REGULAR SEMIGROUPS OF PARTIAL TRANSFORMATIONS PRESERVING A FENCE \( \mathbb{N} \)

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Abstract. Semigroups of order-preserving transformations have been extensively studied for finite chains. We study the monoid \( OP_\mathbb{N} \) of all order-preserving partial transformations on the set \( \mathbb{N} \) of natural numbers, where the partial order is a fence (also called zigzag poset). The monoid \( OP_\mathbb{N} \) is not regular. In this paper, we determine particular maximal regular subsemigroups of \( OP_\mathbb{N} \) and show that \( OP_\mathbb{N} \) has infinitely many maximal regular subsemigroups.

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1. Introduction

Semigroups of transformations on a set \( X \) preserving a partially order set \( (X; \preceq) \) have long been considered in the literature, starting with the works by Aizenštat \(^1\) and Popova \(^3\), in 1962. In the case that \( (X; \preceq) \) is a linearly ordered set, semigroups of order-preserving transformations have been the object of study by several authors and several papers (e.g. \(^6\), \(^7\), \(^8\), \(^9\), \(^10\), \(^11\)). In particular, the semigroup of all transformations preserving a linear order is regular. Semigroups of order-preserving transformations on \( (X; \preceq) \), where \( (X; \preceq) \) is any infinite partially ordered set, have not been extensively studied. For example in \(^12\), the authors investigated rank properties of endomorphisms of infinite partially ordered sets. We consider semigroups of transformations preserving a particular non-linear order on a countable infinite set, where the partial order is a fence. Let us recall that a fence (also called a zigzag poset) is a partially ordered set \( (X; \preceq) \), where the order on \( X \) is

\[
a_1 \prec a_2 \succ a_3 \prec a_4 \succ a_5 \succ \cdots \succ a_{2m-1} \prec a_{2m} \succ a_{2m+1} \prec \cdots
\]

or

\[
a_1 \succ a_2 \prec a_3 \succ a_4 \prec a_5 \succ \cdots \prec a_{2m-1} \succ a_{2m} \prec a_{2m+1} \succ \cdots,
\]

whenever \( X = \{a_1, a_2, \ldots\} \) is finite or countable infinite. The definition of the partial order \( \preceq \) is self-explanatory. Every element of \( X \) is either minimal or

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maximal. The number of order-preserving maps of fences and of crowns as well as of maps between fences and crowns are calculated in [1, 6, 10]. Recently, regular semigroups of transformations preserving a fence were characterized in [2, 13].

Let \( \mathbb{N} \) be the set of natural numbers and let \( \leq \) be the natural order on \( \mathbb{N} \). Throughout the paper, we consider the following binary relation \( \preceq \) on \( \mathbb{N} \):

\[
\begin{align*}
    n \prec n + 1 & \quad \text{if } n \text{ is odd;} \\
    n \succ n + 1 & \quad \text{if } n \text{ is even.}
\end{align*}
\]

It is easy to verify that \((\mathbb{N}; \preceq)\) is a fence. Where there is no possibility of confusion, we will write \( \mathbb{N} \) instead of \((\mathbb{N}; \preceq)\). Let \( \Sigma \) be any subset of \( \mathbb{N} \). We say that \( \Sigma \) is a fence if \((\Sigma; \preceq \cap (\Sigma \times \Sigma))\) is a fence, or in other words, \( x \in \Sigma \) implies either \( x + 1 \in \Sigma \) or \( x + k \notin \Sigma \) for all \( k \in \mathbb{N} \). Note that \( x \perp y \ (x, y \in \mathbb{N}) \) means that \( x \) and \( y \) are incomparable in the fence \( \mathbb{N} \), while \( x|y \) means that \( x \) and \( y \) are incomparable in the fence \( \mathbb{N} \). If \( \Sigma_1 \) and \( \Sigma_2 \) are subsets of \( \mathbb{N} \), we write \( \Sigma_1 \perp \Sigma_2 \) (and \( \Sigma_1 \parallel \Sigma_2 \), respectively) if \( x_1 \perp x_2 \) (and \( x_1 \parallel x_2 \), respectively) for all \( x_1 \in \Sigma_1 \) and all \( x_2 \in \Sigma_2 \).

In this paper, we study regular semigroups of partial transformations preserving the fence \( \mathbb{N} \). Let \( Y \subseteq \mathbb{N} \). A mapping \( \alpha : Y \to \mathbb{N} \) is called a partial transformation on \( \mathbb{N} \) with domain \( Y \) (in symbols: \( \text{dom } \alpha = Y \)). If \( Y = \mathbb{N} \), then \( \alpha \) is called a full transformation on \( \mathbb{N} \). We write transformations to the right of their argument and compose from left to right. The range of a partial transformation \( \alpha \) is denoted by \( \text{ran } \alpha = \{ y : \text{there exists } x \in \text{dom } \alpha \text{ such that } x\alpha = y \} \) and the kernel of \( \alpha \) is denoted by \( \ker \alpha = \{(x, y) : x, y \in \text{dom } \alpha \text{ and } x\alpha = y\alpha \} \). The rank of \( \alpha \) is the size of its range, denoted by \( \text{rank } \alpha = |\ker \alpha| \). For \( \Sigma \subseteq \mathbb{N} \), the set \( \Sigma \alpha^{-1} := \{ x \in \text{dom } \alpha : x\alpha \in \Sigma \} \) denotes the pre-image of \( \Sigma \) under \( \alpha \). If \( \Sigma \) is a singleton set, say \( \Sigma = \{ x \} \), then we write \( x\alpha^{-1} \) rather than \( \{ x \} \alpha^{-1} \). We denote by \( \alpha|_{\Sigma} \) the restriction of \( \alpha \) to \( \Sigma \cap \text{dom } \alpha \) and by \( \text{id}_{\Sigma} \) the identity mapping on \( \Sigma \). A subset \( T \) of \( \text{dom } \alpha \) is said to be a transversal of \( \ker \alpha \) if the intersection of \( T \) and any class of the partition corresponding to the equivalence relation \( \ker \alpha \) is a singleton set.

We say that a partial transformation \( \alpha \) on \( \mathbb{N} \) is order-preserving (or preserving the fence) if \( x \preceq y \) implies \( x\alpha \preceq y\alpha \), for all \( x, y \in \text{dom } \alpha \). Note that \( \text{ran } \alpha \) is a fence, whenever \( \alpha \) is order-preserving and \( \text{dom } \alpha \) is a fence [2]. Of course, the empty transformation \( \emptyset \) and the identity mapping \( \text{id}_{\mathbb{N}} \) on \( \mathbb{N} \) are order-preserving partial transformations on \( \mathbb{N} \). If \( \alpha \) is order-preserving, then \( \alpha|_{\Sigma} \) is also order-preserving for any set \( \Sigma \subseteq \text{dom } \alpha \). We denote by \( OP_{\mathbb{N}} \) the set of all order-preserving partial transformations on \( \mathbb{N} \) and remark that \( OP_{\mathbb{N}} \) is a semigroup under composition. For a subset \( A \) of \( OP_{\mathbb{N}} \), the least subsemigroup of \( OP_{\mathbb{N}} \) containing \( A \) is denoted by \( \langle A \rangle \).

A classical topic in the study of semigroups is the characterization of the regularity. An element \( x \) of a semigroup is said to be regular if there exists \( y \) in this semigroup such that \( xyx = x \). A semigroup is called regular if all its elements are regular. Groups are of course regular semigroup, but the class of regular semigroups is vastly more extensive than the class of groups. In [12], regular elements of some order-preserving transformation semigroups are
investigated. A characterization of the maximal regular subsemigroups of ideals of order-preserving or order-reversing transformations was given in [11]. Note that the semigroup $OP_N$ of all partial transformations preserving the fence $N$ is not regular.

Already in 1972, B. Schein asked for maximal regular subsemigroups of particular semigroups of binary relations. After that many authors studied maximal regular subsemigroups of transformations preserving an order (e.g. [13, 17, 18]). The purpose of this paper is to study (maximal) regular subsemigroups of $OP_N$: a subsemigroup $C$ of $OP_N$ is called a maximal regular subsemigroup if $C$ is regular and any subsemigroup of $OP_N$ having $C$ as proper subsemigroup is not regular. In the next section, we describe the regular elements in $OP_N$ and the surjective transformations in $OP_N$. In Section 3, we determine two maximal regular subsemigroups of $OP_N$ whose union covers all the regular transformations with rank $\leq 2$. The main result of this paper is given in Section 4. It seems almost impossible to determine all maximal regular subsemigroups of $OP_N$ but we classify a countable infinite set of maximal regular subsemigroups of $OP_N$.

