Novi Sad J. Math.
Vol. 46, No. 1, 2016, 191-205

THE CR-GEOMETRY OF THE COMPLEX INDICATRIX

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Abstract. In this paper we study the differential geometry of the complex indicatrix, which is approached as an embedded CR hypersurface of the punctual holomorphic tangent bundle of a complex Finsler space. Following the study of CR submanifolds of a Kähler manifold and using the submanifold formulae we investigate some properties of the complex indicatrix, such as the fact that it is an extrinsic hypersphere of the holomorphic tangent space. The fundamental equations of the indicatrix as a real submanifold of codimension 1 are also determined. Besides this, the CR-structure integrability is studied and the Levi form and characteristic direction of the complex indicatrix are given.

AMS Mathematics Subject Classification (2010): 53B40; 53C60; 53C40; 53B25

Key words and phrases: complex Finsler space; complex indicatrix; CR hypersurface; extrinsic sphere; fundamental equations; Levi form

1. Introduction

In the differential geometry hypersurfaces of real and complex spaces have been investigated by many geometers. A significant role in the geometry of real Finsler spaces is played by the indicatrix, ([3, 4, 10, 14, 19], etc.), mainly because it is a compact and strictly convex set surrounding the origin. Moreover, it is the unit tangent sphere of the Finsler space and its connections with the real or complex unit sphere are analysed.

In the present paper, we will regard the indicatrix of a complex Finsler space as a real hypersurface in the holomorphic tangent bundle of a complex Finsler manifold (M, F). The intrinsic properties of the indicatrix, as an embedded hypersurface in a complex space, will be studied using the holomorphic vector fields which are tangent to the indicatrix.

Firstly, in Section 1, we mention some fundamental notions of the complex Finsler geometry and we introduce the geometry of CR manifolds, since the complex indicatrix is a real hypersurface of a complex space, and thus, an embedded CR manifold. Then, in the second Section, considering a fixed point z_0, we study the complex indicatrix as a CR hypersurface of the holomorphic tangent bundle in z_0, which can be locally viewed as a Kähler manifold. The fundamental equations of the complex indicatrix in a fixed point will be obtained in Section 3 and in the fourth Section we analyse the integrability

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of the CR-structure by considering the characteristic direction of the complex indicatrix and its Levi form.

Now, we will recall the concepts and terminology used in complex Finsler geometry, as in [1, 15]. Let M be an n dimensional complex manifold, $\dim_C M = n$, and $z := (z^k),\ k = 1,..,n$, the complex coordinates on a local chart (U, φ). The complexified form of the real tangent bundle $T_{\mathbb{C}}M$ is expressed as the direct sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$, i.e. $T_{\mathbb{C}}M = T'M \oplus T''M$. The holomorphic tangent bundle $T'M$ is in its turn a $2n$-dimensional complex manifold of local coordinates $u := (z^k, \eta^k),\ k = 1,..,n$.

Definition 1.1. A *complex Finsler space* is a pair (M, F), with $F : T'M \to \mathbb{R}^+,\ F = F(z, \eta)$ a continuous function which satisfies the following conditions:

i. F is a smooth function on $\widetilde{T'M} := T'M \setminus \{0\}$;

ii. $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii. $F(z, \lambda\eta) = |\lambda| F(z, \eta)$, $\forall \lambda \in \mathbb{C}$;

iv. the Hermitian matrix $\big(g_{i\bar{j}}(z, \eta)\big)$ is positive definite, where $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ is the fundamental metric tensor, with $L := F^2$.

The fourth condition is equivalent to the strong pseudoconvexity of the complex indicatrix in a fixed point $I_z M = \{\eta \mid g_{i\bar{j}}(z, \eta)\eta^i \bar{\eta}^j = 1\}$, for any $z \in M$. Moreover, the positivity of $(g_{i\bar{j}})$ from this condition ensures the existence of the inverse $(g^{\bar{j}i})$, with $g^{\bar{j}i} g_{i\bar{k}} = \delta^{\bar{j}}_{\bar{k}}$.

