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ULAM STABILITY OF A BI-RECIPROCAL FUNCTIONAL EQUATION IN QUASI-β-NORMED SPACES

B.V. Senthil Kumar[1], J.M. Rassias[2], and K. Ravi[3]

Abstract. In this paper, we investigate the generalized Hyers-Ulam stability of a bi-reciprocal functional equation in quasi-β-normed spaces.

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1. Introduction

The issue of stability of functional equations has been a very popular subject of investigations for the last 3 decades. In almost all areas of mathematical analysis, we can raise the following fundamental question: When is it true that a mathematical object satisfying a certain property approximately must be close to an object satisfying the property exactly? If we turn our attention to the case of functional equations, we can particularly ask the question when the solutions of an equation differing slightly from a given one must be close to the solution of the given equation.

A stimulating and famous talk presented by S.M. Ulam [45] in 1940, motivated the study of stability problems for various functional equations. He gave wide range of talk before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. The stability problem of functional equations originates from such a fundamental question. In connection with the above question, S. M. Ulam raised a question concerning the stability of homomorphisms as follows:

"Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $a : G_1 \to G_2$ with $d(f(x), a(x)) < \epsilon$ for all $x \in G_1$?"

If the answer is affirmative, we say that the functional equation for homomorphism is stable. In 1941, D.H. Hyers [6] was the first mathematician to present the result concerning the stability of functional equations. He brilliantly

[1]Department of Mathematics, C. Abdul Hakeem College of Engg. & Tech., Melvisharam - 632 509, Tamil Nadu, INDIA, e-mail: bvssree@yahoo.co.in

[2]Pedagogical Department E.E., Section of Mathematics and Informatics, National and Capodistrian University of Athens, 4, Agamemnonos Str., Aghia Paraskevi, Athens, Attikis 15342, GREECE, e-mail: jrassias@primedu.uoa.gr

[3]PG & Research Department of Mathematics, Sacred Heart College, Tirupattur - 635 601, Tamil Nadu, INDIA, e-mail: shckravi@yahoo.co.in

answered the question of Ulam for the cases where G_1 and G_2 are assumed to be Banach spaces. He considered that if $f : G_1 \to G_2$ satisfying the inequality $\|f(x+y) - f(x) - f(y)\| \le \epsilon$, for all $x, y \in G_1$, then the proved that the limit $a(x) = \lim\limits_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in G_1$ and that $a : G_1 \to G_2$ is the unique additive mapping satisfying $\|f(x) - a(x)\| \le \epsilon$.

Using Hyers' result, the additive functional equation $f(x+y) = f(x) + f(y)$ is said to have Hyers-Ulam stability on (G_1, G_2). After Hyers gave an affirmative answer to Ulams question, a large number of papers have been published in connection with various generalizations of Ulams problem and Hyers' theorem.

In 1978, Th.M. Rassias [22] provided a generalized version of Hyers' result by allowing the Cauchy difference to be unbounded. The paper of Th.M. Rassias has provided a lot of influence in the development of the stability of functional equations, and this new concept is known as generalized Hyers-Ulam-Rassias stability or Hyers-Ulam-Rassias stability. Since then, the stability problems have been widely studied and extensively developed by many authors for a number of functional equations. For the past 3 decades, the topic of the generalized Hyers-Ulam stability of functional equations was taken up by a number of mathematicians, and the study of this area has grown to be one of central subjects in mathematical analysis.

During 1982-1989, J.M. Rassias ([17]-[19]) gave a further generalization of the result of D.H. Hyers and proved theorem using weaker conditions controlled by a product of different powers of norms. This type of stability involving a product of powers of norms is called Ulam-Gavruta-Rassias Stability by B. Bouikhalene, E. Elquorachi [1], P. Nakmahachalasint ([12], [13]), C. Park, A. Najati [14], A. Pietrzyk [16] and A. Sibaha et.al. [44].