2. The Regular Elements in $OP_N$

This section is devoted to the set $\operatorname{Reg}(OP_N)$ of all regular elements in $OP_N$. First of all, we give a straightforward description of the elements in $\operatorname{Reg}(OP_N)$. For the sake of completeness, we give a proof. Since the proof uses only the properties of a partially ordered set, the statement is true for any semigroup of order-preserving partial transformations.

**Proposition 2.1.** Let $\alpha \in OP_N \setminus \{\emptyset\}$. Then $\alpha$ is regular if and only if there exists a subset $Y$ of $\text{dom } \alpha$ such that $\alpha|_Y$ is a bijection from $Y$ to $\text{ran } \alpha$ and $(\alpha|_Y)^{-1} \in OP_N$.

**Proof.** Let $\alpha$ be regular. Then there exists $\beta \in OP_N$ such that $\alpha\beta\alpha = \alpha$. It is clear that $\text{ran } \alpha \subseteq \text{dom } \beta$ and $\text{ran } \alpha\beta \subseteq \text{dom } \alpha$. We show that $Y := \text{ran } \alpha\beta$ satisfies the required properties. First, we observe that $\alpha|_{\text{ran } \alpha\beta} : \text{ran } \alpha\beta \to \text{ran } \alpha$ is a bijection from $\text{ran } \alpha\beta$ to $\text{ran } \alpha$. It remains to show that $(\alpha|_{\text{ran } \alpha\beta})^{-1} \in OP_N$. Let $a, b \in \text{ran } \alpha$ with $a \prec b$. Since $\alpha|_{\text{ran } \alpha\beta}$ is a bijection from $\text{ran } \alpha\beta$ to $\text{ran } \alpha$, there exist a unique $a_1 \in \text{ran } \alpha\beta$ and a unique $b_1 \in \text{ran } \alpha\beta$ such that $a_1\alpha|_{\text{ran } \alpha\beta} = a$ and $b_1\alpha|_{\text{ran } \alpha\beta} = b$. Since $a_1, b_1 \in \text{ran } \alpha\beta$, there are $a_2, b_2 \in \text{ran } \alpha \subseteq \text{dom } \beta$ such that $a_2\beta = a_1$ and $b_2\beta = b_1$. Since $a_2, b_2 \in \text{ran } \alpha$, there are $a_3, b_3 \in \text{dom } \alpha$ such that $a_3\alpha = a_2$ and $b_3\alpha = b_2$. Thus, we obtain $a = a_3\alpha\beta\alpha$ as well as $b = b_3\alpha\beta\alpha$ and so

$$a_2 = a_3\alpha = a_3\alpha\beta\alpha = a \prec b = b_3\alpha\beta\alpha = b_3\alpha = b_2.$$  

Since $a_2, b_2 \in \text{ran } \alpha \subseteq \text{dom } \beta$ and $\beta \in OP_N$, we obtain $a_1 = a_2\beta \prec b_2\beta = b_1$. Hence,

$$a(\alpha|_{\text{ran } \alpha\beta})^{-1} = a_1 \prec b_1 = b(\alpha|_{\text{ran } \alpha\beta})^{-1}.$$
Conversely, suppose that there exists a subset $Y$ of $\text{dom } \alpha$ such that $\alpha|_Y$ is a bijection from $Y$ to $\text{ran } \alpha$ and $(\alpha|_Y)^{-1} \in \text{OP}_N$. Then, we have $\alpha(\alpha|_Y)^{-1}\alpha = \alpha \text{id}_{\text{ran } \alpha} = \alpha$, i.e. $\alpha$ is regular. \hfill \Box

Note that the set $Y$ in Proposition 2.1 is nothing but a transversal of $\text{ker } \alpha$. We prefer the setting in Proposition 2.1 since it is more practical for our proofs. We will frequently use this proposition in our proofs without mentioning it.

It is clear that any regular subsemigroup of $\text{OP}_N$ is a closed set (under composition) of regular elements in $\text{OP}_N$ but not conversely, i.e. a closed set (under composition) of regular elements in $\text{OP}_N$ does not need to be a regular subsemigroup of $\text{OP}_N$.

For example, let us consider the set $\text{Sur} := \{ \alpha \in \text{Reg}(\text{OP}_N) : \alpha \text{ is surjective} \}$.

**Proposition 2.2.** $\text{Sur}$ is a semigroup which is not regular.

**Proof.** First, we show that $\text{Sur}$ is closed under composition. Let $\gamma, \beta \in \text{Sur}$. Then $\gamma\beta$ is surjective. Since $\gamma, \beta \in \text{Reg}(\text{OP}_N)$, there exist subsets $Y_1 \subseteq \text{dom } \gamma$ and $Y_2 \subseteq \text{dom } \beta$ such that $\gamma|_{Y_1} : Y_1 \to \text{ran } \gamma = \mathbb{N}$ as well as $\beta|_{Y_2} : Y_2 \to \text{ran } \beta = \mathbb{N}$ are bijective and $(\gamma|_{Y_1})^{-1}, (\beta|_{Y_2})^{-1} \in \text{OP}_N$. It is easy to verify that

$$\gamma\beta|_{Y_2 \gamma^{-1} \cap Y_1} : Y_2 \gamma^{-1} \cap Y_1 \to \text{ran } \gamma\beta$$

is bijective and $(\gamma\beta|_{Y_2 \gamma^{-1} \cap Y_1})^{-1} \in \text{OP}_N$. This shows that $\gamma\beta \in \text{Reg}(\text{OP}_N)$ and hence $\text{Sur}$ is a semigroup. We now consider the partial transformation $\alpha : \mathbb{N} \setminus \{1, 2\} \to \mathbb{N}$ defined by $x\alpha = x - 2$ for all $x \in \mathbb{N} \setminus \{1, 2\}$. It is clear that $\alpha \in \text{Sur}$. Assume that there exists $\delta \in \text{Sur}$ with $\alpha\delta\alpha = \alpha$. Then $\alpha\delta\alpha = \alpha$, where $\delta = \text{ran } \alpha \subseteq \text{dom } \delta$ and we conclude that $\text{dom } \delta = \mathbb{N}$ and so $\text{ran } \delta = \mathbb{N}\delta = \mathbb{N} \setminus \{1, 2\}$ (otherwise, $\delta$ is not a function). It contradicts with the assumption that $\delta$ is surjective. Therefore, there is no $\delta \in \text{Sur}$ such that $\alpha\delta\alpha = \alpha$ and so $\text{Sur}$ is not regular. \hfill \Box

Now, we give a kind of constructive description of the transformations in $\text{Sur}$.

**Proposition 2.3.** $\text{Sur}$ is the set of all partial transformations $\alpha$ on $\mathbb{N}$ such that there are a natural number $k$ and a transformation $\beta \in \text{OP}_N$ with the following properties:

1. $\text{dom } \beta = \{i \in \mathbb{N} : 1 \leq i \leq 2(k - 1)\} \cap \text{dom } \alpha$;
2. $(2(k - 1))\beta \in \{1, 2\}$, whenever $2(k - 1) \in \text{dom } \beta$;
3. $\alpha|_{\text{dom } \beta} = \beta$;
4. $(2(k - 1) + i)\alpha = i$ for all $i \in \mathbb{N}$.

**Proof.** Clearly, a partial transformation $\alpha$ satisfying the given properties is order-preserving and surjective. Let $Y := \{2(k - 1) + i : i \in \mathbb{N}\} \subseteq \text{dom } \alpha$. 

It is easy to verify that $\alpha|_Y : Y \to \mathbb{N}$ is bijective and $(\alpha|_Y)^{-1} \in OP_{\mathbb{N}}$, i.e. $\alpha \in \text{Reg}(OP_{\mathbb{N}})$.

