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SOME RESULTS ON FARTHEST POINTS IN 2-NORMED SPACES

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Abstract. In this paper, we consider the problem of the farthest poinst for bounded sets in a real 2-normed spaces. We investigate some properties of farthest points in the setting of 2-normalised spaces and present various characterizations of b-farthest point of elements by bounded sets in terms of b-linear functional. We also provide some applications of farthest points in the setting of 2- inner product spaces.

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1. Introduction

The concepts of 2-metric spaces and linear 2-normed spaces were first introduced by Gähler in 1963 [8] and have been developed extensively in different subjects by others authors (see [3, 4, 9, 10, 12]). Elumalai, Vijayaragavan and Sistani, Moghaddam in [6, 14] gave some results on the concept best approximation in the context of bounded linear 2-functionals on real linear 2-normed spaces. They established various characterizations of the best approximation elements in these spaces. The concepts of farthest point in normed spaces have been studied by many authors (see [1, 2, 5, 7, 13]). In this paper we study this concept in 2-normed spaces, and obtain some results on characterization and existence of farthest points in normed linear spaces in terms of bounded b-linear functionals. In section 2, we give some preliminary results. In section 3, we give some fundamental concepts of b-farthest points and give characterization of farthest points in 2-normed linear spaces and some basic properties of farthest points. Also we study the farthest point mapping on X by virtue of the Gateaux derivative in 2-normed spaces. We show in the case that 2-normed space is strictly convex there exists a unique farthest points of the closed convex set from each point. In the end, we delineate some applications of farthest points in 2-inner product spaces.

2. Preliminaries

Definition 2.1. Let X be a linear space of dimension greater than 1. Suppose $\|.,.\|$ is a real-valued function on $X \times X$ satisfying the following conditions:

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a) $\|x,z\| \ \geq 0$ and $\|x,z\| = 0$ if and only if x and z are linearly dependent.

b) $\|x,z\| = \|z,x\|$,

d) $\|\alpha x,z\| = \alpha\|x,z\|$ for any scalar $\alpha \in R$,

e) $\|x+x^{'},z\| \leq \|x,z\| + \|x^{'},z\|$.

Then $\|.,.\|$ is called a 2-norm on X and $(X, \|.,.\|)$ is called a linear 2- normed space.

Example 2.2. Let $X = R^3$, and consider the following 2-norm on X:

$$\|x,y\| = |xxy| = |det \begin{bmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} |.$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$. Then X is a 2-normed space.

Example 2.3. Let X be a real linear space having two seminorms $\|.\|_1$ and $\|.\|_2$. Then $(X, \|.\|)$ is a generalized 2-normed space with the 2-norm defined by

$$\|x,y\| = \|x\|_1\|.\|y\|_2, \quad for\ x,y \in X.$$

Every 2-normed space is a locally convex topological vector space. In fact for a fixed $b \in X$, $p_b = \|x,b\|$; $x \in X$ is a semi-norm on X and the family $P = \{p_b : b \in X\}$ of semi-norms generates a locally convex topology.

Definition 2.4. Let $(X, \|.,b\|)$ be a 2-normed linear space, E be a nonempty subset of X. The set E is called b-open if and only if for each $a_0 \in E$, there exists $\varepsilon_{a_0} > 0$ such that for each $c \in E$ with $\|a_0 - c, b\| < \varepsilon_{a_0}$ implies $a_0 - c \in E$. The b-interior of E is denoted $int_b(E)$, is the largest b-open set contained in E.

A sequence $\{x_n\}$ in 2-normed linear space X is said to be a b-convergent if there exists an element $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, b\| = 0$. A set is b-closed if and only if it contains all of its limit points.

Definition 2.5. Let $(X, \|.,.\|)$ be a 2-normed space, $b \in X$ be fixed, then a map $T : X\times <b>\to R$ is called a b-linear functional on $X\times <b>$ whenever

1) $T(a+c,b) = T(a,b) + T(c,b)$ for $a;c,b \in X$ such that ;

2) $T(\alpha a,b) = \alpha T(a,b)$ for $\alpha \in R$.