Condition iii. expresses the homogeneity of the complex Lagrangian associated to the complex Finsler function, L, with respect to the complex norm $L(z, \lambda\eta) = \lambda\bar{\lambda} L(z, \eta)$, $\forall \lambda \in \mathbb{C}$, and by applying Euler's formula we get

$$\frac{\partial L}{\partial \eta^k}\eta^k = \frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L; \quad \frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0 \quad \text{and} \quad L = g_{i\bar{j}}\eta^i \bar{\eta}^j. \tag{1}$$

An immediate consequence of the above homogeneity conditions concerns the Cartan complex tensors $C_{i\bar{j}k} := \frac{\partial g_{i\bar{j}}}{\partial \eta^k}$ and $C_{i\bar{j}\bar{k}} := \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}$, with

$$C_{i\bar{j}k} = C_{k\bar{j}i} \quad ; \quad C_{i\bar{j}\bar{k}} = C_{i\bar{k}\bar{j}} \quad ; \quad C_{i\bar{j}k} = \overline{C_{j\bar{i}\bar{k}}} \quad \text{and} \tag{2}$$

$$C_{i\bar{j}k}\eta^k = C_{i\bar{j}\bar{k}}\bar{\eta}^j = C_{i\bar{j}k}\eta^i = C_{i\bar{j}\bar{k}}\bar{\eta}^k = 0. \tag{3}$$

A first step in the study of geometric objects on the $T'M$ endowed with the Hermitian metric structure given by $g_{i\bar{j}}$ represents the investigation of the sections of its complexified tangent bundle, $T_{\mathbb{C}}(T'M) = T'(T'M) \oplus T''(T'M)$, where $T''_u(T'M) = \overline{T'_u(T'M)}$. Let $V(T'M) \subset T'(T'M)$ be the vertical bundle, locally spanned by $\left\{\frac{\partial}{\partial \eta^k}\right\}$ and let $V(T''M)$ be its conjugate.

A complex nonlinear connection, briefly (c.n.c.), is a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. By considering $N^j_k(z, \eta)$ the coefficients of the (c.n.c.), the horizontal distribution

$H_u(T'M)$ is locally spanned by $\left\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\right\}$. The adapted frame of the (c.n.c.) is the pair $\left\{\delta_k := \frac{\delta}{\delta z^k},\, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\right\}$ and by conjugation everywhere we get an adapted frame $\{\delta_{\bar{k}},\, \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted bases are $\{\mathrm{d}z^k,\, \delta\eta^k := \mathrm{d}\eta^k + N_j^k \mathrm{d}z^j\}$, respectively $\{\mathrm{d}\bar{z}^k,\, \delta\bar{\eta}^k\}$. Further we will denote by $\bar{\eta}^j =: \eta^{\bar{j}}$ a conjugate object.

A classic example is the Chern-Finsler (c.n.c.) ([1, 15]), given by the coefficients $N_j^k = g^{\bar{m}k}\partial_j(g_{l\bar{m}})\eta^l$. Take then the *Chern-Finsler linear connection*, which is locally given by the coefficients $L^i_{jk} = g^{\bar{l}i}\delta_k(g_{j\bar{l}})$, $C^i_{jk} = g^{\bar{l}i}\dot{\partial}_k(g_{j\bar{l}})$, $L^{\bar{\imath}}_{\bar{j}k} = 0$, $C^{\bar{\imath}}_{\bar{j}k} = 0$, which satisfy $L^i_{jk} = \dot{\partial}_j N^i_k$ and $C^i_{jk}\eta^j = C^i_{jk}\eta^k = 0$ from the homogeneity conditions (1).