Conversely, let $\alpha \in \text{Sur}$. Then $\alpha \in \text{Reg}(OP_{\mathbb{N}})$ and ran $\alpha = \mathbb{N}$. Since $\alpha \in \text{Reg}(OP_{\mathbb{N}})$, there exists a subset $Y$ of dom $\alpha$ such that $\alpha|_Y : Y \to \mathbb{N}$ is bijective and $(\alpha|_Y)^{-1} \in OP_{\mathbb{N}}$. Since ran $\alpha = \mathbb{N}$ is a fence and $(\alpha|_Y)^{-1} \in OP_{\mathbb{N}}$, we obtain that $Y$ is a fence. Let $a$ be the least natural number in $Y$ with respect to the natural order of $\mathbb{N}$. Assume that $a \alpha = x$ for some $x \in \mathbb{N}\setminus\{1\}$. Since $x-1, x+1 \in \text{ran} \alpha = \mathbb{N}$ and $\alpha|_Y : Y \to \mathbb{N}$ is bijective, there exist $(x-1)', (x+1)' \in Y$ such that $(x-1)'\alpha = x-1$ and $(x+1)'\alpha = x+1$. Since $(\alpha|_Y)^{-1} \in OP_{\mathbb{N}}$ and either $x-1 < x > x+1$ or $x-1 > x > x+1$, we obtain that either $(x-1)' < a > (x+1)'$ or $(x-1)' > a < (x+1)'$. Because $(x-1)' \neq (x+1)'$, this provides that either $(x-1)' = a-1$ or $(x+1)' = a-1$. It contradicts with $a$ is the least natural number in $Y$. So, we have $a \alpha = 1$. Since $1 < 2$ in ran $\alpha = \mathbb{N}$, from $\alpha|_Y : Y \to \mathbb{N}$ is bijective and $(\alpha|_Y)^{-1} \in OP_{\mathbb{N}}$, we conclude that $a = 1(\alpha|_Y)^{-1} < 2(\alpha|_Y)^{-1}$. This shows that $a$ is odd. We choose $k \in \mathbb{N}$ such that $2(k-1) + 1 = a$. Using the facts that $a \alpha = 1$ and that the image of $\{a, \ldots, a+i\}$ under $\alpha|_Y$ is a fence of size $i + 1$ for all $i \in \mathbb{N}$, we obtain recursively that $(a+i)\alpha = i + 1$ for all $i \in \mathbb{N}$. So, $(2(k-1) + i)\alpha = i$ for all $i \in \mathbb{N}$.

Let us put $\beta := \alpha|_{\{1,\ldots,a-1\}\cap \text{dom} \alpha}$. Clearly, $\beta \in OP_{\mathbb{N}}$ and dom $\beta = \{i \in \mathbb{N} : 1 \leq i \leq 2(k-1)\} \cap \text{dom} \alpha$. Suppose $2(k-1) \in \text{dom} \beta$. Since $2(k-1) > 2(k-1) + 1 = a$ in dom $\alpha$, we have that $(2(k-1))\alpha > (2(k-1) + 1)\alpha = a\alpha = 1$. This implies that either $(2(k-1))\alpha = 1$ or $(2(k-1))\alpha = 2$. \hfill $\square$

Note that $\text{Inj} := \{\alpha \in OP_{\mathbb{N}} : \alpha$ is injective and $\alpha^{-1} \in OP_{\mathbb{N}}\} = \{\alpha \in \text{Reg}(OP_{\mathbb{N}}) : \alpha$ is injective\} is an inverse subsemigroup of $OP_{\mathbb{N}}$. It is well known that a partial transformation $\alpha$ on $\mathbb{N}$ is idempotent if $\alpha|_{\text{ran} \alpha}$ is the identity mapping on ran $\alpha$. Any idempotent element in $OP_{\mathbb{N}}$ is regular in $OP_{\mathbb{N}}$. But $OP_{\mathbb{N}}$ is not orthodox, i.e. the idempotent elements in $OP_{\mathbb{N}}$ do not form a regular subsemigroup.

### 3. Regular Transformations with rank $\leq 2$

In this section, we study regular subsemigroups of $OP_{\mathbb{N}}$ containing regular transformations with rank $\leq 2$. The set

$$I_1 := \{\alpha \in OP_{\mathbb{N}} : \text{rank} \alpha \leq 1\}$$

is a regular subsemigroup of $OP_{\mathbb{N}}$ since $I_1$ is an ideal consisting entirely of regular elements in $OP_{\mathbb{N}}$.

**Remark 3.1.** $I_1$ is contained in all maximal regular subsemigroups of $OP_{\mathbb{N}}$.

**Proof.** Let $C$ be any maximal regular subsemigroup of $OP_{\mathbb{N}}$. Since $I_1$ and $C$ are regular subsemigroups of $OP_{\mathbb{N}}$ and $I_1$ is an ideal of $OP_{\mathbb{N}}$, we obtain that $I_1 \cup C$ is a regular subsemigroup of $OP_{\mathbb{N}}$. Since $C$ is a maximal regular subsemigroup of $OP_{\mathbb{N}}$, we have that $C = I_1 \cup C$ and so $I_1 \subseteq C$. \hfill $\square$
Not all order-preserving partial transformations with rank = 2 are regular, for example the partial transformation $\alpha \in OP_N$ with $\text{dom} \alpha = \mathbb{N} \setminus \{2\}$ defined by $1\alpha = 1$ and $x\alpha = 2$ for all $x \in \mathbb{N} \setminus \{1, 2\}$ is not regular in $OP_N$.

**Proposition 3.2.** Let $\alpha \in OP_N$ with rank $\alpha = 2$, say $\text{ran} \alpha = \{a, b\}$. Then the following statements are equivalent:

(i) $\alpha \in \text{Reg}(OP_N)$;

(ii) $a \parallel b$ if and only if $a\alpha^{-1} \parallel b\alpha^{-1}$.

**Proof.** Suppose that $\alpha \in \text{Reg}(OP_N)$. Then there exists a subset $Y$ of $\text{dom} \alpha$ such that $\alpha|_Y$ is a bijection from $Y$ to $\text{ran} \alpha = \{a, b\}$ and $(\alpha|_Y)^{-1} \in OP_N$. If $a \parallel b$, then $\alpha \in OP_N$ implies $a\alpha^{-1} \parallel b\alpha^{-1}$. If $a\alpha^{-1} \parallel b\alpha^{-1}$, then $(\alpha|_Y)^{-1} \in OP_N$ implies $a \parallel b$.

Conversely, suppose that $a \parallel b$ if and only if $a\alpha^{-1} \parallel b\alpha^{-1}$. If $a \parallel b$, then we let $Y$ be a transversal of ker $\alpha$. If $a \perp b$, then there are $a' \in a\alpha^{-1}$, $b' \in b\alpha^{-1}$ such that $a' \perp b'$ and we set $Y := \{a', b'\}$. It is easy to verify that in both cases $\alpha|_Y$ is a bijection from $Y$ to $\text{ran} \alpha$ and $(\alpha|_Y)^{-1} \in OP_N$, i.e. $\alpha \in \text{Reg}(OP_N)$.

In the remainder of this section, we will frequently use the following well known fact.

**Remark 3.3.** Let $\alpha, \beta \in OP_N$. Then $\text{rank} \alpha \beta \leq \min\{\text{rank} \alpha, \text{rank} \beta\}$, where $\min\{\text{rank} \alpha, \text{rank} \beta\}$ means the least one of both cardinals $\text{rank} \alpha$ and $\text{rank} \beta$.

Let now $K_1 := \{\alpha \in \text{Reg}(OP_N) : \text{ran} \alpha$ is a fence and rank $\alpha = 2\} \cup I_1$.

**Proposition 3.4.** $K_1$ is a regular subsemigroup of $OP_N$.

**Proof.** First, we show that $K_1$ is a subsemigroup of $OP_N$. Let $\alpha, \beta \in K_1$. By Remark 4.3, we have that rank $\alpha \beta \leq 2$. If rank $\alpha \beta \leq 1$, then $\alpha \beta \in I_1 \subseteq K_1$. Suppose that rank $\alpha \beta = 2$. Then rank $\alpha = \text{rank} \beta = 2$ and $\text{ran} \alpha = \text{ran} \beta$ as well as $\text{dom} \alpha \beta = \text{dom} \alpha$. Suppose that ran $\alpha = \{a_1, a_2\}$ with $a_1 \prec a_2$.

