LOCALLY $\phi$-QUASICONFORMALLY SYMMETRIC SASAKIAN FINSLER STRUCTURES ON TANGENT BUNDLES

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Abstract. In this study, the notion of locally $\phi$-quasiconformally symmetric Sasakian Finsler structures on the distributions of tangent bundles is introduced and its various geometric properties are studied with an example in dimension 3.

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1. Introduction

Miron $^3$, used the vector bundle approach in Finsler geometry. Sinha and Yadav $^7$, defined almost contact structures on vector bundles and studied their integrability condition. In $^8$, Yaliniz and Caliskan analysed almost contact and Sasakian Finsler structures on vector bundles and extended their characteristics with curvature properties and some structure theorems. Massamba and Mbatakou $^4$, approved pulled-back bundles to construct Sasakian Finsler structures. In this paper, tangent bundle approach is chosen to clarify locally $\phi$-quasiconformally symmetry property of Sasakian Finsler structures. On the other hand, quasiconformal curvature tensor appears in the literature with Yano and Sawaki $^9$. Also, $\phi$-quasiconformal flatness and $\phi$-quasiconformal symmetry features of several manifolds, like $^2, 3$, are studied quite frequently. Here, we are interested in locally $\phi$-quasiconformally symmetric Sasakian Finsler structures on tangent bundles.

In this section, a brief account of Sasakian Finsler structures on tangent bundles is given:

Let $M$ be an $m = (2n + 1)$-dimensional smooth manifold. In this manner, $T_xM$ is denoted as the tangent space at $x \in M$ where $x = (x^1, \ldots, x^m)$ are the local coordinates of $M$ and $y = y^i \frac{\partial}{\partial x^i} \in T_xM$. Then $u = (x, y) \in TM$ where $TM$ is the tangent bundle.

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Definition 1.1. The function $F : TM \rightarrow [0, \infty[$, the Hessian $G$ and the manifold $F^m = (M, F)$ are called "Finsler norm", "Finsler metric" and "Finsler manifold", respectively, if the following relations hold \([\mathbb{H}]\):

1. $F$ is smooth on the slit tangent bundle $TM$,
2. $F(x, \lambda y) = |\lambda|F(x, y)$, for $\lambda \in \mathbb{R}$ and $u = (x, y) \in TM$,
3. $g_{ij}(x, y) = \frac{1}{2} \left[ \frac{\partial^2 F^2}{\partial y^i \partial y^j} \right]$ is positive definite on $TM$.

Assume that $(x^i, y^j)$ and $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\}$ denote the local coordinates of $TM$ and natural bases of $T_u TM$, respectively. If $\pi : TM \rightarrow M$ is the projection map, the differential map $\pi_* : T_u TM \rightarrow T_{\pi(u)}M$ satisfies $\pi_u \in \pi_* (X_u)$. So, $\ker(\pi) = VTM$.

The non-linear connection $HTM = (N^j_i (x, y))$ is the complementary distribution of $VTM$ for $TTM$ i.e. $TTM = HTM \oplus VTM$, where $N^j_i = \frac{\partial N^j}{\partial y^i}$ are obtained via the spray coefficients $N^j_i = \frac{1}{4} g^{ik} \left( \frac{\partial^2 F^2}{\partial y^k \partial x^i} y^j - \frac{\partial F^2}{\partial x^i} \right)$ \([\mathbb{S}]\).

For every $u \in TM$ and $X \in T_u TM$, by using non-linear connections, $X = (X^i \frac{\partial}{\partial x^i} - N^j_i (x, y) X^i \frac{\partial}{\partial y^j}) + ((N^j_i (x, y) X^i + X^j) \frac{\partial}{\partial y^i}) = X^H + X^V$ unique decomposition is obtained as the horizontal part and the vertical part of vector field $X$ where $X^H \in T^H u TM$ and $X^V \in T^V u TM$ and $T^H u TM$ and $T^V u TM$ are spanned by $\{\frac{\delta}{\delta x^i}\}$ and $\{\frac{\delta}{\delta y^j}\}$, respectively. In addition, their dual bases are $\{dx^i\}$ and $\{dy^j = dy^j + N^j_i dx^i\}$, respectively.

