transcendental meromorphic (resp. entire) function,* $q \in \mathbb{C} \setminus \{0\}$, $\eta \in \mathbb{C}$. *Then for* $n > \Gamma_1 + 4$ *(resp.* $n > \Gamma_1$*),* $P(f)f(qz+\eta) = \alpha(z)$ *has infinitely many solutions, where* $\alpha(z) \in S(f) \setminus \{0\}$.

In that paper the authors also investigated the uniqueness problems of q-shift of entire functions and obtained the following result.

Theorem H. *Let* f *and* g *be two transcendental entire functions of zero order, and let* $q \in \mathbb{C} \setminus \{0\}$, $\eta \in \mathbb{C}$. *If* $P(f)f(qz+\eta)$ *and* $P(g)g(qz+\eta)$ *share* 1 *CM and* $n > 2\Gamma_2 + 1$ *be an integer, then one of the following results hold:*
(i) $f \equiv tg$ *for a constant* t *such that* $t^d = 1$;
(ii) f *and* g *satisfy the algebraic equation* $R(f,g) = 0$, *where* $R(w_1, w_2) = P(w_1)w_1(qz+\eta) - P(w_2)w_2(qz+\eta)$;
(iii) $fg \equiv \mu$, *where* μ *is a complex constant satisfying* $a_n^2 \mu^{n+1} \equiv 1$.

To state the next result of Xu, Liu and Cao [18] we need the following definition of weighted sharing.

Definition 1.1. ([8, 9]) Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if $m > k$. If $E_k(a; f) = E_k(a; g)$, then we say that f, g share the value a with weight k.

Theorem I. *Let* f *and* g *be two transcendental entire functions of zero order, and let* $q \in \mathbb{C} \setminus \{0\}$, $\eta \in \mathbb{C}$. *If* $E_l(1; P(f)f(qz+\eta)) = E_l(1; P(g)g(qz+\eta))$ *and* l, m, n *are integers satisfying one of the following conditions :*
(i) $l \geq 3$, $n > 2\Gamma_2 + 1$;
(ii) $l = 2$, $n > \Gamma_1 + 2\Gamma_2 + 2 - \lambda$;
(iii) $l = 1$, $n > 2\Gamma_1 + 2\Gamma_2 + 3 - 2\lambda$;
(iv) $l = 0$, $n > 3\Gamma_1 + 2\Gamma_2 + 4 - 3\lambda$.
Then the conclusions of Theorem H hold, where $\lambda = \min\{\Theta(0, f), \Theta(0, g)\}$.

Regarding Theorems G, H and I, it is natural to ask the following questions which are the motive of the present paper.

Question 1.2. What can we get about the zeros of $(P(f)f(qz+\eta))^{(k)} - \alpha(z)$, where $\alpha(z) \in S(f) \setminus \{0\}$ and k is any positive integer?

Question 1.3. What happen if one replace the difference polynomials $P(f)f(qz+\eta)$ by $(P(f)f(qz+\eta))^{(k)}$ in Theorems H and I, where k is any positive integer?

In the paper, our aim is to find out the possible answer of the above questions. We prove following results first one of which extends Theorem G and another one improves Theorems H and I. The following are the main results of the paper.

Theorem 1.4. *Let f be a transcendental entire function of zero order and $\alpha(z) \in S(f) \setminus \{0\}$. Suppose that η is a nonzero complex constant, n and k are positive integers. Then for $n > \Gamma_1 + km_2$, $(P(f)f(qz+\eta))^{(k)} - \alpha(z) = 0$ has infinitely many solutions.*

The following example shows that the zero order growth restriction in Theorem 1.4 can not be extended to finite order.

Example 1.5. Let $P(z) = z^n$, $f(z) = e^z$, $q = -n$ *and* $\alpha(z) = -ae^{\eta}$ where a is a nonzero constant. Then for any integer k, $(P(f(z))f(qz+\eta))^{(k)} - \alpha(z)$ has no zero.