A b-linear functional $T : X\times <b>\to R$ is said to be bounded if there exists a real number $M > 0$ such that $|T(x,b)| < M\|x,b\|$ for every $x \in X$.. The norm of the b-linear functional $T : X\times <b>\to R$ is defined by

$$\|T\| = \sup\{\|T(x,b)\| : \ \|x,.b\| \neq 0\}.$$

3. Farthest points in 2-normed spaces

Let X be a 2-normed vector space. For a nonempty subset G of X and $x \in X$, define

$$f_G(x,b) = \sup_{g \in G} \|x - g, b\|. \tag{3.1}$$

Recall that a point $g_0 \in G$ is called a b-farthest point for $x \in X$ if

$$\|x - g_0, b\| = f_G(x,b). \tag{3.2}$$

The set of all b-farthest points to x from G is denoted by $\mathbf{F}_G(x,b)$. Let

$$R_b(G) = \{x \in X : \mathbf{F}_G(x,b) \neq \varnothing\}.$$

The set G is said to be a b-remotal set if $R_b(G) = X$.

Corollary 3.1. *Let X be a 2-normed vector space and G be a nonempty bounded subset of X. Then for any x, z of X*

i) $\mid f_G(x,b) - f_G(z,b) \mid \leq \|x - z, b\|$.

ii) $\|x - z, b\| \leq f_G(x,b) + f_G(z,b)$.

Proof. i) Let $y \in \mathbf{F}_G(z,b)$. By the definition of b-farthest points, we have

$$f_G(x,b) \geq \|x - y, b\| = \|x - z + z - y, b\| \geq \|x - z, b\| - \|z - y, b\|$$

$$f_G(x,b) - f_G(z,b) \geq \|x - z, b\|.$$

Interchanging x and y, we get

$$f_G(z,b) - f_G(x,b) \geq \|x - z, b\|.$$

Hence $\mid f_G(x,b) - f_G(z,b) \mid \leq \|x - z, b\|$.

ii) It's proof is similar to that of (i).

□

Theorem 3.2. *Let G is a closed bounded b-remotal set in a 2-normed space X. Then $\mathbf{F}_G(x,b) \cap int_b(G) = \varnothing$.*

Proof. Suppose $e \in G$ such that $e \in \mathbf{F}_G(x,b) \cap int_b(G)$. There exists a number $r > 0$ such that $\{y \in X : \|y - e, b\| < r\} \subseteq G$. Put $u = e - \dfrac{r}{2\|x - e, b\|}(x - e)$. Then $\|u - e, b\| = \dfrac{r}{2} \leq r$, and hence $u \in G$ and

$$\begin{aligned} \|x - u, b\| &= \|x - e + \frac{r}{2\|x - e, b\|}(x - e), b\| \\ &= \|(1 + \frac{r}{2\|x - e, b\|})(x - e), b\| \\ &= (1 + \frac{r}{2\|x - e, b\|})\|(x - e), b\| > \|(x - e), b\|. \end{aligned}$$

This is a contradiction. □

Theorem 3.3. *A nonvoid bounded set G in a 2-normed space X is b-remotal if and only if the following associated set*

$$K_d = G + CB^b_d(0)$$

is closed for $d > 0$, where $CB^b_d(0) = \{y \in X : \|x, b\| \geq d\|\}$.

Proof. Let x be an adherent element of $G + CB^b_d(0)$, i.e. there exist a sequence $(x_n)_{\in N}$ which converges to x and a sequence $(u_n)_{n\in N} \subset G$ such that for all $n \in N$ $\|x_n - u_n, b\| \geq d$. Thus, for every $\varepsilon > 0$ there exists $n_\varepsilon \in N$ such that $\|x_n - u_n, b\| > d - \varepsilon$ for all $n \geq n_\varepsilon$. Now, if G is b-remotal, taking an element $g' \in \mathbf{F}_G(x, b)$ we obtain that $\|x - g', b\| \geq \|x_n - u_n, b\|$ for all $n \geq n_\varepsilon$ and so $\|x - g', b\| \geq d - \varepsilon$, for every $\varepsilon > 0$. Consequently $\|x - g', b\| \geq d$ i.e. $x \in G + CB^b_d(0)$. Conversely, for an arbitrary element $x \in X$ we take $d = f_G(x, b)$. We can suppose $d > 0$ since $f_G(x, b) = 0$ if and only if $G = \{x\}$. When G is b-remotal obviously for every $n \in N$ exist $u_n \in G$ such that $\|x - u_n, b\| \geq d - \frac{1}{n}$. But, we have

$$\frac{1}{n}(d - \frac{1}{n})^{-1}(x - u_n) + x \in u_n + CB^b_d(0) \subset G + CB^b_d(0),$$

for all $n \in N$ such that $n > 1$. Since $(u_n)_{n\in N}$ is bounded, by passing to the limit we get $x \in \overline{G + CB^b_d(0))}$. Therefore, if $G + CB^b_d(0)$ is closed there exists $g' \in G$ such that $\|x - g', b\| \geq d$ i.e. $g' \in \mathbf{F}_G(x, b)$. Hence the set G is b-remotal. □

Some characterizations of farthest points in 2-normed spaces are provided in following theorems.