Similarly, for $\eta \in (T_u TM)^*$, $\eta = \tilde{\eta}_i dx^i + \eta_j dy^j = \eta^H + \eta^V$ is obtained where $\eta^H \in (T^H u TM)^*$ and $\eta^V \in (T^V u TM)^*$.

The Sasaki-Finsler metric $G$ on $TM$ is defined as follows:

\[ G = G^H + G^V \] in the type of \(\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\) and \(\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\) on $TM^H$ and $TM^V$, respectively. Thus, Sasaki Finsler metric structures $(\phi^H, \xi^H, \eta^H, G^H)$ and $(\phi^V, \xi^V, \eta^V, G^V)$ can be constructed on either $TM^H$ or $TM^V$, respectively where; $\phi$ denotes the tensor field of type \(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\), $\xi$ is the structure vector field of type \(\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\), $\eta$ is the 1-form of type \(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\)$, $\nabla$ is the Finsler connection with respect to $G$ on $TM$, $L$ is the Lie differential operator, $R$ is the Riemann curvature tensor field of type \(\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}\), $S$ is the Ricci tensor field of type \(\begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}\), for $X^H, Y^H, \xi^H \in T^H u TM$ and $X^V, Y^V, \xi^V \in T^V u TM$, respectively.

The following relations hold for $m$-dimensional Sasaki Finsler metric manifolds $(TM^H, \phi^H, \xi^H, \eta^H, G^H)$ and $(TM^V, \phi^V, \xi^V, \eta^V, G^V)$ \([\mathbb{S}]\):

\begin{align*}
(1.1) & \quad \phi \phi = -I + \eta^H \otimes \xi^H + \eta^V \otimes \xi^V \\
(1.2) & \quad \phi \xi^H = 0, \phi \xi^V = 0
\end{align*}
(1.3) $\eta^H(\xi^H) = 1, \eta^V(\xi^V) = 1$

(1.4) $\eta^H(\phi X^H) = 0, \eta^V(\phi X^V) = 0, \eta^H(\phi X^V) = 0$

(1.5) $\Omega(X^H, Y^H) = 2(\nabla^H_X \eta)(Y^H) = -2(\nabla^H_Y \eta)(X^H)$

(1.6) $G(X^H, Y^H) = G(\phi^H X^H, \phi^H Y^H) + \eta^H(X^H)\eta^H(Y^H)$

(1.7) $G(X^H, \xi^H) = \eta^H(X^H), G(X^V, \xi^V) = \eta^V(X^V)$

(1.8) $G(\phi^H X^H, Y^H) = -G(X^H, \phi^H Y^H)$

(1.9) $\Omega(X^H, Y^H) = 2(\nabla^H_X \eta)(Y^H) = -2(\nabla^H_Y \eta)(X^H)$

(1.10) $G(X^V, Y^V) = G(\phi^V X^V, \phi^V Y^V) + \eta^V(X^V)\eta^V(Y^V)$

(1.11) $G(X^H, \phi^H Y^H) = d\eta^H(X^H, Y^H), G(X^V, \phi^V Y^V) = d\eta^H(X^H, Y^H)$

(1.12) $\nabla^H_X \xi^H = -\frac{1}{2} \phi^H X^H, \nabla^V_X \xi^V = -\frac{1}{2} \phi^V X^V$

(1.13) $\nabla^H_X \phi^H Y^H = \frac{1}{2}[G(X^H, Y^H)\xi^H - \eta^H(Y^H)X^H]$  

(1.14) $\nabla^V_X \phi^V Y^V = \frac{1}{2}[G(X^V, Y^V)\xi^V - \eta^V(Y^V)X^V]$

(1.15) $R(X^H, Y^H)\xi^H = \frac{1}{4}[\eta^H(Y^H)X^H - \eta^H(X^H)Y^H]$  

(1.16) $R(X^V, Y^V)\xi^V = \frac{1}{4}[\eta^V(Y^V)X^V - \eta^V(X^V)Y^V]$  

(1.17) $R(X^H, \xi^H)Y^H = \frac{1}{4}[\eta^H(Y^H)X^H - G(X^H, Y^H)\xi^H]$  

(1.18) $R(X^V, \xi^V)Y^V = \frac{1}{4}[\eta^V(Y^V)X^V - G(X^V, Y^V)\xi^V]$
\[ S(X^H, \xi^H) = \frac{n}{2} \eta^H(X^H), S(X^V, \xi^V) = \frac{n}{2} \eta^V(X^V) \]