The notion of Cauchy-Riemann structure, briefly (CR) structure, was introduced firstly in an attempt to describe intrinsically the property of being a real hypersurface in a complex space by studying the properties of holomorphic vector fields which are tangent to the hypersurface. So, a CR manifold is a differentiable manifold together with a CR structure, and it can be considered as an embedded CR manifold (hypersurfaces and edges of wedges in complex space) or an abstract CR manifold (given by the distributions of its tangent bundle).

The theory of Cauchy-Riemann (CR) submanifolds of an almost Hermitian or Kähler manifold was introduced by A. Bejancu ([5, 6, 7]) and was generalized to the Finsler geometry by S. Dragomir ([11, 12]). An n-dimensional real submanifold $\tilde{M}$ of the $2m$-dimensional real almost Hermitian Finsler space (M, g), is said to be a *CR submanifold* if it carries a pair of complementary Finslerian distributions with respect to the restriction of g to $\tilde{M}$, $\mathcal{D} : u \to \mathcal{D}_u \subset T_u\tilde{M}$, $\dim_{\mathbb{R}} \mathcal{D}_u = 2p$, and $\mathcal{D}^\perp : u \to \mathcal{D}^\perp_u \subset T_u\tilde{M}$, $\dim_{\mathbb{R}} \mathcal{D}^\perp = n - 2p$, such that $\mathcal{D}$ is invariant, i.e. $J(\mathcal{D}_u) = \mathcal{D}_u$, while $\mathcal{D}^\perp$ is anti-invariant, i.e. $J(\mathcal{D}^\perp_u) \subset (T_u\tilde{M})^\perp$, for each $u \in \tilde{M}$. Here J is an almost complex structure on $\tilde{M}$ and $(T_u\tilde{M})^\perp$ is the normal space to $\tilde{M}$ at u. Thus $\tilde{M}$ is a CR-manifold endowed with a CR-structure of $(p, n-2p)$ type and $\mathcal{D}_u$ is called the Levi distribution in u.

We call the CR-submanifold $\tilde{M}$ to be *invariant* or *complex* if $2p = n$, *anti-invariant* if $p = 0$ and *proper* if it is neither holomorphic nor complex. Any real hypersurface $\tilde{M}$ of M is a CR submanifold, where we define $\mathcal{D}^\perp : u \to \mathcal{D}^\perp_u = J(T_u\tilde{M})^\perp$ and take $\mathcal{D}$ the complementary orthogonal distribution of $\mathcal{D}^\perp$ in $T\tilde{M}$.

2. The complex indicatrix as a CR hypersurface

Considering a complex Finsler manifold (M, F), we denote by T'_zM the corresponding holomorphic tangent space of M and by F_z the Finsler metric in an arbitrary fixed point $z \in M$. Then, (T'_zM, F_z) can be regarded as a complex Minkowski space, with the complex coordinate system $\eta = (\eta^i) = \eta^i \frac{\partial}{\partial z^i}|_z$. Let g be the Hermitian structure on $T'(\widetilde{T'M})$ associated to F_z, i.e. $g_\eta(Z, W) = g_{j\bar{k}}(z, \eta)Z^j\overline{W^k}$ for any $\eta \in T'_zM$, $Z = Z^j \frac{\partial}{\partial \eta^j}|_\eta$, $W = W^k \frac{\partial}{\partial \eta^k}|_\eta \in T'_\eta(\widetilde{T'_zM})$.