Theorem 3.4. *Let G be a subset of a 2-norm space X and $x \in X \setminus M + < b >$, then $g_0 \in \mathbf{F}_G(x, b)$, if and only if there exists a b-bilinear function p such that*

$$p(x - g_0, b) = \sup_{g\in G} \|x - g, b\| \ \text{and} \ \|p\| = 1. \tag{3.3}$$

Proof. Suppose that there is a b-bilinear function p which satisfies (3.3), then

$$\|x - g_0, b\| = \|x - g_0, b\|\|p\| \geq | p(x - g_0, b) | = \sup_{g\in G} \|x - g, b\| \geq \|x - g, b\|.$$

Conversely, let $g_0 \in \mathbf{F}_G(x, b)$, by Hahn-Banach theorem in the context of 2-normed spaces (see Theorem 2.2 [11]) there exists a b-bilinear function p such that $\|p\| = 1$, $p(x - g_0, b) = \|x - g_0, b\| = \sup_{g\in G} \|x - g, b\|$. □

Theorem 3.5. *Let G be a subset of a 2-norm space X and $x \in X \setminus M + < b >$. Then the following statements are equivalent.*

i) $g_0 \in \mathbf{F}_G(x, b)$.

ii) There is a b-bilinear function p on X which satisfies

$$| p(x - g_0, b) | = \sup_{g \in G} \|x - g, b\| \ \ and \ \ \|p\| = 1, \tag{3.4}$$

$$| p(x - g_0, b) | \geq | p(x - g, b) | \,. \tag{3.5}$$

iii) There is a b-bilinear function p on X which satisfies (3.4) and

$$p(g_0 - g, b)p(g_0 - x, b) \geq 0. \tag{3.6}$$

Proof. Let $g_0 \in \mathbf{F}_G(x, b)$. Then by Theorem 3.4 we have (3.4) and

$$| p(x - g_0, b) | = \sup_{g \in G} \|x - g, b\| \geq \|x - g, b\| \geq | p(x - g, b) |,$$

which proves (3.5). Thus, $(i) \Rightarrow (ii)$.
$(ii) \Rightarrow (iii)$. Suppose that there is a b-bilinear function p on X satisfying (3.4), (3.5) then

$$\begin{aligned} |p(x - g_0, b)|^2 \geq |p(x - g, b)|^2 &= |p(x - g_0, b)|^2 + |p(g - g_0, b)|^2 \\ &+ 2p(g_0 - g, b)p(g_0 - x, b) \\ &\geq |p(x - g_0, b)|^2 + 2p(g_0 - g, b)p(g_0 - x, b), \end{aligned}$$

whence it follows that $p(g_0 - g, b)p(g_0 - x, b) \geq 0$.
$(iii) \Rightarrow (i)$ It is a consequence of Theorem 3.4. □

Definition 3.6. A linear 2-normed space $(X, \|.,.\|)$ is said to be strictly convex if $\|x + y, c\| = \|x, c\| + \|y, c\|$ and $c \notin \mathrm{Span}\{x, y\}$ imply that $x = \alpha y$ for some $\alpha > 0$.

Definition 3.7. A real-valued function f on $X\times < b >$ is said to be b-Gateaux differentiable at a point x of X if there is a b-linear functional df_x such that, for each $y \in X$,

$$df_x(y, b) = \lim_{t \to 0} \frac{f(x + ty, b) - f(x, b)}{t},$$

and we call df_x the b-Gateaux derivative of f at x.