\[ S(\xi^H, \xi^H) = \frac{n}{2}, S(\xi^V, \xi^V) = \frac{n}{2} \]

\[ S(X^H, Y^H) = G(QX^H, Y^H), S(X^H, Y^H) = G(QX^V, Y^V) \]

\[ Q(X^H) = \sum_{i=1}^{2n+1} R(E_i^H, X^H)E_i^H, Q(X^V) = \sum_{i=1}^{2n+1} R(E_i^V, X^V)E_i^V \]

\[ r = \sum_{i=1}^{2n+1} (S(E_i^H, E_i^H) + S(E_i^u, E_i^u)) \]

Above-stated formulas can be used to construct Sasakian Finsler structures on both \( TM^H \) and \( TM^V \). But in this paper, in second and third sections, locally \( \phi \)-quasiconformal symmetry of \( TM^H \) and 3-dimensional \( TM^H \) is discussed briefly.

2. Locally \( \phi \)-quasiconformally symmetric Sasakian Finsler structures on \( TM^H \)

**Definition 2.1.** Let \( TM^H \) be a Sasakian Finsler manifold, then it is locally \( \phi \)-symmetric if and only if

\[ \phi^2((\nabla^H_w R)(X^H, Y^H)Z^H) = 0 \]

for all \( X^H, Y^H, Z^H, W^H \in T^H_u TM \).

**Definition 2.2.** Let \( TM^H \) be a Sasakian Finsler manifold, then it is locally \( \phi \)-symmetric if and only if

\[ \phi^2((\nabla^H_W C^*)(X^H, Y^H)Z^H) = 0 \]

for all vector fields \( X^H, Y^H, Z^H, W^H \in T^H_u TM \) and where the quasiconformal curvature tensor \( C^* \) is defined by


\[ -\frac{r}{2n+1} \left( \frac{a}{2n} + 2b \right) (G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H) \]

for all \( X^H, Y^H, Z^H, W^H \in T^H_u TM \) and the constants \( a, b \).

If \( a = 1 \) and \( b = \frac{1}{2n-1} \), (2.3) can be expressed as follows:
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\[ C^*(X^H, Y^H)Z^H = R(X^H, Y^H)Z^H + \frac{1}{2n-1}S(Y^H, Z^H)X^H - \frac{r}{(2n)(2n-1)}(G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H) = C(X^H, Y^H)Z^H \]

where $C$ is Weyl conformal curvature tensor.

Calculating the covariant differentiation of (2.3), the following equality is obtained:

\[

\[
\quad - \frac{dr(W^H)}{2n+1}(\frac{a}{2n} + 2b)(G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H).
\]

(2.4)

If $S(Y^H, W^H) = \lambda G(W^H, Y^H)$ is satisfied, where $\lambda$ is a constant and $X^H, Y^H \in T^u TM$, the manifold $TM^H$ is called an Einstein manifold, where $QX^H = \lambda X^H$.

By using (1.1), (2.2) takes the following form:

\[-(\nabla^H_W C)(X^H, Y^H)Z^H + \gamma^H(\nabla^H_W C)(X^H, Y^H)Z^H)\xi^H = 0.\]

By virtue of (2.3), we obtain

\]

\[\quad + a\gamma^H(\nabla^H_W R)(X^H, Y^H)Z^H)\xi^H + b(\nabla^H_W S)(Y^H, Z^H)\eta^H(\nabla^H_W Q)X^H)\xi^H - bG(Y^H, Z^H)\eta^H(\nabla^H_W Q)Y^H)\xi^H + \frac{dr(W^H)}{2n+1}(\frac{a}{2n} + 2b)\eta^H(\nabla^H_W Q)Y^H)\xi^H.
\]