There are $x_1, x_2 \in \text{dom} \alpha = \text{dom} \alpha \beta$ with $x_1 \alpha = a_1$ and $x_2 \alpha = a_2$ such that $x_1 \prec x_2$ by Proposition 4.2. Since $\alpha, \beta \in OP_N$, we obtain $x_1 \alpha \beta < x_2 \alpha \beta$, where ran $\alpha \beta = \text{ran} \beta = \{a_1 \beta, a_2 \beta\} = \{x_1 \alpha \beta, x_2 \alpha \beta\}$. This provides that $\alpha \beta \in \text{Reg}(OP_N)$ by Proposition 4.2 and so $\alpha \beta \in K_1$.

Now, we show that $K_1$ is a regular subsemigroup of $OP_N$. Since $I_1$ is a regular semigroup and $I_1 \subset K_1$, we have only to consider the elements in $K_1 \setminus I_1$. Let $\alpha \in K_1 \setminus I_1$. Assume now that ran $\alpha = \{b_1, b_2\}$ with $b_1 \prec b_2$. Since $\alpha \in \text{Reg}(OP_N)$, there are $a_1, a_2 \in \text{dom} \alpha$ with $a_1 \alpha = b_1, a_2 \alpha = b_2$, and $a_1 \prec a_2$ by Proposition 4.2. Let us consider the partial transformation $\beta : \{b_1, b_2\} \rightarrow \{a_1, a_2\}$ defined by $b_1 \beta = a_1$ and $b_2 \beta = a_2$. It is easy to verify that $\beta \in K_1$ and $\alpha \beta \alpha = \alpha \text{id}_{\text{ran} \alpha} = \alpha$. This shows that $\alpha$ is regular in $K_1$. Altogether, we have that $K_1$ is a regular subsemigroup of $OP_N$.

As an immediately consequence, we obtain the following:

**Corollary 3.5.** $K_1 \cup \{\text{id}_N\}$ is a regular subsemigroup of $OP_N$. 
**Proof.** The monoid $K_1 \cup \{id_N\}$ corresponding to the regular semigroup $K_1$ is regular.

In order to prove that $K_1 \cup \{id_N\}$ is a maximal regular subsemigroup of $OP_N$, we need two technical lemmas. The first one states that the union of the semigroups $K_1$ and $Sur$ is also a semigroup. The second one shows that $id_N$ is the only bijection from $N$ to $N$ which preserves the fence $N$.

**Lemma 3.6.** $K_1 \cup Sur$ is a subsemigroup of $OP_N$.

**Proof.** Proposition 2.2 and Theorem 3.3 show that both $Sur$ and $K_1$ are closed. Let $\alpha \in K_1$ and $\beta \in Sur$. By Remark 3.2, we have that rank $\alpha \beta \leq 2$ and rank $\beta \alpha \leq 2$. If rank $\alpha \beta$, rank $\beta \alpha \leq 1$, then $\alpha \beta, \beta \alpha \in I_1 \subseteq K_1$.

Assume now that rank $\alpha \beta = 2$. Then rank $\alpha = 2$ and we have that $\alpha \subseteq \text{dom } \beta$ and $\text{dom } \alpha \beta = \text{dom } \alpha$. Since $\text{ran } \alpha \subseteq \text{dom } \beta$ and $\text{ran } \alpha$ is a fence, we have that $\alpha \beta = (\text{ran } \alpha) \beta$ is a fence. It remains to show that $\alpha \beta \in \text{Reg}(OP_N)$. Suppose that $\text{ran } \alpha = \{b_1, b_2\}$ and $\alpha \beta = \{x_1, x_2\}$ with $b_1 < b_2$ and $x_1 < x_2$. Then there exist $a_1 \in b_1 \alpha^{-1}$ and $a_2 \in b_2 \alpha^{-1}$ such that $a_1 < a_2$ by Proposition 3.2. We conclude that $a_1 \alpha \beta = x_1$ and $a_2 \alpha \beta = x_2$, i.e. $a_1 \in x_1(\alpha \beta)^{-1}$ and $a_2 \in x_2(\alpha \beta)^{-1}$, since $x_1 < x_2$, $b_1 < b_2$, and $\alpha \beta = \{x_1, x_2\}$.

We obtain $\alpha \beta \in \text{Reg}(OP_N)$ by Proposition 3.2.

Next, we assume that rank $\beta \alpha = 2$. Since $\text{ran } \beta = N$, we have that $\text{dom } \alpha \subseteq \text{ran } \beta$ and $\text{ran } \beta \alpha = \text{ran } \alpha$, say $\alpha = \{x_1, x_2\}$ with $x_1 < x_2$. It remains to show that $\beta \alpha \in \text{Reg}(OP_N)$. Since $\alpha \in K_1$, there exist $b_1 \in x_1 \alpha^{-1}$ and $b_2 \in x_1 \alpha^{-1}$ such that $b_1 < b_2$ by Proposition 3.2. Because $\beta \in \text{Reg}(OP_N)$, there exists a subset $Y$ of $\text{dom } \beta$ such that $\beta|_Y$ is a bijection from $Y$ to $\text{ran } \beta = N$ and $(\beta|_Y)^{-1} \in OP_N$. Since $b_1 < b_2$ in $\text{dom } \alpha \subseteq \text{ran } \beta$, we obtain $b_1(\beta|_Y)^{-1} < b_2(\beta|_Y)^{-1}$, where $b_1(\beta|_Y)^{-1} \in x_1(\alpha \beta)^{-1} \subseteq \text{dom } \beta \alpha$ and $b_2(\beta|_Y)^{-1} \in x_2(\alpha \beta)^{-1} \subseteq \text{dom } \beta \alpha$. Together with ran $\beta \alpha = \{x_1, x_2\}$ and $x_1 < x_2$, we conclude that $\beta \alpha \in \text{Reg}(OP_N)$ by Proposition 3.2.

**Lemma 3.7.** Let $\alpha \in OP_N$ be a bijection from $N$ to $N$. Then $\alpha = id_N$.

**Proof.** Assume that $\alpha \neq id_N$. Then there exists the least natural number $n \in N$ with respect to the natural order on $N$ such that $n \alpha \neq n$, i.e. $n \alpha > n$. If $n \geq 2$, then $n - 1 \in \text{dom } \alpha$. Since $n - 1 \perp n$ in $\text{dom } \alpha$ but $(n - 1)\alpha = n - 1||n\alpha$, we obtain $\alpha \not\in OP_N$, a contradiction. Suppose now that $n = 1$. Then $a \alpha > 1$ and there are $a_1, \ldots, a_{1\alpha-1} \in \text{dom } \alpha$ with $\{a_1 \alpha, \ldots, a_{1\alpha-1} \alpha\} = \{1, \ldots, 1\alpha - 1\}$. Let $b := \max\{a_1, \ldots, a_{1\alpha-1}\}$ (the maximal element with respect to the natural order on $N$). Because of $ba < 1\alpha < (b + 1)\alpha$, we get that $ba||((b + 1)\alpha$, a contradiction with $\alpha \in OP_N$.

Now, we are able to prove that the regular subsemigroup $K_1 \cup \{id_N\}$ of $OP_N$ is a maximal one.

**Theorem 3.8.** $K_1 \cup \{id_N\}$ is a maximal regular subsemigroup of $OP_N$. 

Proof. $K_1 \cup \{id_N\}$ is a regular subsemigroup of $OP_N$ by Corollary 3.3. It remains to show the maximality. Assume that there exists a regular semigroup $S$ such that

$$K_1 \cup \{id_N\} \subseteq S.$$ 

Then there is $\beta \in S \setminus (K_1 \cup \{id_N\})$.