Theorem 1.6. *Let f and g be two transcendental entire functions of zero order, and let $q \in \mathbb{C} \setminus \{0\}$, $\eta \in \mathbb{C}$. If $E_l(1; (P(f)f(qz+\eta))^{(k)}) = E_l(1; (P(g)g(qz+\eta))^{(k)})$ and l, m, n are integers satisfying one of the following conditions :*
(i) $l \geq 2$, $n > 2\Gamma_2 + 2km_2 + 1$;
(ii) $l = 1$, $n > \frac{1}{2}(\Gamma_1 + 4\Gamma_2 + 5km_2 + 3)$;
(iii) $l = 0$, $n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 4$.
Then one of the following results hold:
(i) $f \equiv tg$ for a constant t such that $t^d = 1$;
(ii) f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(w_1, w_2) = P(w_1)w_1(qz+\eta) - P(w_2)w_2(qz+\eta)$;
(iii) $fg \equiv \mu$, where μ is a complex constant satisfying $a_n^2\mu^{n+1} \equiv 1$.

The following example exhibits that Theorem 1.6 improves Theorems H and I respectively by relaxing the nature of sharing and by reducing the lower bound of n.

Example 1.7. Let $P(z) = (z-1)^4(z+1)^4z^7$, $f(z) = \sin z$, $g(z) = \cos z$, $q = 1$, $k = 0$, *and* $\eta = 2\pi$. It immediately yields that $n > 2\Gamma_2 + 1$ and $P(f)f(qz+\eta) = P(g)g(qz+\eta)$. Therefore $P(f)f(qz+\eta)$ and $P(g)g(qz+\eta)$ share 1 CM and hence they share $(1,2)$.

Here f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(w_1, w_2) = P(w_1)w_1(qz+\eta) - P(w_2)w_2(qz+\eta)$.

2 Lemmas

In this section, we state some lemmas which will be needed in the sequel. We denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

where F and G are nonconstant meromorphic functions defined in the complex plane $\mathbb{C}$.

Lemma 2.1. *([21]) Let f be a nonconstant meromorphic function and $P(f) = a_n f^n + a_{n-1} f^{n-1} + \ldots + a_0$, where $a_n (\neq 0)$, a_{n-1}, ... , a_0 are complex constants. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2.2. *([23]) Let f be a nonconstant meromorphic function, and p, k be two positive integers. Then*

$$N_p\left(r, 0; f^{(k)}\right) \leq T\left(r, f^{(k)}\right) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{2.1}$$

$$N_p\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2.2}$$

Lemma 2.3. *([9]) Let f and g be two nonconstant meromorphic functions. If $E_2(1; f) = E_2(1; g)$ then one of the following cases hold:*
(i) $T(r) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r)$,
(ii) $f = g$,
(iii) $fg = 1$,
where $T(r) = \max\{T(r, f), T(r, g)\}$ and $S(r) = o\{T(r)\}$.

Lemma 2.4. *([1]) Let F and G be two nonconstant meromorphic functions. If $E_1(1; F) = E_1(1; G)$ and $H \not\equiv 0$ then*
$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \frac{1}{2}\overline{N}(r, 0; F) + \frac{1}{2}\overline{N}(r, \infty; F) + S(r, F) + S(r, G)$;
the same inequality holds for $T(r, G)$.

Lemma 2.5. *([1]) Let F and G be two nonconstant meromorphic functions sharing 1 IM and $H \not\equiv 0$. Then*
$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + S(r, F) + S(r, G)$;
the same inequality holds for $T(r, G)$.