Theorem 3.8. *Let G be a subset of a 2-norm space X, $x \in X$ and $y \in \mathbf{F}_G(x, b)$. Suppose that the functional $df_{x,b}$ is the Gateaux derivative of the function $f_G(., b)$ at the point x. Then*

$$df_x(x - y, b) = \|x - y, b\| \ \ and \ \ \|df_x\| = 1.$$

Proof. If G is a single point this is clear. Otherwise $x \neq y$ and $\|x - y, b\| = f_G(x, b)$, for $0 < t < 1$,

$$\begin{aligned} f_G(x, b) + t\|x - y, b\| &= (1 + t)\|x - y, b\| = \|x + t(x - y) - y, b\| \\ &\leq f_G(x + t(x - y), b) \leq f_G(x, b) + t\|x - y, b\|. \end{aligned}$$

As above and Corollary 3.1 so omitted holds throughout, and

$$df_x(x-y,b) = \lim_{t\to 0} \frac{f_G(x+t(y-x),b) - f_G(x,b)}{t} = \|x-y,b\|.$$

Corollary 3.1 implies that $\|df_{x,b}\| \leq 1$, so this also show that $\|df_x\| = 1$. □

Theorem 3.9. *Let G be a convex subset of a strictly convex 2-normed space X, $x \in X \setminus G$ and $b \notin \mathrm{Span}\{x, G\}$. Suppose that the functional $df_{x,b}$ is the Gateaux derivative of the function $f_G(.,b)$ at the point x. Then there is at most one b-farthest point in G to x.*

Proof. Suppose that y, z of $\mathbf{F}_G(x,b)$. Theorem 3.8 shows that

$$df_x(x-y,b) = \|x-y,b\| = \|x-z,b\| = df_x(x-z,b).$$

$$\begin{aligned} f_G(x,b) = \frac{1}{2}(\|x-y,b\| + \|x-z,b\|) &= \frac{1}{2}(df_x(x-y,b) + df_x(x-z,b)) \\ &= df_x(x - \frac{y+z}{2}, b) \leq \|x - \frac{y+z}{2}, b\| \\ &\leq f_G(x,b). \end{aligned}$$

Hence equality must hold throughout these inequalities. Since X is strictly convex 2-normed space and $b \notin \mathrm{Span}\{x, G\}$, it follows that $\mathbf{F}_G(x,b)$ has at most one element. □

The properties of linear 2-normed spaces have been extensively studied by many authors. The same properties also hold in 2-inner product spaces, which were introduced by Diminnie et al [4].

Definition 3.10. Let X be a linear space. Suppose that $\langle .|.\rangle$ is a R valued function defined on $X \times X \times X$ satisfying the following conditions:

a) $\langle x, x|z\rangle \geq 0$ and $\langle x, x|z\rangle = 0$ if and only if x and z are linearly dependent.

b) $\langle x, x|z\rangle =< z, z|x\rangle$,

c) $\langle x, y|z\rangle = \langle y, x|z\rangle$,

d) $\langle \alpha x, x|z\rangle = \alpha\langle x, x|z\rangle$ for any scalar $\alpha \in R$,

e) $\langle x + x', y|z\rangle = \langle x, y|z\rangle + \langle x', y|z\rangle$.

$\langle ., .|, \rangle$ is called a 2-inner product and $(X, \langle ., , |.\rangle)$ is called a 2-inner product space (or a 2-perHilbert space).

In any given 2-inner product space $(X, (., .|.))$, we can define a function $\|., .\|$ on $X \times X$ by

$$\|x, z\| = \langle x, x \mid z\rangle^{\frac{1}{2}}.$$

Using the above properties, we can prove the Cauchy-Schwarz inequality

$$|\langle x, y \mid z\rangle|^{\frac{1}{2}} \leq \langle x, x \mid z\rangle\langle y, y \mid z\rangle.$$

Theorem 3.11. *Let G be a bounded subset of 2-inner product space X, $x \in X$, and $y_0 \in G$. If $\langle x-y, y_0-y|b\rangle \leq 0$ for all $y \in G$, then $y_0 \in \mathbf{F}_G(x,b)$.*

Proof. Suppose that $\langle x-y, y_0-y|b\rangle \leq 0$ for all $y \in G$, then

$$\begin{aligned}\|x-y,b\|^2 &= \langle x-y, x-y|b\rangle = \langle x-y, x-y_0+y_0-y|b\rangle \\ &= \langle x-y, x-y_0|b\rangle + \langle x-y, y_0-y|b\rangle \\ &\leq \langle x-y, x-y_0|b\rangle \leq \|x-y,b\|\|x-y_0,b\|.\end{aligned}$$

Hence $\|x-y,b\|^2 \leq \|x-y_0,b\|$ i.e. $y_0 \in \mathbf{F}_G(x,b)$. □

Definition 3.12. A set A in a 2-normed space X is said to be b-strongly convex with constant $r > 0$ if there exists a set $A_1 \subset E$ such that

$$A = \cap_{a\in A_1} B_r^b(a),$$

where $B_r^b(a) = \{y \in X : \ \|x-a,b\| \leq r\|\}$.
A set A is called a b-strongly convex set of radius $R > 0$ if this set is the intersection of balls of radius R.