For $U^H \in T^u TM$, the last equality is expressed by

\[ 0 = -aG((\nabla^H_W R)(X^H, Y^H)Z^H, U^H) - b(\nabla^H_W S)(Y^H, Z^H)G(X^H, U^H) + b(\nabla^H_W S)(X^H, Z^H)G(Y^H, U^H) - bG(Y^H, Z^H)(\nabla^H_W Q)X^H)G(X^H, U^H) + \frac{dr(W^H)}{2n+1}(\frac{a}{2n} + 2b)\eta^H(\nabla^H_W Q)Y^H)\xi^H + \frac{dr(W^H)}{2n+1}(\frac{a}{2n} + 2b)\eta^H(\nabla^H_W Q)Y^H)\xi^H.
\]

Putting $X^H = U^H = E^H_i$, where $\{E^H_i\}, i = 1, 2, \ldots, 2n + 1$ is an orthonormal basis of $T^u TM$, and taking summation over $i$, we have
0 = (-a - b(2n + 1))(\nabla^H_W S)(Y^H, Z^H) + b(\nabla^H_W S)(E^H_i, Z^H)G(Y^H, E^H_i) \\
- G(Y^H, Z^H)[bG((\nabla^H_W Q)E^H_i, E^H_i) + dr(W^H)(\frac{a}{2n} + 2b)] + bG((\nabla^H_W Q)Y^H, Z^H) \\
- \frac{dr(W^H)}{2n+1}(\frac{a}{2n} + 2b)G(E^H_i, Z^H)G(Y^H, E^H_i) \\
+ a\eta^H((\nabla^H_W R)(E^H_i, Y^H)Z^H)\eta^H(E^H_i) \\
(2.5)

In (2.4) an\eta^H((\nabla^H_W R)(E^H_i, Y^H)Z^H)\eta^H(E^H_i) is expressed by

\eta^H((\nabla^H_W R)(E^H_i, Y^H)Z^H) = G(\nabla^H_W R)(E^H_i, Y^H)\eta^H(E^H_i, Y^H) \\
- G(\nabla^H_W E^H_i, Y^H)\xi^H, \xi^H) \\
(2.6) \\
- G(R(E^H_i, \nabla^H_W Y^H)\xi^H, \xi^H) - G(R(E^H_i, Y^H)\nabla^H_W \xi^H, \xi^H)

Owing to the fact that \( E^H_i \) is an orthonormal basis, it is easily seen that
\( \nabla^H_W E^H_i = 0 \).

By virtue of (1.13), it is possible to obtain below relation:

\[ 0 = G(R(E^H_i, \nabla^H_W Y^H)\xi^H, \xi^H) = \frac{1}{4}[G(\nabla^H_W Y^H, \xi^H)G(E^H_i, \xi^H) - G(E^H_i, \xi^H)G(\nabla^H_W Y^H, \xi^H)] \]
(2.7)

By using these equalities, the second and third terms of the right part of
(2.6) vanish. Thus (2.6) takes this form:

\[ G((\nabla^H_W R)(E^H_i, Y^H)\xi^H, \xi^H) = G((\nabla^H_W R)(E^H_i, Y^H)\xi^H, \xi^H) \\
- G(R(E^H_i, Y^H)\nabla^H_W \xi^H, \xi^H). \]

Due to \( G((\nabla^H_W R)(E^H_i, Y^H)\xi^H, \xi^H) + G(R(E^H_i, Y^H)\xi^H, \nabla^H_W \xi^H) = 0, \) (2.7)

can be expressed as follows:

\[ 0 = G((\nabla^H_W R)(E^H_i, Y^H)\xi^H, \xi^H) \\
- G((R)(E^H_i, Y^H)\xi^H, \nabla^H_W \xi^H) + G(R(E^H_i, Y^H)\xi^H, \nabla^H_W \xi^H) \]

In consequence of these calculations and by putting \( Z^H = \xi^H \) in (2.3) we
have the following:

\[ (-a - b(2n + 1))(\nabla^H_W S)(Y^H, \xi^H) + b(\nabla^H_W S)(\xi^H, \xi^H)\eta^H(Y^H) \\
- \eta^H(Y^H)[bG((\nabla^H_W Q)\xi^H, \xi^H) \\
+ dr(W^H)(\frac{a}{2n} + 2b)] + bG((\nabla^H_W Q)Y^H, \xi^H) = 0 \]
(2.8)