Lemma 2.6. *([18]) Let f be a transcendental meromorphic function of zero order and q, η two nonzero complex constants. Then*

$$T(r, f(qz + \eta)) = T(r, f(z)) + S(r, f),$$

$$N(r,\infty;f(qz+\eta)) \leq N(r,\infty;f(z)) + S(r,f),$$

$$N(r,0;f(qz+\eta)) \leq N(r,0;f(z)) + S(r,f),$$

$$\overline{N}(r,\infty;f(qz+\eta)) \leq \overline{N}(r,\infty;f(z)) + S(r,f),$$

$$\overline{N}(r,0;f(qz+\eta)) \leq \overline{N}(r,0;f(z)) + S(r,f).$$

Lemma 2.7. *([18]) Let f be a transcendental meromorphic function of zero order, and $q(\neq 0)$, η complex constants. Then*

$$(n-1)T(r,f) + S(r,f) \leq T(r, P(f)f(qz+\eta)) \leq (n+1)T(r,f) + S(r,f).$$

In addition, if f is a transcendental entire function of zero order, then

$$T(r, P(f)f(qz+\eta)) = T(r, P(f)f(z)) + S(r,f) = (n+1)T(r,f) + S(r,f).$$

Lemma 2.8. *Let f and g be two entire functions, n, k be two positive integers, $q(\neq 0)$, η complex constants and let*

$$F = (P(f)f(qz+\eta))^{(k)},\ G = (P(g)g(qz+\eta))^{(k)}.$$

If there exists two nonzero constants c_1 and c_2 such that $\overline{N}(r,c_1;F) = \overline{N}(r,0;G)$ and $\overline{N}(r,c_2;G) = \overline{N}(r,0;F)$, then $n \leq 2\Gamma_1 + 2km_2 + 1$.

Proof. We put $F_1 = P(f)f(qz+\eta)$ and $G_1 = P(g)g(qz+\eta)$. By the second main theorem of Nevanlinna we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}(r,c_1;F) + S(r,F) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F). \end{aligned} \tag{2.3}$$

Using (2.1), (2.2), (2.3), Lemmas 2.1, 2.6 and 2.7 we get

$$\begin{aligned} (n+1)T(r,f) &\leq T(r,F) - \overline{N}(r,0;F) + N_{k+1}(r,0;F_1) + S(r,f) \\ &\leq \overline{N}(r,0;G) + N_{k+1}(r,0;F_1) + S(r,f) \\ &\leq N_{k+1}(r,0;F_1) + N_{k+1}(r,0;G_1) + S(r,f) + S(r,g) \\ &\leq N_{k+1}(r,0;P(f)) + N_{k+1}(r,0;P(g)) + N(r,0;f(qz+\eta)) \\ &\quad + N(r,0;g(qz+\eta)) + S(r,f) + S(r,g) \\ &\leq (m_1 + m_2 + km_2 + 1)(T(r,f) + T(r,g)) \\ &\quad + S(r,f) + S(r,g). \end{aligned} \tag{2.4}$$

Similarly,

$$\begin{aligned} (n+1)T(r,g) &\leq (m_1 + m_2 + km_2 + 1)(T(r,f) + T(r,g)) \\ &\quad + S(r,f) + S(r,g). \end{aligned} \tag{2.5}$$

In view of (2.4) and (2.5) we have

$$(n - 2m_1 - 2m_2 - 2km_2 - 1)(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g),$$

which gives $n \leq 2\Gamma_1 + 2km_2 + 1$. This proves the lemma. □

3 Proof of Theorems

Proof of Theorem 1.4. Let $F_1 = P(f)f(qz+\eta)$. Then F_1 is a transcendental entire function. In contrary, we may assume that $F_1^{(k)} - \alpha(z)$ has only finitely many zeros. Then

$$N(r,\alpha;F_1^{(k)}) = O\{\log r\} = S(r,f). \tag{3.1}$$

Using (2.1), (3.1) and Nevanlinna's theorem for three small functions we deduce

$$\begin{aligned} T(r,F_1^{(k)}) &\leq \overline{N}(r,0;F_1^{(k)}) + \overline{N}(r,\alpha;F_1^{(k)}) + S(r,f) \\ &\leq T(r,F_1^{(k)}) - T(r,F_1) + N_{k+1}(r,0;F_1) + S(r,f). \end{aligned} \tag{3.2}$$

By Lemma 2.7 we obtain from (3.2)

$$\begin{aligned} (n+1)T(r,f) &\leq N_{k+1}(r,0;F_1) + S(r,f) \\ &\leq N_{k+1}(r,0;P(f)) + N(r,0;f(qz+\eta)) + S(r,f) \\ &\leq (m_1+m_2+km_2+1)T(r,f) + S(r,f). \end{aligned}$$