A generalized and modified form of the theorem evolved by Th.M. Rassias was advocated by P. Gavruta [5] who replaced the unbounded Cauchy difference by driving into study a general control function within the viable approach designed by Th.M. Rassias. This type of stability is called Generalized Hyers-Ulam Stability.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2], [3], [4], [7], [9], [10], [20], [21], [46]).

In 2008, J.M. Rassias et al. [23] discussed the stability of quadratic functional equation

$$f(mx+y) + f(mx-y) = 2f(x+y) + 2f(x-y) + 2\left(m^2 - 2\right) f(x) - 2f(y)$$

for any arbitrary but fixed real constant m with $m \ne 0$; $m \ne \pm 1$; $m \ne \pm\sqrt{2}$ using mixed product-sum of powers of norms. This type of stability is called J.M. Rassias stability involving mixed product-sum of powers of norms by K. Ravi et al. ([23], [24], [29], [30]).

Several mathematicians have remarked interesting applications of the Hyers-Ulam-Rassias Stability theory to various mathematical problems. Stability theory is applied in in fixed point theory to find the expression of the asymptotic derivative of a nonlinear operator.

S.M. Jung [8] investigated the Hyers-Ulam Stability for Jensens equation on a restricted domain and he applied the result to the study of an interesting asymptotic property of additive mappings.

The stability properties of different functional equations can have applications to unrelated fields. For instance, Zhou [47] used a stability property of the functional equation $f(x+y+f(x-y)=2f(x)$ to prove a conjecture of Z. Ditzian about the relationship between the smoothness of a mapping and the degree of its approximation by the associated Bernstein polynomials. These stability results can be applied in stochastic analysis [11], financial and actuarial mathematics, as well as in psychology and sociology.

W.G. Park and J.H. Bae [15] obtained the general solution and the stability of the functional equation

$$\begin{aligned}
&f(x+y+z,u+v+w)+f(x+y-z,u+v+w)\\
&\qquad +2f(x,u,-w)+2f(y,v,-w)\\
&\quad =f(x+y,u+w)+f(x+y,v+w)+f(x+z,u+w)\\
&\qquad +f(x-z,u+v+w)+f(y+z,v+w)+f(y-z,u+v-w).
\end{aligned}\tag{1.1}$$

The function $f(x,y)=x^3+ax+b-y^2$ having level curves as elliptice curves is a solution of the above functional equation (1.1). The mapping $f(x,y)=x^3+ax+b-y^2$ is useful in studying cryptography and their applications. In the same paper, they have presented an application of the stability of the equation (1.1) by showing the canonical height function of an elliptice curve over $\mathbb{Q}$ is a solution of the equation (1.1).

In the year 2010, K. Ravi and B.V. Senthil Kumar [25] investigated the generalized Hyers-Ulam stability for the reciprocal functional equation

$$g(x+y)=\frac{g(x)g(y)}{g(x)+g(y)}\tag{1.2}$$

where $g:X\to\mathbb{R}$ is a mapping with X as the space of non-zero real numbers. The reciprocal function $g(x)=\frac{c}{x}$ is a solution of the functional equation (1.2). The functional equation (1.2) holds good for the “Reciprocal formula” of any electric circuit with two resistors connected in parallel.

Recently K. Ravi and B.V. Senthil Kumar [43] obtained the generalized Hyers-Ulam stability of the system of bi-reciprocal functional equations

$$r(x+u,y)=\frac{r(x,y)r(u,y)}{r(x,y)+r(u,y)},$$

$$r(x,y+v)=\frac{r(x,y)r(x,v)}{r(x,y)+r(x,v)}$$

in Fréchet spaces.

The results regarding stability results of various forms of reciprocal type functional equations can be referred in ([26]-[42]).

In this paper, we investigate the generalized Hyers-Ulam stability of a bi-reciprocal functional equation of the type

$$r(x+u, y+v) = \frac{r(x,y)r(u,v)}{r(x,y)+r(x,v)+r(u,y)+r(u,v)} \tag{1.3}$$

in quasi-β-normed spaces.