We calculate \( (\nabla^H_W S)(\xi^H, \xi^H) = 0 \) and \( G((\nabla^H_W Q)\xi^H, \xi^H) = 0 \) and additionally
\( G((\nabla^H_W Q)Y^H, \xi^H) = 0. \)

So, (2.3) is expressed by

\[ (\nabla^H_W S)(Y^H, \xi^H) = dr(W^H)(-\frac{a + 4bn}{(2n + 1)(a + (2n - 1)b)})\eta^H(Y^H) \]
(2.9)

where \( a + 4bn \neq 0 \). Because if \( a + 4bn = 0 \) from (2.3), we get \( C^* = aC \). By
putting \( Y^H = \xi^H \) in (2.9), we find the following:
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$$(\nabla^SH)(\xi^H, \xi^H) = dr(W^H)(-\frac{a+4bn}{(2n+1)(a+(2n-1)b)})$$

$$0 = dr(W^H).$$

This implies $r$ is constant. So we find $(\nabla^SH)(Y^H, \xi^H) = 0$.

By the virtue of (1.14) and (1.16), from (2.9) we have

$$S(Y^H, \phi W^H) = \frac{n}{2}G(W^H, \phi Y^H)$$

By putting $\phi W^H$ instead of $W^H$, we find $S(Y^H, W^H) = \frac{n}{2}G(W^H, Y^H)$. If we get $\frac{n}{2} = \lambda$ this means that a $\phi$-quasiconformally symmetric manifold $TM^H$ is an Einstein manifold. Then it is possible to have the following theorem:

**Theorem 2.3.** If a Sasakian Finsler manifold $TM^H$ is locally $\phi$-quasiconformally symmetric, then it is an Einstein manifold.

If we get $S(X^H, Y^H) = \lambda G(X^H, Y^H)$ in (2.9), the below relation is found.

$$C^*(X^H, Y^H)Z^H = (a + 4bn)$$

$$-\frac{4r}{2n+1}(\frac{a}{2n} + 2b))R(X^H, Y^H)Z^H$$

From (2.9), it is possible to say that $TM^H$ is locally $\phi$-quasiconformally symmetric because $C^*$ satisfies $\phi^2(\nabla^H W^H C^*(X^H, Y^H)Z^H) = 0$ for all vector fields $X^H, Y^H, Z^H \in T^uT^MT$. Also $\phi^2(\nabla^H W R)(X^H, Y^H)Z^H = 0$ implies that $TM^H$ is locally $\phi$-symmetric. So, it enables to state the following corollary:

**Corollary 2.4.** Let $TM^H$ be locally $\phi$-quasiconformally symmetric. Then it is locally $\phi$-symmetric.

3. **Locally $\phi$-quasiconformally symmetric Sasakian Finsler structures on 3-dimensional $TM^H$**

In a 3-dimensional $TM^H$, due to $C = 0$ [1], we have

$$R(X^H, Y^H)Z^H = [S(Y^H, Z^H)X^H - S(X^H, Z^H)Y^H + G(Y^H, Z^H)QX^H$$

$$-G(X^H, Z^H)QY^H] - \frac{r}{2}(G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H)$$

(3.1)

Putting $Z^H = \xi^H$ in(3.1), by the virtue of (1.4) and (1.10), we find

$$\frac{1}{4} - \frac{r}{2}[\eta^H(Y^H)X^H - \eta^H(X^H)Y^H] = [\eta^H(X^H)QY^H$$

$$-\eta^H(Y^H)QX^H].$$

(3.2)

Changing $Y^H = \xi^H$ in (3.2), we get
we have

\[ (3.3) \quad QX^H = \left( \frac{r}{2} - \frac{1}{4} \right)X^H + \left( \frac{3}{4} - \frac{r}{2} \right)\eta^H(X^H)\xi^H. \]

By using (3.3), we have

\[ (3.4) \quad S(X^H, Y^H) = \left( \frac{r}{2} - \frac{1}{4} \right)G(X^H, Y^H) + \left( \frac{3}{4} - \frac{r}{2} \right)\eta^H(X^H)\eta^H(Y^H). \]