This gives

$$(n-m_1-m_2-km_2)T(r,f) \leq S(r,f),$$

a contradiction to the assumption that $n > \Gamma_1 + km_2$. This proves the theorem. $\square$

Proof of Theorem 1.6. Let $F_1 = P(f)f(qz+\eta)$, $G_1 = P(g)g(qz+\eta)$, $F = F_1^{(k)}$ and $G = G_1^{(k)}$. Then F and G are transcendental entire functions satisfying $E_l(1;F) = E_l(1;G)$. Using (2.1) and Lemma 2.7 we get

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;(F_1)^{(k)}) + S(r,f) \\ &\leq T(r,(F_1)^{(k)}) - T(r,F_1) + N_{k+2}(r,0;F_1) + S(r,f) \\ &= T(r,F) - (n+1)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f). \end{aligned}$$

From this we get

$$(n+1)T(r,f) \leq T(r,F) - N_2(r,0;F) + N_{k+2}(r,0;F_1) + S(r,f). \tag{3.3}$$

Again from (2.2) we have

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;(F_1)^{(k)}) + S(r,f) \\ &\leq N_{k+2}(r,0;F_1) + S(r,f). \end{aligned} \tag{3.4}$$

We now discuss the following three cases separately.

Case 1. Let $l \geq 2$. Suppose, if possible, that (i) of Lemma 2.3 holds. Then using (3.4) we obtain from (3.3)

$$\begin{aligned}(n+1)T(r,f) &\leq N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\\ &\quad +N_{k+2}(r,0;F_1)+S(r,f)+S(r,g)\\ &\leq N_{k+2}(r,0;F_1)+N_{k+2}(r,0;G_1)+S(r,f)+S(r,g)\\ &\leq (m_1+2m_2+km_2+1)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g). \end{aligned} \tag{3.5}$$

Similarly,

$$\begin{aligned}(n+1)T(r,g) &\leq (m_1+2m_2+km_2+1)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g). \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) we obtain

$$(n-2m_1-4m_2-2km_2-1)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

contradicting to the fact that $n > 2\Gamma_2+2km_2+1$. Therefore, by Lemma 2.3, we have either $FG=1$ or $F=G$. Let $FG=1$. Then

$$(P(f)f(qz+\eta))^{(k)}(P(g)g(qz+\eta))^{(k)}=1. \tag{3.7}$$

If possible, we assume that $P(z)=0$ has m roots u_1, u_2, ..., u_m with multiplicities s_1, s_2, ..., s_m. Then we have $s_1+s_2+...+s_m=n$. Then

$$\begin{aligned}&[a_n(f-u_1)^{s_1}(f-u_2)^{s_2}...(f-u_m)^{s_m}f(qz+\eta)]^{(k)}[a_n(g-u_1)^{s_1}\\ &\qquad (g-u_2)^{s_2}...(g-u_m)^{s_m}g(qz+\eta)]^{(k)}=1. \end{aligned} \tag{3.8}$$

Since f and g are entire functions, from (3.8), we see that $u_1=u_2=...=u_m=0$. In fact, from (3.8), we can get that u_1, u_2, ... , u_m are Picard exceptional values. If $m \geq 2$ and $u_i \neq 0$ $(i=1,2,...,m)$, by Picard's theorem of the entire function, we can get that Picard's exceptional values of f are at least three, a contradiction. Next we assume that $P(z)=0$ has only one root. Then $P(f)=a_n(f-a)^n$ and $P(g)=a_n(g-a)^n$, where a is any complex constant. Then from (3.7) we get