We claim that there exists an element in $S \setminus (K_1 \cup \{id_N\})$ which is not in $Sur$. If $\beta \notin Sur$, then $\beta$ is the required element. Assume now that $\beta \in Sur$. Since $S$ is regular and $\beta \in S$, there exists $\alpha \in S$ such that $\beta \alpha \beta = \beta$. Assume that $\alpha = N$. Since $\beta \alpha \beta = \beta$, we have $N = \text{ran } \beta \subseteq \text{dom } \alpha$ and so $\text{dom } \alpha = N$. Since $\text{ran } \alpha = N = \text{dom } \alpha$ and $\beta \alpha \beta = \beta$, we obtain that $\alpha : N \to N$ is a bijection from $N$ to $N$. Therefore, $\alpha = id_N$ by Lemma 3.2. Then $\beta \alpha \beta = \beta$ implies that $\beta^2 = \beta$. But $\beta$ is idempotent means that $\beta|_N = id_N$, i.e. $\beta = id_N$, a contradiction. Thus, $\text{ran } \alpha \neq N$ and so $\alpha \notin Sur$. Since $\beta \alpha \beta = \beta$, we obtain $\text{rank } \alpha > 2$ by Remark 3.3, i.e. $\alpha \notin K_1$. Consequently, we have shown that $\alpha \in S \setminus (K_1 \cup \{id_N\})$ and $\alpha \notin Sur$. Then $\alpha$ is the required element and so we have the claim.

So, there is $\gamma \in S \setminus (K_1 \cup \{id_N\})$ with $\gamma \notin Sur$. If $\text{ran } \gamma = N \setminus \{1, 2, \ldots, i\}$ for some $i \in N$, then we put $b_1 := i + 1$, $b_2 := i + 3$, and $b_3 := i$. If $\text{ran } \gamma = \{i, i + 1, i + 2, \ldots, j\}$ with $2 \leq i + 1 < j \in N$, then we put $b_1 := j$, $b_2 := j - 2$, and $b_3 := j + 1$. If $\text{rank } \gamma = 2$ and $\text{ran } \gamma$ is not a fence, say $\text{ran } \gamma = \{c, d\}$, then we put $b_1 := c$, $b_2 := d$, and $b_3 := c + 1$. If $\text{ran } \gamma$ is not a fence with $\text{rank } \gamma \geq 3$, then there exist $c, d \in \text{ran } \gamma$ with $c < d$ and $c + 1 \notin \text{ran } \gamma$. In this case we put $b_1 := c$, $b_2 := d$, and $b_3 := c + 1$. It is easy to verify that we have covered all the possibilities for $\text{ran } \gamma$. Now, we define a partial transformation $\alpha : \{b_1, b_2, b_3\} \to \{b_1, b_3\}$ by

$$x \alpha := \begin{cases} b_1 & \text{if } x = b_1 \\ b_3 & \text{if } x = b_2, b_3. \end{cases}$$

It is clear that $\alpha \in K_1$. Then $\gamma \alpha \in S$. Now, we have $\text{ran } \gamma \alpha = \{b_1, b_3\}$. Since $b_1 || b_2$ in $\text{ran } \gamma$, we obtain $b_1 \gamma^{-1} || b_2 \gamma^{-1}$. Since $b_1 (\gamma \alpha)^{-1} = b_1 \alpha^{-1} \gamma^{-1} = b_1 \gamma^{-1}$ and $b_3 (\gamma \alpha)^{-1} = b_3 \alpha^{-1} \gamma^{-1} = \{b_2, b_3\} \gamma^{-1} = b_2 \gamma^{-1}$, we get $b_1 (\gamma \alpha)^{-1} || b_3 (\gamma \alpha)^{-1}$. By Proposition 3.2, we obtain $\gamma \alpha \notin \text{Reg}(OP_N)$. It contradicts that $S$ is regular.

Altogether, we obtain that $K_1 \cup \{id_N\}$ is a maximal regular subsemigroup of $OP_N$.

Let $K_2 := \{\alpha \in \text{Reg}(OP_N) : a \perp b \Rightarrow a \alpha^{-1} \perp b \alpha^{-1} \text{ for all } a \neq b \in \text{ran } \alpha\}$. It is easy to verify that any $\alpha \in \text{Reg}(OP_N)$ with rank $\alpha = 2$ such that $\text{ran } \alpha$ is not a fence is contained in $K_2$. Moreover, we can observe that $x_1 \prec x_2$ for all $x_1 \in a_1 \alpha^{-1}$ and all $x_2 \in a_2 \alpha^{-1}$, whenever $\alpha \in K_2$ and $a_1, a_2 \in \text{ran } \alpha$ with $a_1 \prec a_2$. We will show that $K_2$ is a maximal regular subsemigroup of $OP_N$. Since we will need it in the proof of the next theorem, let us note that $K_2$ contains both the inverse semigroups $\text{Inj}$ and $I_1$. 


Theorem 3.9. $K_2$ is a maximal regular subsemigroup of $OP_\mathbb{N}$.

Proof. First, we show that $K_2$ is a subsemigroup of $OP_\mathbb{N}$. Let $\alpha, \beta \in K_2$. Since $\alpha \in \text{Reg}(OP_\mathbb{N})$, there exists a subset $Y$ of $\text{dom} \alpha$ such that $\alpha|_Y$ is a bijection from $Y$ to $\text{ran} \alpha$ and $(\alpha|_Y)^{-1} \in OP_\mathbb{N}$. Since $\beta \in K_2$, it is easy to verify that $\beta|_{\text{ran} \alpha} \in K_2$. Since $\beta|_{\text{ran} \alpha} \in \text{Reg}(OP_\mathbb{N})$, there exists $Z \subset \text{dom} \beta|_{\text{ran} \alpha}$ such that $\beta|_{\text{ran} \alpha}$ is a bijection from $Z$ to $\text{ran} \beta|_{\text{ran} \alpha} = \text{ran} \alpha$. Let

$$Y' := Z(\alpha|_Y)^{-1}.$$ 

It is easy to check that $\alpha \beta|_{Y'}$ is a bijection from $Y'$ to $\text{ran} \alpha \beta$. Let $a_1, a_2 \in \text{ran} (\alpha \beta|_{Y'})$ with $a_1 \prec a_2$. Then there are $x_1, x_2 \in Y'$ with $x_1 \alpha \beta = a_1$ and $x_2 \alpha \beta = a_2$ and thus $x_1 \alpha \in a_1 \beta^{-1}$ and $x_2 \alpha \in a_2 \beta^{-1}$. This provides $x_1 \alpha \prec x_2 \alpha$ and so $x_1 \prec x_2$ since $\beta \in K_2$ and $\alpha \in K_2$, respectively. Consequently, $(\alpha \beta|_{Y'})^{-1} \in OP_\mathbb{N}$, i.e. $\alpha \beta \in \text{Reg}(OP_\mathbb{N})$. Now, we show that $\alpha \beta \in K_2$. Let $b_1, b_2 \in \text{ran} \alpha \beta$ with $b_1 \perp b_2$. Assume that there are $a_1 \in b_1(\alpha \beta)^{-1}$ and $a_2 \in b_2(\alpha \beta)^{-1}$ such that $a_1 \parallel a_2$. This implies $a_1 \alpha \parallel a_2 \alpha$ since $\alpha \in K_2$, where $a_1 \alpha, a_2 \alpha \in \text{dom} \beta$. Thus, $\beta \in K_2$ implies $a_1 \alpha \parallel a_2 \alpha \beta$. This means $b_1 \parallel b_2$, a contradiction. Hence, $b_1(\alpha \beta)^{-1} \perp b_2(\alpha \beta)^{-1}$. This shows $\alpha \beta \in K_2$.

Next, we show that $K_2$ is a regular subsemigroup of $OP_\mathbb{N}$. Let $\alpha \in K_2$ and let $Y$ be a transversal of $\ker \alpha$. Then $(\alpha|_Y)^{-1}$ is a bijection from $\text{ran} \alpha$ to $Y$ with $(\alpha|_Y)^{-1} \in OP_\mathbb{N}$, which is an immediate consequence of the fact that $\alpha \in K_2$. On the other hand, $\alpha \in OP_\mathbb{N}$ implies $((\alpha|_Y)^{-1})^{-1} = \alpha|_Y \in OP_\mathbb{N}$. Then $(\alpha|_Y)^{-1} \in \text{Inj} \subseteq K_2$, where $\alpha(\alpha|_Y)^{-1} \alpha = \text{id}_{\text{ran} \alpha} = \alpha$.