In the following, we study uniqueness problem for a point of closed bounded set that is the farthest point from a given point in 2-inner product spaces.

Lemma 3.13. *Let G be a b-strongly convex set of radius $r > 0$ in the 2-inner product space X.Then the inequality*

$$\|a_1-a_2,b\|^2 \leq R\langle a_1-a_2, p_2-p_1|b\rangle,$$

holds for vectors p_1, p_2 such that $\|p_1,b\|, \|p_2,b\| \geq 1$.

Proof. We fix vectors p_1, p_2. According to the definition of strongly convex sets, we have

$$G \subseteq B_r^b(a_1 - R\frac{p_1}{\|p_1,b\|}) \cap B_r^b(a_2 - R\frac{p_2}{\|p_2,b\|}),$$

which implies the inequalities

$$\|a_2-a_1+R\frac{p_1}{\|p_1,b\|},b\|^2 \leq R^2, \qquad \|a_1-a_2+R\frac{p_2}{\|p_2,b\|},b\|^2 \leq R^2$$

and hence

$$\|a_2-a_1+R\frac{p_1}{\|p_1,b\|},b\|^2 = \langle a_2-a_1+R\frac{p_1}{\|p_1,b\|}, a_2-a_1+R\frac{p_1}{\|p_1,b\|}|b\rangle,$$

$$= \langle a_2-a_1, a_2-a_1|b\rangle + \langle R\frac{p_1}{\|p_1,b\|}, R\frac{p_1}{\|p_1,b\|}|b\rangle + 2\langle a_2-a_1, R\frac{p_1}{\|p_1,b\|}|b\rangle \leq R^2,$$

and hence

$$\|a_1-a_2,b\|^2 \leq 2R\langle a_1-a_2, -p_1|b\rangle$$

$$\|a_1-a_2,b\|^2 \leq 2R\langle a_1-a_2, p_2|b\rangle.$$

We sum the last two inequalities and obtain the desired inequality. □

For a set G in a 2-normed space X and a number $r > 0$, we define the set

$$T_r^b(G) = \{x \in X : \ f_G(x,b) > r\}.$$

Theorem 3.14. *Let G be a b-strongly convex set of radius $r > 0$ in the 2-inner product space X. Then for $x_1, x_2 \in T_R^b(G)$ the inequality*

$$\|f_b(x_1) - f_b(x_2), b\|^2 \leq \frac{r}{R-r}\|x_1 - x_2, b\|, \tag{3.7}$$

holds for any $R > r$ and $f_b(x_i) \in \mathbf{F}_G(x_i, b)$, $i = 1, 2$.

Proof. We choose a number $R > r$, and introduce the vectors

$$p_i = \frac{1}{R}(f_b(x_i) - x_i), i = 1, 2.$$

From Lemma 3.13, we obtain

$$\begin{aligned} &\|f_b(x_1) - f_b(x_2), b\|^2 \\ &\leq \ r\langle f_b(x_1) - f_b(x_2), p_2 - p_1 | b\rangle \\ &= \ r\langle f_b(x_1) - f_b(x_2), \frac{1}{R}(f_b(x_2) - x_2) - \frac{1}{R}(f_b(x_1) - x_1), | b\rangle \\ &= \ \frac{r}{R}\|f_b(x_1) - f_b(x_2), b\|^2 - \frac{r}{R}\langle f_b(x_1) - f_b(x_2), x_2 - x_1 | b\rangle. \end{aligned}$$

Hence by Cauchy-Schwarz inequality we get

$$(1 - \frac{r}{R})\|f_b(x_1) - f_b(x_2), b\|^2 \leq \frac{r}{R}\|f_b(x_1) - f_b(x_2), b\|\|x_1 - x_2, b\|.$$

which implies formula (3.7). □

Corollary 3.15. *Let G be a b-strongly convex set of radius $r > 0$ in the 2-inner product space X, $x \in T_R^b(G)$ and $b \notin \mathrm{Span}\{G\}$. Then there is at most one b-farthest point in G to x.*

Proof. It is a consequence of Theorem 3.14. □

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