Writing (3.3) and (3.4) in (2.2), we get the following:

\[ R(X^H, Y^H)Z^H = \left( \frac{r}{2} - \frac{1}{4} \right)(G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H) + \left( \frac{3}{4} - \frac{r}{2} \right)\eta^H(Y^H)\eta^H(Z^H)X^H - \eta^H(X^H)\eta^H(Z^H)Y^H \]
\[ + G(Y^H, Z^H)\eta^H(X^H)\xi^H - G(X^H, Z^H)\eta^H(Y^H)\xi^H. \]

Using (1.3), (2.2) and (3.3) in (2.2), we obtain

\[ C^*(X^H, Y^H)Z^H = \left[ \frac{1}{3} \frac{(a+b)r}{3} - \frac{1}{2} \frac{(a+b)}{3} \right] (G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H) \]
\[ + \left( \frac{3}{4} - \frac{r}{2} \right) \frac{(a+b)}{3} G(Y^H, Z^H)\eta^H(X^H)\xi^H - G(X^H, Z^H)\eta^H(Y^H)\xi^H \]
\[ + \left( \frac{3}{4} - \frac{r}{2} \right) \frac{(a+b)}{3} \eta^H(Y^H)\eta^H(Z^H)X^H - \eta^H(X^H)\eta^H(Z^H)Y^H \]

By calculating covariant differentiation of both sizes of (3.3)

\[ -(\nabla^H_W C^*)(X^H, Y^H)Z^H = \left[ \frac{(a+b)}{3} dr(W^H) (G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H) \right] \]
\[ + \left( \frac{a+b}{3} - \frac{1}{2} \frac{(a+b)}{3} \right) (\nabla^H_W G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H) \]
\[ + \left( \frac{3}{4} - \frac{r}{2} \right) (a+b) \left( \nabla^H_W (G(Y^H, Z^H)\eta^H(X^H)\xi^H) - \nabla^H_W (G(X^H, Z^H)\eta^H(Y^H)\eta^H(Z^H)X^H) - \nabla^H_W (\eta^H(X^H)\eta^H(Z^H)Y^H) \right). \]

Then we can write the following relation:

\[ (\nabla^H_W C)(X^H, Y^H)Z^H = \left[ \frac{(a+b)}{3} dr(W^H) (G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H) \right] \]
\[ + \left( \frac{a+b}{3} - \frac{1}{2} \frac{(a+b)}{3} \right) (\nabla^H_W G(Y^H, Z^H)) \nabla^H_W(\eta^H(X^H))\xi^H - G(X^H, Z^H)\nabla^H_W(\eta^H(X^H))\xi^H \]
\[ - G(X^H, Z^H)\nabla^H_W(\eta^H(Y^H)\eta^H(Z^H)X^H) - \nabla^H_W(\eta^H(X^H)\eta^H(Z^H)Y^H). \]

Because \( X^H, Y^H, Z^H \in T_u \) are orthogonal to \( \xi^H \), by using (1.3) we get \( \phi^2(\nabla^H_W C(X^H, Y^H)Z^H) = -\nabla^H_W C(X^H, Y^H)Z^H + \eta^H(\nabla^H_W C(X^H, Y^H)Z^H) \)

from which we have

\[ \phi^2(\nabla^H_W C(X^H, Y^H)Z^H) = -\left( \frac{a+b}{3} \right) dr(W^H) (G(Y^H, Z^H)X^H) \]
\[ - G(X^H, Z^H)Y^H \]

Due to \( \phi^2(\nabla^H_W C(X^H, Y^H)Z^H) = 0 \) if we take \( a+b = 0 \) and \( a = -b \) in (2.2), we have \( C(X^H, Y^H)Z^H = aC(X^H, Y^H)Z^H \). Because of \( C = 0 \), in (3.7) we find \( dr(W^H) = 0 \). This means that the curvature \( r \) is constant. Then it is possible to state the following theorem:
Theorem 3.1. Let $TM^H$ be a 3-dimensional Sasakian Finsler manifold. A necessary and sufficient condition to be locally $\phi$-quasiconformally symmetric is that $r$ is constant.

Corollary 3.2. Let $TM^H$ be a 3-dimensional Sasakian Finsler manifold. A necessary and sufficient condition to be $\phi$-symmetric is that $r$ is constant.