Finally, we show that $K_2$ is a maximal regular subsemigroup of $OP_\mathbb{N}$. Assume that there is a regular semigroup $S$ such that $K_2 \subseteq S$. Then there exists $\beta \in S \setminus K_2$. Since $\beta \not\in K_2$, there are $a, b \in \text{ran} \beta$ with $a \perp b$ such that there are $x \in a \beta^{-1}$ and $y \in b \beta^{-1}$ with $x \parallel y$. Let $\alpha$ be the identity mapping on $\{x, y\}$. It is clear that $\alpha \in K_2$ and so $\alpha \beta \in S$. Now, we have that $\text{dom} \alpha \beta = \{x, y\}$ and $\text{ran} \alpha \beta = \{a, b\}$. Thus, $a \perp b$ and $x \parallel y$ implies $\alpha \beta \not\in \text{Reg}(OP_\mathbb{N})$ by Proposition 3.2. It contradicts that $S$ is a regular semigroup. Consequently, $K_2$ is a maximal regular subsemigroup of $OP_\mathbb{N}$.

We finish this section with the remark that the union of both maximal regular subsemigroups $K_1 \cup \{\text{id}_\mathbb{N}\}$ and $K_2$ covers all regular partial transformations with rank $\leq 2$. In the next section, we will show that there are countably infinitely many maximal regular subsemigroups of $OP_\mathbb{N}$.

4. Maximal Regular Subsemigroups

It seems almost impossible to classify all (maximal) regular subsemigroups of $OP_\mathbb{N}$. But we are able to determine countably infinitely many maximal regular subsemigroups of $OP_\mathbb{N}$.

Definition 4.1. Let $a \in \mathbb{N} \setminus \{1\}$. Then let $C_a^*$ be the set of all $\alpha \in \text{Reg}(OP_\mathbb{N})$ with

(i) $a \not\in \text{ran} \alpha$ or $\text{ran} \alpha \subseteq \{a - 1, a, a + 1\}$;
Suppose now a ran are bijective and a

(i) Since C X = Y
Proof.

Let a transversal of ker Proof.

C also the remaining elements in
It is clear that
Lemma 4.3.

Corollary 4.2. Let a, b ∈ N \ {1} with a ≠ b. Then C_a * ≠ C_b *.

Proof. Let a, b ∈ N \ {1} with a ≠ b. Without loss of generality, we can suppose that a = b + i for some i ∈ N. Let α be the identity mapping on {a, a + 2}. It is clear that α ∈ C_b *. Since a ∈ dom α and ran α = {a, a + 2} ⊄ {a - 1, a, a + 1}, we conclude that α ∉ C_a * and so C_a * ≠ C_b *.

We show now that C_a := C_a * ∪ {id_N} is a maximal regular subsemigroup of OP_N. First, we observe that any α ∈ C_a with a ∉ dom α belongs to K_2. But also the remaining elements in C_a * are related to the semigroup K_2.

Lemma 4.3. Let a ∈ N \ {1}. If α ∈ C_a * \ {∅}, then there exists a subset Y of dom α such that α|_Y is a bijection from Y to ran α with

(a) α|_Y ∈ K_2 :
(b) a ∉ Y if a ∉ dom α or | aαα^-1 | ≥ 2 :
(c) Y ⊆ {a - 1, a, a + 1} if ran α = ran (α|_{a-1,a,a+1}) and | aαα^-1 | = 1.

Proof. Let α ∈ C_a * \ {∅}. If a ∉ dom α or | aαα^-1 | ≥ 2, then let Y be a transversal of ker α with a ∉ Y. Since α|_{dom α \ \{a\}} ∈ K_2, we obtain that α|_Y ∈ K_2. Assume now that ran α = ran (α|_{a-1,a,a+1}) and | aαα^-1 | = 1. Let Y ⊆ {a - 1, a, a + 1} be a transversal of ker α. Then | aαα^-1 | = 1 implies a ∈ Y and so Y is a fence. Hence, α|_Y ∈ K_2 is clear.

In the next step, we verify that C_a is a semigroup which is regular.

Lemma 4.4. Let a ∈ N \ {1}. Then C_a is a regular subsemigroup of OP_N.

Proof. First, we show that C_a is a subsemigroup of OP_N. Let α, β ∈ C_a. If αβ = ∅, then αβ ∈ C_a. Moreover, α = id_N or β = id_N gives immediately αβ ∈ C_a. Assume now that α, β ≠ id_N and αβ ≠ ∅. By Lemma 13, there are sets X ⊆ dom α and Y ⊆ dom β such that α|_X : X → ran α and β|_Y : Y → ran β are bijective and α|_X, β|_Y ∈ K_2. Let

X' := (ran α ∩ Y)α^{-1} ∩ X ⊆ dom α.

It is clear that αβ|_X' : X' → ran αβ is bijective and (αβ|_X')^-1 ∈ OP_N, i.e. αβ ∈ Reg(OP_N). It remains to show that αβ satisfies the three properties of C_a *.

(i) Since a ∉ ran β or ran β ⊆ {a - 1, a, a + 1}, we obtain that a ∉ ran αβ or ran αβ ⊆ ran β ⊆ {a - 1, a, a + 1}.
(ii) If a ∉ dom α or | aαα^-1 | ≥ 2, then a ∉ dom αβ or | a(αβ)(αβ)^{-1} | ≥ 2. Suppose now

ran α = ran (α|_{a-1,a,a+1}).
This implies 
\[(\text{ran } \alpha)\beta = (\text{ran } \alpha|_{\{a-1,a,a+1\}})\beta,\]
where 
\[(\text{ran } \alpha)\beta = \text{ran } \alpha \beta \]
and
\[(\text{ran } \alpha|_{\{a-1,a,a+1\}})\beta = \text{ran } (\alpha|_{\{a-1,a,a+1\}})\beta = \text{ran } (\alpha|_{\{a-1,a,a+1\}}).\]

Altogether, we obtain that 
\[\text{ran } \alpha \beta = \text{ran } (\alpha|_{\{a-1,a,a+1\}}).\]

(iii) Let \(b_1, b_2 \in \text{ran } (\alpha|_{\text{dom } \alpha|_{\{a\}}})\) be such that \(b_1 \neq b_2\) and \(b_1 \perp b_2\). Assume that there are \(a_1 \in b_2(\alpha|_{\text{dom } \alpha|_{\{a\}}}^{-1})^{-1}\) and \(a_2 \in b_2(\alpha|_{\text{dom } \alpha|_{\{a\}}}^{-1})^{-1}\) with \(a_1 \parallel a_2\). Since \(a_1 \parallel a_2\) in \(\text{dom } \alpha \setminus \{a\}\) and \(\alpha|_{\text{dom } \alpha \setminus \{a\}} \in K_2\), we have \(a_1 \alpha \parallel a_2\alpha\). If \(a \not\in \text{ran } \alpha\), then \(a_1 \alpha \parallel a_2\alpha\) in \(\text{dom } \beta \setminus \{a\}\). Assume now that \(\text{ran } \alpha \subseteq \{a-1,a,a+1\}\). If \(a_1 \alpha = a\) or \(a_2\alpha = a\), then \(a_1 \alpha \perp a_2\alpha\), a contradiction. Hence, \(a_1 \alpha, a_2\alpha \neq a\) and \(a_1 \alpha \parallel a_2\alpha\) in \(\text{dom } \beta \setminus \{a\}\). Since \(\beta|_{\text{dom } \beta \setminus \{a\}} \in K_2\), we obtain \(a_1 \alpha \beta \parallel a_2\alpha \beta\). This means that \(b_1||b_2\), a contradiction. Thus,
\[
b_1(\alpha|_{\text{dom } \alpha|_{\{a\}}}^{-1})^{-1} \perp b_2(\alpha|_{\text{dom } \alpha|_{\{a\}}}^{-1})^{-1}.
\]

Altogether, we have shown that \(\alpha|_{\text{dom } \alpha|_{\{a\}}} \in K_2\).

Next, we check that \(C_a\) is a regular subsemigroup of \(OP_N\). Let \(\alpha \in C_a\). If \(\alpha \in \{\emptyset, \text{id}_N\}\), then \(\alpha\) is regular in \(C_a\). Assume now that \(\alpha \in C_a \setminus \{\emptyset, \text{id}_N\}\). By Lemma 4.3, there exists a subset \(Y\) of \(\text{dom } \alpha\) such that \(\alpha|_Y : Y \to \text{ran } \alpha\) is bijective and \(\alpha|_Y \in K_2\) satisfying the properties (a), (b), and (c). Let \(\beta := (\alpha|_Y)^{-1}\). It is clear that \(\alpha \beta \alpha = \text{id}_{\text{ran } \alpha} = \alpha\). We still have to show that \(\beta \in C_a^*\).