As usual in Hermitian geometry we extend g to a complex bilinear form $\mathcal{G}$ on $\widetilde{T'_zM}$ by $\mathcal{G}(Z,\bar{W}) = g(Z,W)$, $\mathcal{G}(Z,W) = \mathcal{G}(\bar{Z},\bar{W}) = 0$, $\mathcal{G}(\bar{Z},W) = \overline{\mathcal{G}(Z,\bar{W})}$, $\forall Z,W \in T'(T'_zM)$ and is given by:

$$\mathcal{G} := \frac{\partial^2 F_z^2}{\partial\eta^j\partial\bar{\eta}^k}\mathrm{d}\eta^j \otimes \mathrm{d}\bar{\eta}^k = g_{j\bar{k}}(z,\eta)\mathrm{d}\eta^j \otimes \mathrm{d}\bar{\eta}^k. \tag{4}$$

According to [15], p.12, a linear connection on M extends by linearity to T_CM, which is isomorphic to $V_C(T'M)$ via vertical lift. We require ∇ to be a compatible complex connection with respect to J, i.e. $\nabla J = 0$, such that ∇ conserves the holomorphic tangent space. Here by J we have denoted the natural complex structure

$$J(\dot{\partial}_k) = \mathrm{i}\dot{\partial}_k,\ J(\dot{\partial}_{\bar{k}}) = -\mathrm{i}\dot{\partial}_{\bar{k}}. \tag{5}$$

We can choose ∇ to be the Levi-Civita connection, which is a metrical and symmetric connection and using (2) we get the following components:

$$\begin{aligned}
\Gamma^i_{jk} &= \frac{1}{2}g^{\bar{h}i}\left(\dot{\partial}_k g_{j\bar{h}} + \dot{\partial}_j g_{k\bar{h}}\right) = g^{\bar{h}i}C_{j\bar{h}k} =: C^i_{jk}(\eta),\\
\Gamma^{\bar{\imath}}_{\bar{j}k} &= \frac{1}{2}g^{\bar{\imath}h}\left(\dot{\partial}_k g_{h\bar{j}} - \dot{\partial}_h g_{k\bar{j}}\right) = 0,\\
\Gamma^i_{\bar{j}k} &= \frac{1}{2}g^{\bar{h}i}\left(\dot{\partial}_{\bar{j}} g_{k\bar{h}} - \dot{\partial}_{\bar{h}} g_{k\bar{j}}\right) = 0,\\
\Gamma^{\bar{\imath}}_{jk} &= 0,
\end{aligned}$$

Since $\Gamma^i_{\bar{j}k} = \Gamma^{\bar{\imath}}_{jk} = 0$, it implies that the Levi-Civita connection is Hermitian, and has only the following non-zero coefficients:

$$C^i_{jk} := \Gamma^i_{jk} = \overline{\Gamma^{\bar{\imath}}_{\bar{j}\bar{k}}} = g^{\bar{h}i}C_{j\bar{h}k} = g^{\bar{h}i}\dot{\partial}_k g_{j\bar{h}},$$

with $C^i_{jk} = C^i_{kj}$ and $C^i_{jk}\eta^j = C^i_{jk}\eta^k = 0$. Taking into consideration that the Levi-Civita connection considered above is equivalent to the linear Chern connection on $\pi^*T'M = span\{\frac{\partial}{\partial z^i}\}$ [2], where $\pi : T'M \to M$ is the natural projection, and since $C^i_{jk} - C^i_{kj} = 0$, we get that $(\widetilde{T'_zM}, F_z)$ is Kählerian and thus ∇ is Kählerian connection, i.e. $\nabla_X(JY) = J\nabla_X Y$.

For a fixed point z on M, the unit sphere in (T'_zM, F_z) is the *complex indicatrix* in z and is defined as:

$$I_zM = \{\eta \in T'_zM \mid F(z,\eta) = 1\}.$$

Since the Hermitian matrix $\left(g_{i\bar{j}}(z,\eta)\right)$ is positive definite, we get that $L = F^2$ is convex and I_zM is a strictly pseudoconvex submanifold. Considering that we have only one defining equation which makes use of the real valued Finsler function F, the indicatrix I_zM is a real hypersurface of the punctured holomorphic tangent bundle, thus a CR hypersurface in T'_zM, for any fixed point $z \in M$.