Throughout this paper, let X be a quasi-β-normed space and let Y be a quasi-β-Banach space with a quasi-β-norm $\|\cdot\|_Y$. For a given mapping $r : X \to Y$, let us define the difference operator $D_r : X \times X \times X \times X \to Y$ by

$$D_r(x,u,y,v) = r(x+u, y+v) - \frac{r(x,y)r(u,v)}{r(x,y)+r(x,v)+r(u,y)+r(u,v)}$$

for all $x, u, y, v \in X$.

2. Preliminaries

In this section, we present some preliminaries concerning quasi-β-normed spaces. Let β be a fixed real number with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. Let X be a linear space over K. A quasi-β-norm $\|\cdot\|$ is a real-valued function on X satisfying the following conditions:

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\| = |\lambda|^\beta \cdot \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

(iii) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$.

Definition 2.2. A quasi-β-Banach space is a complete quasi-β-normed space.

Definition 2.3. A quasi-β-norm $\|\cdot\|$ is called a (β, p)-norm $(0 < p < 1)$ if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space.

3. Generalized Hyers-Ulam stability of equation (1.3)

In this section, we investigate the generalized Hyers-Ulam stability of bi-reciprocal functional equation (1.3) in quasi-β-normed spaces. We also present the pertinent stability results of Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by the mixed product-sum of powers of norms.

Theorem 3.1. *Let $\varphi : X \times X \times X \times X \to [0, \infty)$ be a mapping satisfying*

$$\sum_{i=0}^{\infty} K^i 4^{i\beta} \varphi\left(2^i x, 2^i u, 2^i y, 2^i v\right) < \infty \tag{3.1}$$

for all $x, u, y, v \in X$. Let $r : X \times X \to Y$ be a mapping such that

$$\|D_r(x, u, y, v)\|_Y \leq \varphi(x, u, y, v) \tag{3.2}$$

for all $x, u, y, v \in X$. Then there exists a unique bi-reciprocal mapping $R : X \times X \to Y$ satisfying (1.3) and

$$\|R(x, y) - r(x, y)\|_Y \leq 4^\beta K \sum_{i=0}^{\infty} \left(4^\beta K\right)^i \varphi\left(2^i x, 2^i x, 2^i y, 2^i y\right) \tag{3.3}$$

for all $x, y \in X$. The mapping $R(x, y)$ is defined by

$$R(x, y) = \lim_{n\to\infty} 4^n r\left(2^n x, 2^n y\right) \tag{3.4}$$

for all $x, y \in X$.

Proof. Plugging (x, u, y, v) into (x, x, y, y) in (3.2) and multiplying by 4^β, we obtain

$$\|4r(2x, 2y) - r(x, y)\|_Y \leq 4^\beta \varphi(x, x, y, y) \tag{3.5}$$

for all $x, y \in X$. Replacing (x, y) by $(2x, 2y)$ in (3.5) and multiplying by 4^β, we get

$$\left\|4^2 r\left(2^2 x, 2^2 y\right) - 4r(2x, 2y)\right\|_Y \leq 4^{2\beta} \varphi(2x, 2x, 2y, 2y) \tag{3.6}$$

for all $x, y \in X$. Combining (3.5) with (3.6), and since $K \geq 1$,

$$\begin{aligned}
&\left\|4^2 r\left(2^2 x, 2^2 y\right) - r(x, y)\right\|_Y \\
&\qquad = \left\|4^2 r\left(2^2 x, 2^y\right) - 4r(2x, 2y) + 4r(2x, 2y) - r(x, y)\right\|_Y \\
&\qquad \leq K \left(\left\|4^2 r\left(2^2 x, 2^2 y\right) - 4r(2x, 2y)\right\|_Y + \|4r(2x, 2y) - r(x, y)\|_Y\right) \\
&\qquad \leq K \left(4^\beta \varphi(x, x, y, y) + 4^{2\beta} \varphi(2x, 2x, 2y, 2y)\right) \\
&\qquad \leq K 4^\beta \varphi(x, x, y, y) + K^2 4^{2\beta} \varphi(2x, 2x, 2y, 2y) \\
&\qquad \leq K 4^\beta \sum_{i=0}^{1} \left(K 4^\beta\right)^i \varphi\left(2^i x, 2^i x, 2^i y, 2^i y\right)
\end{aligned}$$