(i) If \(a \not\in \text{dom } \alpha\) or \(|a \alpha \alpha^{-1}| \geq 2\), then \(a \not\in Y = \text{ran } \beta\) by (b). If \(\alpha = \text{ran } \alpha|_{\{a-1,a,a+1\}}\) and \(|a \alpha \alpha^{-1}| = 1\), then we obtain that \(\text{ran } \beta = Y \subseteq \{a-1,a,a+1\}\) by (c).

(ii) If \(a \not\in \text{ran } \alpha\), then \(a \not\in \text{dom } \beta\). If \(\alpha \subseteq \{a-1,a,a+1\}\) and \(\beta = \text{ran } \alpha \subseteq \{a-1,a,a+1\}\) and so \(\beta = \beta|_{\{a-1,a,a+1\}}\), i.e. \(\text{ran } \beta = \beta|_{\{a-1,a,a+1\}}\).

(iii) Since \(\beta^{-1} = (\alpha|_Y)^{-1} = \alpha|_Y \in K_2 \subseteq OP_N\) is bijective, we conclude that \(\beta \in \text{Inj} \subseteq K_2\). Hence, \(\beta|_{\text{dom } \beta \setminus \{a\}} \in K_2\).

Now we show that \(C_a\) is a maximal regular subsemigroup of \(OP_N\). In order to prove it we need three technical lemmas.

**Lemma 4.5.** Let \(\alpha \in \text{Sur}\). If there exists \(x \in \mathbb{N}\) such that \(x \alpha = x\) and \(|x \alpha^{-1}| = 1\), then \(\alpha = \text{id}_N\).

**Proof.** By Proposition 4.3, there is \(k \in \mathbb{N}\) such that \((2(k-1)+i)\alpha = i\) for all \(i \in \mathbb{N}\). Since \((2(k-1)+x)\alpha = x\) and \(x \alpha^{-1} = \{x\}\), we obtain that \((2(k-1)+x) = x\) and so \(k = 1\). Therefore, \((2(k-1)+i)\alpha = i\alpha = i\) for all \(i \in \mathbb{N}\). This shows that \(\alpha = \text{id}_N\).
Lemma 4.6. Let $a \in \mathbb{N} \setminus \{1\}$, let $C$ be a regular subsemigroup of $OP_{\mathbb{N}}$ with $C_a \subseteq C$ and let $\alpha \in \text{Reg}(OP_{\mathbb{N}})$. If $a \in \text{ran} \alpha$ and if there is $b \in \text{ran} \alpha$ with $a\|b$ and $b + 1 \notin \text{ran} \alpha$ or $b - 1 \in \mathbb{N} \setminus \text{ran} \alpha$, then $\alpha \notin C$.

Proof. Let $a, b \in \text{ran} \alpha$ with $a\|b$ such that $b + 1 \notin \text{ran} \alpha$ or $b - 1 \in \mathbb{N} \setminus \text{ran} \alpha$. Assume that $\alpha \in C$. If $b + 1 \notin \text{ran} \alpha$, then we put $d := b + 1$. If $b + 1 \in \text{ran} \alpha$, then $b - 1 \in \mathbb{N} \setminus \text{ran} \alpha$ and we put $d := b - 1$. We define a partial transformation $\beta : \{a, d\} \to \{d, b\}$ by

\[ z\beta := \begin{cases} d & \text{if } z = a, d \\ b & \text{if } z = b. \end{cases} \]

It is clear that $\beta \in C_a$ and so $\alpha \beta \in C$. Now, we have that $\text{dom} \alpha \beta = aa^{-1} \cup ba^{-1}$, $\text{ran} \alpha \beta = \{d, b\}$, and $d \perp b$. Since $a\|b$ in $\text{ran} \alpha$, we get $x\|y$ for all $x \in aa^{-1} = d(\alpha \beta)^{-1}$ and for all $y \in ba^{-1} = b(\alpha \beta)^{-1}$. By Proposition 4.6, we obtain that $\alpha \beta \notin \text{Reg}(OP_{\mathbb{N}})$ and so $\alpha \beta \notin C$, a contradiction. Hence, $\alpha \notin C$. \□

Lemma 4.7. Let $a \in \mathbb{N} \setminus \{1, 2, 3\}$ and let $C$ be a regular subsemigroup of $OP_{\mathbb{N}}$ with $C_a \subseteq C$. If $\alpha \in C \setminus C_a$, then $|aaa^{-1}| \neq 1$ or $\text{ran} \alpha = \alpha|_{\{a-1, a, a+1\}}$.

Proof. Let $\alpha \in C \setminus C_a$. Assume that $|aaa^{-1}| = 1$ and $\text{ran} \alpha \neq \alpha|_{\{a-1, a, a+1\}}$. Then there exists $b \in \text{ran} \alpha$ with $a - 1, a, a + 1 \notin ba^{-1}$ and $a\|b$ since $\alpha \notin \text{Reg}(OP_{\mathbb{N}})$. We will consider the both cases $aa = a$ and $aa \neq a$.

Case 1: $aa = a$. Assume that $\text{ran} \alpha = \mathbb{N}$. Then $\alpha \in \text{Sur}$ with $aa = a$ and $|aa^{-1}| = 1$. Hence, we obtain $\alpha = \text{id}_{\mathbb{N}} \in C_a$ by Lemma 3.5, a contradiction. Thus, $\text{ran} \alpha \neq \mathbb{N}$. If $b+1 \notin \text{ran} \alpha$ or $b-1 \in \mathbb{N} \setminus \text{ran} \alpha$, then we obtain $\alpha \notin C$ by Lemma 3.4, a contradiction. Suppose that $b+1 \in \text{ran} \alpha$ and either $b-1 \notin \mathbb{N}$ or $b-1 \in \text{ran} \alpha$. Since $\text{ran} \alpha \neq \mathbb{N}$, there exists $c \in \mathbb{N} \setminus \text{ran} \alpha$ such that $c\|b$. Then we define a partial transformation $\gamma : \{a, b, c\} \to \{b, c\}$ by

\[ y\gamma := \begin{cases} c & \text{if } y = a, c \\ b & \text{if } y = b. \end{cases} \]

It is clear that $\gamma \in C_a$ and so $\alpha \gamma \in C$. Now, we have that $\text{dom} \alpha \gamma = ba^{-1} \cup \{a\}$ and $\text{ran} \alpha \gamma = \{c, b\}$, where $aa \gamma = c$ and $z \alpha \gamma = b$ for all $z \in ba^{-1}$. Since $\alpha \gamma \in C$ and $C$ is a regular subsemigroup of $OP_{\mathbb{N}}$, there exists $\beta \in C$ such that $(\alpha \gamma) \beta(\alpha \gamma) = \alpha \gamma$. Then $c \beta = a$ and $b \beta = x$ for some $x \in ba^{-1}$. Note that $c \neq a$ (since $a \in \text{ran} \alpha$ and $c \notin \text{ran} \alpha$), $b \neq a$ (since $aa = a$ and $a\|b$), and $a\|x$ (since $x \in ba^{-1}$ and $a - 1, a, a + 1 \notin ba^{-1}$). Let $\delta$ be the identity mapping on $\{b, c\}$. It is clear that $\delta \in C_a$ and so $\delta \beta \in C$. Now, we have ran $\delta \beta = \{x, a\}$ and $x\|a$.

We obtain $\delta \beta \notin C$ by Lemma 3.4, a contradiction.