$$[a_n(f-a)^n f(qz+\eta)]^{(k)}[a_n(g-a)^n g(qz+\eta)]^{(k)}=1. \tag{3.9}$$

Since f and g are transcendental entire functions, by Picard's theorem, we can get that $f-a=0$ and $g-a=0$ do not have zeros. Then, we obtain that $f(z)=e^{\alpha(z)}+a$ and $g(z)=e^{\beta(z)}+a$, $\alpha(z)$, $\beta(z)$ being nonconstant polynomials. From (3.9), we also see that $f(qz+\eta)\neq 0$ and $g(qz+\eta)\neq 0$ and therefore $a=0$. Thus $f(z)=e^{\alpha(z)}$, $g(z)=e^{\beta(z)}$, $P(z)=a_n z^n$ and

$$[a_n e^{n\alpha(z)+\alpha(qz+\eta)}]^{(k)}[a_n e^{n\beta(z)+\beta(qz+\eta)}]^{(k)}=1. \tag{3.10}$$

If $k=0$, then from (3.10) we have

$$a_n^2 e^{n(\alpha(z)+\beta(z))+\alpha(qz+\eta)+\beta(qz+\eta)}=1.$$

Then we must have $\alpha+\beta\equiv c$, where c is a constant. From this, we can easily see that $f(z)=e^{\alpha(z)}$, $g(z)=\mu e^{-\alpha(z)}$ where μ is a constant satisfying $a_n^2\mu^{n+1}=1$. If $k\geq 1$ then we deduce

$$[a_n e^{n\alpha(z)+\alpha(qz+\eta)}]^{(k)}=a_n e^{n\alpha(z)+\alpha(qz+\eta)}P(\alpha',\alpha'_\eta,...,\alpha^{(k)},\alpha^{(k)}_\eta),$$

where $\alpha_\eta=\alpha(qz+\eta)$. Obviously, $P(\alpha',\alpha'_\eta,...,\alpha^{(k)},\alpha^{(k)}_\eta)$ has infinitely many zeros, a contradiction with (3.10). Next we assume that $F=G$. Then

$$(P(f)f(qz+\eta))^{(k)}=(P(g)g(qz+\eta))^{(k)}.$$

Integrating once we obtain

$$(P(f)f(qz+\eta))^{(k-1)}=(P(g)g(qz+\eta))^{(k-1)}+c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1}\neq 0$, from Lemma 2.8, it follows that $n\leq 2\Gamma_1+2km_2+1$, contrary with the fact that $n>2\Gamma_2+2km_2+1$ and $\Gamma_2\geq\Gamma_1$. Hence we must have $c_{k-1}=0$. Repeating the process k-times, we deduce that

$$P(f)f(qz+\eta)=P(g)g(qz+\eta).$$

Then by a similar argument as in Case 2 in the proof of Theorem 11 [18] we obtain either $f=tg$ for a constant t such that $t^d=1$, $d=\gcd(\lambda_0,\lambda_1,...,\lambda_n)$, or f and g satisfy the algebraic equation $R(f,g)=0$, where $R(w_1,w_2)=P(w_1)w_1(qz+\eta)-P(w_2)w_2(qz+\eta)$.

Case 2. Let $l=1$ and $H\not\equiv 0$. Using Lemma 2.4 and (3.4) we obtain from (3.3)

$$\begin{aligned}(n+1)T(r,f) &\leq N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+\frac{1}{2}\overline{N}(r,0;F)\\ &\quad+\frac{1}{2}\overline{N}(r,\infty;F)+N_{k+2}(r,0;F_1)+S(r,f)+S(r,g)\\ &\leq N_{k+2}(r,0;F_1)+N_{k+2}(r,0;G_1)+\frac{1}{2}N_{k+1}(r,0;F_1)\\ &\quad+S(r,f)+S(r,g)\\ &\leq \frac{1}{2}[3m_1+(3k+5)m_2+3]T(r,f)+[m_1+(k+2)m_2+1]\\ &\quad T(r,g)+S(r,f)+S(r,g)\\ &\leq \frac{1}{2}[5m_1+(5k+9)m_2+5]T(r)+S(r),\end{aligned}$$

where $T(r)$ and $S(r)$ are same as in Lemma 2.3. Similarly, we obtain

$$(n+1)T(r,g)\leq\frac{1}{2}[5m_1+(5k+9)m_2+5]T(r)+S(r).$$

From the above two inequalities, we have

$$\left(n-\frac{5m_1+(5k+9)m_2+3}{2}\right)T(r)\leq S(r),$$

contradicting with the fact that $n > \frac{1}{2}(\Gamma_1 + 4\Gamma_2 + 5km_2 + 3)$. We now assume that $H = 0$. Then