for all $x, y \in X$. Using induction arguments on a positive integer n, we conclude that

$$\|4^n r\left(2^n x, 2^n y\right) - r(x, y)\|_Y \leq 4^\beta K \sum_{i=0}^{n-1} \left(K 4^\beta\right)^i \varphi\left(2^i x, 2^i x, 2^i y, 2^i y\right) \tag{3.7}$$

for all $x, y \in X$. From (3.6), we have

$$\left\|4^{i+1} r\left(2^{i+1} x, 2^{i+1} y\right)-4^{i} r\left(2^{i} x, 2^{i} y\right)\right\|_{Y} \leq 4^{i \beta} 4^{\beta} \varphi\left(2^{i} x, 2^{i} x, 2^{i} y, 2^{i} y\right)$$

for all $x, y \in X$. For $m>l$,

$$\begin{aligned} & \left\|4^{m} r\left(2^{m} x, 2^{m} y\right)-4^{l} r\left(2^{l} x, 2^{l} y\right)\right\|_{Y} \\ & \leq \sum_{i=l}^{m-1}\left\|4^{j+1} r\left(2^{j+1} x, 2^{j+1} y\right)-4^{j} r\left(2^{j} x, 2^{j} y\right)\right\|_{Y} \\ & \leq 4^{\beta} K \sum_{i=l}^{m-1} K^{i} 4^{i \beta} \varphi\left(2^{i} x, 2^{i} x, 2^{i} y, 2^{i} y\right) \end{aligned} \tag{3.8}$$

for all $x, y \in X$. The right-hand side of the above inequality (3.8) tends to 0 as $m \rightarrow \infty$. Hence $\left\{4^{n} r\left(2^{n} x, 2^{n} y\right)\right\}$ is a Cauchy sequence in Y. Therefore, we may define

$$R(x, y)=\lim _{n \rightarrow \infty} 4^{n} r\left(2^{n} x, 2^{n} y\right)$$

for all $x, y \in X$. Since $K \geq 1$, replacing (x, y) by $\left(2^{n} x, 2^{n} y\right)$ and multiplying by 4^{β} in (3.2), we have

$$4^{n \beta}\left\|D_{r}\left(2^{n} x,{ }^{n} u, 2^{n} y, 2^{n} v\right)\right\|_{Y} \leq 4^{n \beta} K^{n} \varphi\left(2^{n} x, 2^{n} u, 2^{n} y, 2^{n} v\right) \tag{3.9}$$

for all $x, y \in X$. By taking $n \rightarrow \infty$, the definition of R implies that R satisfies (1.3) for all $x, u, y, v \in X$. Thus R is a bi-reciprocal mapping. Also, the inequality (3.7) implies the inequality (3.3). Now, it remains to show the uniqueness. Assume that there exists $R^{\prime}: X \times X \rightarrow Y$ satisfying (1.3) and (3.3). It is easy to show that for all $x, y \in X$, $R^{\prime}\left(2^{n} x, 2^{n} y\right)=\frac{1}{4^{n}} R^{\prime}(x, y)$ and $R\left(2^{n} x, 2^{n} y\right)=\frac{1}{4^{n}} R(x, y)$. Then

$$\begin{aligned} \left\|R^{\prime}(x, y)-R(x, y)\right\|_{Y} & =\left\|4^{n} R^{\prime}\left(2^{n} x, 2^{n} y\right)-4^{n} R\left(2^{n} x, 2^{n} y\right)\right\|_{Y} \\ & =4^{n \beta}\left\|R^{\prime}\left(2^{n} x, 2^{n} y\right)-R\left(2^{n} x, 2^{n} y\right)\right\|_{Y} \\ & \leq 4^{n \beta} K\Big(\left\|R^{\prime}\left(2^{n} x, 2^{n} y\right)-r\left(2^{n} x, 2^{n} y\right)\right\|_{Y} \\ & \qquad +\left\|r\left(2^{n} x, 2^{n} y\right)-R\left(2^{n} x, 2^{n} y\right)\right\|_{Y}\Big) \\ & \leq 2 \cdot 4^{\beta} K^{2} \sum_{i=0}^{\infty}\left(4^{\beta} K\right)^{n+i} \varphi\left(2^{n+1} x, 2^{n+1} x, 2^{n+1} y, 2^{n+1} y\right) \\ & \leq 2 \cdot K^{2} 4^{\beta} \sum_{i=0}^{\infty}\left(4^{\beta} K\right)^{n+i} \varphi\left(2^{n+1} x, 2^{n+1} x, 2^{n+1} y, 2^{n+1} y\right) \end{aligned}$$