Case 2: $aa \neq a$. Since $\alpha \in C \subseteq \text{Reg}(OP_{\mathbb{N}})$, there exists $\beta \in C$ such that $\alpha \beta \alpha = \alpha$. Then $(aa) \beta = a$ and $b \beta = x$ for some $x \in ba^{-1}$. Note that $a\|x$ (since $x \in ba^{-1}$ and $a - 1, a, a + 1 \notin ba^{-1}$). If $x+1 \notin \text{ran} \beta$ or $x-1 \in \mathbb{N} \setminus \text{ran} \beta$, then we obtain $\beta \notin C$ by Lemma 3.4, a contradiction. Suppose that $x+1 \in \text{ran} \beta$ and either $x-1 \notin \mathbb{N}$ or $x-1 \in \text{ran} \beta$. If $a \neq b$, then we let $\gamma$ be the identity mapping on $\{aa, b\}$. It is clear that $\gamma \in C_a$ and so $\gamma \beta \in C$. Now, we have ran $\gamma \beta = \{a, x\}$ as well as $a\|x$ and obtain $\gamma \beta \notin C$ by Lemma 3.4, a contradiction. Suppose
now that \( a = b \). If \( x - 1 \not\in \mathbb{N} \), then \( x = 1 \). Since \( x + 1 = 2 \in \text{ran} \beta \) and \( a\beta = 1 \), there exists \( y \in 2\mathbb{Z}^+ \setminus \{a\} \). In this case, we put \( c := y \) and \( d := 2 \), where \( a\|d \)
\( \text{since } a \geq 4 \). If \( x - 1 \in \text{ran} \alpha \), then there exists \( l \in \{x - 1, x + 1\} \) such that \( a\|l \). Since \( b\beta = x \) and \( b = a \), there exists \( y' \in l\beta^{-1} \) such that \( y' \neq a \). In this case, we put \( c := y' \) and \( d := l \). Note that \( a\|d \) and \( a\alpha\|c \) (since \( a\alpha \in a\beta^{-1} \) and \( c \in d\beta^{-1} \)). Let \( \delta \) be the identity mapping on \( \{a\alpha, c\} \). It is clear that \( \delta \in C_a \) and so \( \delta \beta \in C \). Now, we have \( \text{ran} \delta \beta = \{a, d\} \) with \( a\|d \) and obtain \( \delta \beta \not\in C \) by Lemma 4.3., a contradiction.

Now, we are able to prove the main result of this paper.

**Theorem 4.8.** Let \( a \in \mathbb{N} \setminus \{1, 2, 3\} \). Then \( C_a \) is a maximal regular subsemigroup of \( OP_\mathbb{N} \).

**Proof.** \( C_a \) is a regular subsemigroup of \( OP_\mathbb{N} \) by Lemma 4.3. It remains to show that \( C_a \) is maximal. Let \( C \) be a regular subsemigroup of \( OP_\mathbb{N} \) with \( C_a \subseteq C \). Assume that \( C_a \) is a proper subsemigroup of \( C \), i.e. \( C \setminus C_a \neq \emptyset \). Then there exists \( a \in C \setminus C_a \).

By Lemma 4.3, we obtain that \( |a\alpha^{-1}| \neq 1 \) (i.e. \( a \not\in \text{dom} \alpha \) or \( a\alpha^{-1} \geq 2 \)) or \( \text{ran} \alpha = \text{ran} (\alpha|_{a-1,a+a+1}) \).

Next, we show that \( a \not\in \text{ran} \alpha \) or \( \text{ran} \alpha \subseteq \{a - 1, a, a + 1\} \). Assume that \( a \in \text{ran} \alpha \) and \( \text{ran} \alpha \subseteq \{a - 1, a, a + 1\} \). Then there exists \( b \in \text{ran} \alpha \setminus \{a - 1, a, a + 1\} \) and so \( a\|b \). Assume that \( |a\alpha^{-1}| = 1 \). Then \( \text{ran} \alpha = \text{ran} (\alpha|_{a-1,a,a+1}) \).

Without loss of generality, we suppose that \( a \) is odd. Since \( a\alpha^{-1} \neq 1 \) and \( a\|b \) in \( \text{ran} \alpha \), there exists \( c \in \text{ran} \alpha \setminus \{a, b\} \) such that \( \{a - 1, a, a + 1\} \alpha = \{a, b, c\} \). Since \( \{a - 1, a, a + 1\} \) is a fence and \( a\|b \) in \( \text{ran} \alpha \), we can conclude that \( a\alpha = c \).

If \( (a - 1)\alpha = a \), then \( (a - 1) \succ a \) implies \( a = (a - 1)\alpha \succ a\alpha = c \), a contradiction with \( a \) is odd. If \( (a + 1)\alpha = a \), then we obtain a contradiction by the same arguments. Thus, \( |a\alpha^{-1}| \neq 1 \) and so there exist \( x \in a\alpha^{-1} \) and \( y \in b\alpha^{-1} \) such that \( x \neq a \) and \( y \neq a \). Let \( \delta \) be the identity mapping on \( \{x, y\} \). It is clear that \( \delta \in C_a \) and so \( \delta \alpha \in C \). Note that \( \text{ran} \delta \alpha = \{a, b\} \) and \( a\|b \). We obtain \( \delta \alpha \not\in C \) by Lemma 4.3., a contradiction. Hence, \( a \not\in \text{ran} \alpha \) or \( \text{ran} \alpha \subseteq \{a - 1, a, a + 1\} \).

Up to now, we have shown that \( \alpha \in C \subseteq \text{Reg}(OP_\mathbb{N}) \) satisfies the conditions (i) and (ii) in Definition 4.3. Since \( \alpha \not\in C_a \), we conclude that \( \alpha|_{\text{dom} \alpha \setminus \{a\}} \not\in K_2 \). Then there are \( x_1, x_2 \in \text{ran} \alpha|_{\text{dom} \alpha \setminus \{a\}} \) with \( x_1 \perp x_2 \) as well as \( a_1 \in x_1 (\alpha|_{\text{dom} \alpha \setminus \{a\}})^{-1} \) and \( a_2 \in x_2 (\alpha|_{\text{dom} \alpha \setminus \{a\}})^{-1} \) such that \( a_1\|a_2 \). Let \( \gamma \) be the identity mapping on \( \{a_1, a_2\} \). It is clear that \( \gamma \in C_a \) and so \( \gamma \alpha \in C \). Now, we have that \( \text{dom} \gamma \alpha = \{a_1, a_2\} \) and \( \text{ran} \gamma \alpha = \{x_1, x_2\} \). Since \( x_1 \perp x_2 \) and \( a_1\|a_2 \), we obtain \( \gamma \alpha \not\in \text{Reg}(OP_\mathbb{N}) \) by Proposition 4.3. Therefore, \( \gamma \alpha \not\in C \), a contradiction.

Consequently, we obtain \( C \setminus C_a = \emptyset \) and thus \( C = C_a \), i.e. \( C_a \) is a maximal regular subsemigroup of \( OP_\mathbb{N} \).

After all together with Corollary 4.3., we obtain immediately:

**Corollary 4.9.** There are countably infinitely many maximal regular subsemigroups of \( OP_\mathbb{N} \).
Proof. By Theorem 4.8 and Corollary 4.2, we have that \( \{C_a : a \in \mathbb{N} \setminus \{1, 2, 3\}\} \) is a countably infinite set of maximal regular subsemigroups of \( OP_N \).

We finish this section with a remark about the cardinalities of any semigroup \( C_a \).

**Proposition 4.10.** Let \( a \in \mathbb{N} \setminus \{1\} \). Then \( |C_a| > \aleph_0 \).

**Proof.** Let \( A := \{a + 2n : n \in \mathbb{N}\}, B := \{a + 1, a + 3\} \), and let \( B^A \) be the set of all mappings from \( A \) to \( B \). Further, let \( \alpha \in B^A \). Clearly, \( \alpha \) is a partial transformation on \( \mathbb{N} \). If rank \( \alpha = 1 \), then \( \alpha \in C_a \) by Remark 3.1. Suppose now that rank \( \alpha = \{a + 1, a + 3\} \). Since \( a + 2n \parallel a + 2m \) for all \( n, m \in \mathbb{N} \) with \( n \neq m \), we get immediately that \( \alpha \in OP_N \). Moreover, \( a + 1 \parallel a + 3 \) provides \( \alpha \in Reg(OPT) \) by Proposition 3.2. From \( \text{ran} \alpha = \{a + 1, a + 3\} \) with \( a + 1 \parallel a + 3 \), we obtain \( \alpha|_{\text{dom} \alpha \setminus \{a\}} \in K_2 \). Together with \( a \notin \text{dom} \alpha \) and \( a \notin \text{ran} \alpha \), we conclude that \( \alpha \in C_a \). Thus, \( B^A \subseteq C_a \), where \( |C_a| \geq |B|^{|A|} = 2^{\aleph_0} > \aleph_0 \).

Proposition 4.10 shows us that \( C_a \) is an uncountably infinite set for all \( a \in \mathbb{N} \setminus \{1\} \).

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**References**


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