Corollary 3.3. Let $TM^H$ be a 3-dimensional Sasakian Finsler manifold. A necessary and sufficient condition to be locally $\phi$-quasiconformally symmetric is to be locally $\phi$-symmetric.

Example 3.4. Suppose $T(TM) = \{TM, \pi, M\}$ is the tangent bundle with $M = R^3$, where $u \in TM$ is defined by $(x^1, x^2, x^3, y^1, y^2, y^3)$. Assume the adapted local frames of $T_u^HTM$ and $T_u^VTM$ are $(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta \pi^i}, \frac{\delta}{\delta y^i})$ and $(\frac{\delta}{\delta \pi^i}, \frac{\delta}{\delta \pi^i}, \frac{\delta}{\delta y^i})$, respectively. Then the orthonormal frame of $T_uTM$ is

$$E_j = E^i_j \frac{\delta}{\delta x^i} + E_{j} \frac{\partial}{\partial y^i} = E^1_j \frac{\delta}{\delta x^i} + E^2_2 \frac{\delta}{\delta x^i} + E^3_3 \frac{\delta}{\delta x^i} + E^1_1 \frac{\partial}{\partial y^i} + E^2_2 \frac{\partial}{\partial y^i} + E^3_3 \frac{\partial}{\partial y^i},$$

where

$$E_1 = - \frac{\delta}{\delta x^i} - y^3 \frac{\partial}{\partial y^i};$$

$$E_2 = -(x^2)^2 \frac{\delta}{\delta x^i} + x^1 \frac{\delta}{\delta x^i} + (y^2)^2 \frac{\partial}{\partial y^i};$$

$$E_3 = \delta \frac{\delta}{\delta x^i} + \frac{\partial}{\partial y^i}.$$

Let $\eta = \tilde{\eta}_1 dx^1 + \eta_2 dx^2 + \eta_3 dx^3 + \tilde{\eta}_1 dy^1 + \tilde{\eta}_2 dy^2 + \tilde{\eta}_3 dy^3 = \eta^H + \eta^V$ be defined by $\eta = \frac{x^1}{(x^2)^2} dx^2 + dx^3 - \frac{y^3}{(y^2)^2} dy^2 + \delta y^3$.

Suppose that $\phi = \phi^H + \phi^V$ is a tensor field such that its coefficients are tensor fields $\phi^H$ and $\phi^V$ with the type of $(1,1)$. Their matrix forms are:

$$\phi^H = \begin{bmatrix} 0 & -\frac{1}{(x^2)^2} & 0 \\ (x^2)^2 & 0 & 0 \\ -x^1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \phi^V = \begin{bmatrix} 0 & -\frac{1}{(y^2)^2} & 0 \\ (y^2)^2 & 0 & 0 \\ -y^1 & 0 & 0 \end{bmatrix}.$$

The Sasaki-Finsler metric is defined by the matrix forms:

$$G^H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{(x^2)^2} & \frac{x^1}{(x^2)^2} \\ 0 & \frac{x^1}{(x^2)^2} & 1 \end{bmatrix} \quad \text{and} \quad G^V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{(y^2)^2} & \frac{y^1}{(y^2)^2} \\ 0 & \frac{y^1}{(y^2)^2} & 1 \end{bmatrix}.$$

It is possible to construct Sasakian Finsler manifolds on both horizontal and vertical distributions. In this example, it is shown that 3-dimensional $TM^H$ admits the Sasakian Finsler structure $(\phi^H, \xi^H, \eta^H, G^H)$.

We calculate

$$\phi^H(\xi^H) = 0, \phi^H(E^H_1) = -E^H_2, \phi^H(E^H_2) = E^H_1,$$

so relation (1.2) is satisfied. Similarly, (1.3) holds. Also it is possible to see that
\[ \phi^H(\phi^H(Z^H)) = -a_1E_1^H - b_1E_2^H = -Z^H + \eta^H(Z^H)\zeta^H, \]

for any \( Z^H = a_1E_1^H + b_1E_2^H + c_1E_3^H \in T_u^uTM \). Hence, it is shown that (1.1) is true.

If \( \eta^H(\phi Z^H) = 0 \), then (1.4) is satisfied. Thus, \( (\phi^H, \zeta^H, \eta^H) \) is an almost contact Finsler structure on \( TM^H \).