for all $x, y \in X$. By letting $n \rightarrow \infty$, we immediately have the uniqueness of R. □

Corollary 3.2. *Let $\epsilon>0$ be fixed. If $r: X \times X \times X \times X \rightarrow Y$ satisfies*

$$\left\|D_{r}(x, u, y, v)\right\|_{Y} \leq \epsilon$$

for all $x, u, y, v \in X$, then there exists a unique bi-reciprocal mapping $R : X \times X \to Y$ such that

$$\|R(x,y) - r(x,y)\|_Y \leq \frac{4K}{1-4^\beta K}\epsilon$$

for all $x, y \in X$.

Proof. Letting $\varphi(x,u,y,v) = \epsilon$, for all $x, u, y, v \in X$ in Theorem 3.1, we lead to

$$\begin{aligned}\|R(x,y) - r(x,y)\|_Y &\leq 4^\beta K \sum_{i=0}^{\infty} \left(4^\beta K\right)^i \epsilon \\ &\leq 4^\beta K \epsilon \left(1 - 4^\beta K\right)^{-1} \\ &\leq \frac{4K}{1-4^\beta K}\epsilon\end{aligned}$$

for all $x, y \in X$. □

Corollary 3.3. *Let $c_1 \geq 0$ be fixed and $p < -2$. If a mapping $r : X \times X \times X \times X \to Y$ satisfies the inequality*

$$\|D_r(x,u,y,v)\|_Y \leq c_1 \left(\|x\|^p + \|u\|^p + \|y\|^p + \|v\|^p\right)$$

for all $x, u, y, v \in X$, then there exists a unique bi-reciprocal mapping $R : X \times X \to Y$ satisfying (1.3) and

$$\|R(x,y) - r(x,y)\|_Y \leq \left(\frac{2c_1 K 4^\beta}{1 - 2^{(2+p)\beta} K}\right) \left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in X$.

Proof. Considering $\varphi(x,u,y,v) = c_1 \left(\|x\|^p + \|u\|^p + \|y\|^p + \|v\|^p\right)$, for all $x, u, y, v \in X$ in Theorem 3.1, we get

$$\begin{aligned}\|R(x,y) - r(x,y)\|_Y &\leq 4^\beta K \sum_{i=0}^{\infty} \left(4^\beta K\right)^i 2c_1 2^{\beta p i} \left(\|x\|^p + \|y\|^p\right) \\ &\leq 2c_1 K 4^\beta \sum_{i=0}^{\infty} K^i 2^{(2+p)\beta i} \left(\|x\|^p + \|y\|^p\right) \\ &\leq 2c_1 K 4^\beta \sum_{i=0}^{\infty} \left(2^{(2+p)\beta} K\right)^i \left(\|x\|^p + \|y\|^p\right) \\ &\leq 2c_1 K 4^\beta \left(1 - K 2^{(2+p)\beta}\right)^{-1} \left(\|x\|^p + \|y\|^p\right) \\ &\leq \left(\frac{2c_1 K 4^\beta}{1 - 2^{(2+p)\beta} K}\right) \left(\|x\|^p + \|y\|^p\right)\end{aligned}$$

for all $x, y \in X$. □

Corollary 3.4. *Let $c_2 \geq 0$ be fixed and $p < -\frac{1}{2}$. If a mapping $r : X \times X \times X \times X \to Y$ satisfies the inequality*

$$\|D_r(x,u,y,v)\|_Y \leq c_2 \|x\|^p \|u\|^p \|y\|^p \|v\|^p$$

for all $x,u,y,v \in X$, then there exists a unique bi-reciprocal mapping $R : X \times X \to Y$ satisfying (1.3) and

$$\|R(x,y) - r(x,y)\|_Y \leq \left(\frac{c_2 4^\beta K}{1 - 2^{(2+4p)\beta}K}\right)\left(\|x\|^{2p} + \|y\|^{2p}\right)$$

for all $x,y \in X$.