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.11}$$

where $A(\neq 0)$ and B are constants. From (3.11) it is obvious that F, G share the value 1 CM and so they share $(1, 2)$. Hence we have $n > 2\Gamma_2 + 2km_2 + 1$. Now we discuss the following three subcases.

Subcase 1. Let $B \neq 0$ and $A = B$. Then from (3.11) we get

$$\frac{1}{F-1} = \frac{BG}{G-1}. \tag{3.12}$$

If $B = -1$, then from (3.12) we obtain $FG = 1$, from which we get $f(z) = e^{\alpha(z)}$, $g(z) = \mu e^{-\alpha(z)}$ where μ is a constant satisfying $a_n^2 \mu^{n+1} = 1$, as in Case 1. If $B \neq -1$, from (3.12), we have $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and so $\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r, 0; F)$. Now from the second main theorem of Nevanlinna, we get using (2.1) and (2.2) that

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq N_{k+1}(r,0;F_1) + T(r,G) + N_{k+1}(r,0;G_1) \\ &\quad -(n+1)T(r,g) + S(r,g). \end{aligned}$$

This gives

$$(n+1)T(r,g) \leq (m_1 + (k+1)m_2 + 1)\{T(r,f) + T(r,g)\} + S(r,g).$$

Thus we obtain

$$(n - 2m_1 - 2(k+1)m_2 - 1)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

a contradiction as $n > 2\Gamma_2 + 2km_2 + 1$.

Subcase 2. Let $B \neq 0$ and $A \neq B$. Then from (3.11) we obtain $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and therefore $\overline{N}(r, \frac{B-A+1}{B+1}; G) = \overline{N}(r, 0; F)$. Proceeding similarly as in Subcase 1, we can get a contradiction.

Subcase 3. Let $B = 0$ and $A \neq 0$. Then from (3.11) we get $F = \frac{G+A-1}{A}$ and $G = AF - (A-1)$. If $A \neq 1$, we have $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, 1 - A; G) = \overline{N}(r, 0; F)$. Then by Lemma 2.8, it follows that $n \leq 2\Gamma_1 + 2km_2 + 1$, a contradiction. Thus $A = 1$ and then $F = G$. Now the result follows from the proof of Case 1.

Case 3. Let $l = 0$ and $H \not\equiv 0$. Using Lemma 2.5 and (3.4) we obtain from (3.3)

$$\begin{aligned}
(n+1)T(r,f) &\leq N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + 2\overline{N}(r,0;F) \\
&\quad +\overline{N}(r,0;G) + N_{k+2}(r,0;F_1) + 2\overline{N}(r,\infty;F) \\
&\quad +\overline{N}(r,\infty;G) + S(r,f) + S(r,g) \\
&\leq N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + 2N_{k+1}(r,0;F_1) \\
&\quad +N_{k+1}(r,0;G_1) + S(r,f) + S(r,g) \\
&\leq [3m_1 + (3k+4)m_2 + 3]T(r,f) + [2m_1 + (2k+3)m_2 + 2] \\
&\quad T(r,g) + S(r,f) + S(r,g) \\
&\leq [5m_1 + (5k+7)m_2 + 5]T(r) + S(r).
\end{aligned}$$

Similarly it follows that

$$(n+1)T(r,g) \leq [5m_1 + (5k+7)m_2 + 5]T(r) + S(r).$$

From the above two inequalities we obtain

$$(n - 5m_1 - (5k+7)m_2 - 4)T(r) \leq S(r),$$

contradicts with the assumption that $n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 4$. Therefore $H = 0$ and then proceeding in a manner similar to Case 2, the result follows.

This completes the proof of theorem 1.6. $\square$

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