Due to

\[ \eta^H(Z^H) = c_1 = G^H(Z^H, \zeta^H) \]

for any \( Z^H \in T_u^uTM \), thus (1.7) holds.

Because of

\[ G^H(\phi Z^H, \phi W^H) = a_1a_2 + b_1b_2 = G^H(Z^H, W^H) - \eta^H(Z^H)\eta^H(W^H), \]

it can be seen that (1.10) holds. This implies \( (\phi^H, \zeta^H, \eta^H, G^H) \) is an almost contact Finsler metric structure.

On the other hand,

\[ [E_1^H, E_2^H] = -E_3^H, [E_1^H, E_3^H] = 0, [E_2^H, E_3^H] = 0. \]

Finsler connection \( \nabla = \nabla^H + \nabla^V \) of metric \( G = G^H + G^V \) can be expressed by the Koszul formula:

\[
\]

This yields

\[
\nabla^H_{E_1^H} E_3^H = \frac{1}{2} E_2^H, \quad \nabla^H_{E_1^H} E_2^H = -\frac{1}{2} E_3^H, \quad \nabla^H_{E_1^H} E_1^H = 0, \\
\nabla^H_{E_2^H} E_3^H = -\frac{1}{2} E_1^H, \quad \nabla^H_{E_2^H} E_2^H = 0, \quad \nabla^H_{E_2^H} E_1^H = \frac{1}{2} E_3^H, \\
\nabla^H_{E_3^H} E_3^H = 0, \quad \nabla^H_{E_3^H} E_2^H = -\frac{1}{2} E_1^H, \quad \nabla^H_{E_3^H} E_1^H = \frac{1}{2} E_2^H.
\]

In consequence of these calculations,

\[ \nabla^H_Z \zeta^H = -\frac{1}{2}(-a_1E_2^H + b_1E_1^H) = -\frac{1}{2}\phi Z^H \]

is satisfied, so (1.12) holds.

Due to

\[
(\nabla^H_Z \phi)W^H = \frac{1}{2}\{ -a_1c_2 E_1^H - b_1c_2 E_2^H + (a_1a_2 + b_1b_2) E_3^H \} = \frac{1}{2}[G^H(Z^H, W^H)\zeta^H - \eta^H(W^H)Z^H],
\]

it can be seen that (1.13) holds.

Because of

\[ \nabla^H_Z \eta^H(W^H) = \frac{1}{2}(a_1b_2 - b_1a_2) = \frac{1}{2}G^H(Z^H, \phi W^H), \]

(1.3) and (1.8) hold. Hence, \( (\phi^H, \zeta^H, \eta^H, G^H) \) is a Sasakian Finsler structure on \( TM^H \).

We can verify the following results:
Locally $\phi$-quasiconformally symmetric Sasakian Finsler structures...

$$R(E_1^H, E_2^H)E_1^H = \frac{3}{4}E_2^H, R(E_1^H, E_2^H)E_2^H = -\frac{3}{4}E_1^H,$$

$$R(E_1^H, E_3^H)E_3^H = 0, R(E_3^H, E_1^H)E_1^H = \frac{1}{4}E_3^H,$$

$$R(E_1^H, E_3^H)E_2^H = 0, R(E_1^H, E_3^H)E_3^H = \frac{1}{4}E_1^H, R(E_2^H, E_3^H)E_1^H = 0,$$

$$R(E_2^H, E_3^H)E_2^H = -\frac{1}{4}E_3^H, R(E_2^H, E_3^H)E_3^H = \frac{1}{4}E_2^H$$

and

$$S(E_1^H, E_1^H) = -\frac{1}{2}, S(E_2^H, E_2^H) = -\frac{1}{2}, S(E_3^H, E_3^H) = \frac{1}{2}$$

and also (1.17) holds and we get $r = -\frac{1}{2}$. Consequently, the scalar curvature $r$ is constant and by virtue of Corollary 3.2 and Corollary 3.3, $TM^H$ is locally $\phi$-quasiconformally symmetric. It is possible to verify that $TM^V$ is locally $\phi$-quasiconformally symmetric, similarly.

References


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