Proof. Choosing $\varphi(x,u,y,v) = c_2 \|x\|^p \|u\|^p \|y\|^p \|v\|^p$, for all $x,u,y,v \in X$ in Theorem 3.1, we obtain

$$\begin{aligned}
\|R(x,y) - r(x,y)\|_Y &\leq 4^\beta K \sum_{i=0}^{\infty} \left(4^\beta K\right)^i c_2 2^{4p\beta i} \|x\|^{2p} \|y\|^{2p} \\
&\leq c_2 4^\beta K \sum_{i=0}^{\infty} 2^{(2\beta+4p\beta)i} K^i \|x\|^{2p} \|y\|^{2p} \\
&\leq c_2 4^\beta K \sum_{i=0}^{\infty} \left(2^{(2+4p)\beta}\right)^i \|x\|^{2p} \|y\|^{2p} \\
&\leq \left(\frac{c_2 4^\beta K}{1 - 2^{(2+4p)\beta}K}\right)\left(\|x\|^{2p} + \|y\|^{2p}\right)
\end{aligned}$$

for all $x,y \in X$. □

Corollary 3.5. *Let $c_3 \geq 0$ be fixed and $p < -\frac{1}{2}$. If a mapping $r : X \times X \times X \times X \to Y$ satisfies the inequality*

$$\|D_r(x,u,y,v)\|_Y \leq c_3\Big(\|x\|^p \|u\|^p \|y\|^p \|v\|^p + \left(\|x\|^{4p} + \|u\|^{4p} + \|y\|^{4p} + \|v\|^{4p}\right)\Big)$$

for all $x,u,y,v \in X$, then there exists a unique bi-reciprocal mapping $R : X \times X \to Y$ satisfying (1.3) and

$$\|R(x,y) - r(x,y)\|_Y \leq \left(\frac{c_3 4^\beta K}{1 - 2^{(2+4p)\beta}K}\right)\left(\|x\|^{2p} \|y\|^{2p} + 2\left(\|x\|^{4p} + \|y\|^{4p}\right)\right)$$

for all $x,y \in X$.

Proof. Taking

$$\varphi(x,u,y,v) = c_3\Big(\|x\|^p \|u\|^p \|y\|^p \|v\|^p + \left(\|x\|^{4p} + \|u\|^{4p} + \|y\|^{4p} + \|v\|^{4p}\right)\Big),$$

for all $x, u, y, v \in X$ in Theorem 3.1, we attain

$$\begin{aligned}
&\|R(x,y) - r(x,y)\|_Y \\
&\quad \leq 4^\beta K \sum_{i=0}^{\infty} \left(4^\beta K\right)^i c_3 2^{4p\beta i}\Big(\|x\|^{2p} \|y\|^{2p} + 2 \left(\|x\|^{4p} + \|y\|^{4p}\right) \Big) \\
&\quad \leq c_3 4^\beta K \sum_{i=0}^{\infty} \left(2^{(2+4p)\beta} K\right)^i \left(\|x\|^{2p} \|y\|^{2p} + 2 \left(\|x\|^{4p} + \|y\|^{4p}\right) \right) \\
&\quad \leq \left(\frac{c_3 4^\beta K}{1 - 2^{(2+4p)\beta} K}\right) \left(\|x\|^{2p} \|y\|^{2p} + 2 \left(\|x\|^{4p} + \|y\|^{4p}\right) \right)
\end{aligned}$$

for all $x, y \in X$. □

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