Let $(u^1, ..., u^{2n-1})$ be local real coordinates on I_zM and let

$$\eta^j = \eta^j(u^1, .., u^{2n-1}), \quad \forall j \in \{1, .., n\}$$

be the equations of inclusion $i : I_zM \hookrightarrow \widetilde{T'_zM}$. Set $l^j = \frac{1}{F}\eta^j$ and $l_j = g_{j\bar{k}}l^{\bar{k}}$, which can be equivalently written as $l_j = \frac{1}{F}\frac{\partial L}{\partial \eta^j}$ or $l_j = 2\frac{\partial F}{\partial \eta^j}$ by using the following relations

$$\frac{\partial F}{\partial \eta^j} = \frac{\eta_j}{2F} := \frac{1}{2}l_j \ ; \qquad\qquad \frac{\partial F}{\partial \bar{\eta}^j} = \frac{\eta_{\bar{j}}}{2F} := \frac{1}{2}l_{\bar{j}}.$$

Since $F(z, \eta(u)) = 1$, we get that $L(z, \eta(u)) = 1$, which yields by derivation to $\frac{\partial L}{\partial \eta^j}\frac{\partial \eta^j}{\partial u^\alpha} + \frac{\partial L}{\partial \bar{\eta}^j}\frac{\partial \bar{\eta}^j}{\partial u^\alpha} = 0$ and so

$$l_j \frac{\partial \eta^j}{\partial u^\alpha} + l_{\bar{j}} \frac{\partial \eta^{\bar{j}}}{\partial u^\alpha} = 0, \quad \alpha \in \{1, \ldots, 2n-1\}, \ j \in \{1, \ldots, n\}. \tag{6}$$

Considering the inclusion map i, its tangent map $i_* : T_R(I_zM) \to T_C(\widetilde{T'_zM})$ acts on the tangent vectors of the complex indicatrix as

$$i_*\left(\frac{\partial}{\partial u^\alpha}\right) = X_\alpha := \frac{\partial \eta^k}{\partial u^\alpha}\frac{\partial}{\partial \eta^k} + \frac{\partial \bar{\eta}^k}{\partial u^\alpha}\frac{\partial}{\partial \bar{\eta}^k}.$$

Note that X_α is still a tangent vector of the indicatrix, but expressed in terms of tangent vectors of the complexified tangent bundle of $T'M$. Then, from (6), we can set (cf. [12])

$$N = l^j \frac{\partial}{\partial \eta^j} + l^{\bar{j}} \frac{\partial}{\partial \bar{\eta}^j}. \tag{7}$$

Thus we obtain that $G_R(X_\alpha, N) = 0$, where G_R represents the Riemannian metric applied to real vector fields and is given by

$$G_R(X, Y) = \operatorname{Re} \mathcal{G}(X', \overline{Y'}), \tag{8}$$

where X' and $\overline{Y'}$ are the holomorphic part and, respectively, the anti-holomorphic part of tangent vectors X and Y of the complexified tangent bundle of $T'M$, obtained by

$$X' = \frac{1}{2}(X - \mathrm{i}JX), \qquad \overline{Y'} = \frac{1}{2}(Y + \mathrm{i}JY), \qquad \mathrm{i} = \sqrt{-1}.$$

Consequently $N \in T_R(I_zM)^\perp$, so that N is the normal vector of the indicatrix bundle. Also, the normal vector has unit length, i.e. $G_R(N, N) = 1$.

Remark. A similar result with respect to the normal vector of the complex indicatrix was obtained in [20] by considering the holomorphic tangent bundle in a fixed point T'_zM as a $2n$-dimensional vector space $V_\mathbb{R}$, via the diffeomorphism $\mathcal{I} : V_\mathbb{R} \to T'_zM$, $(x^j, y^j) \mapsto (\eta^j) = (x^j + \sqrt{-1}y^j)$. Then, g induces a Riemannian metric on $V_\mathbb{R}\backslash\{0\}$, which is defined by

$$\hat{g}_v(X, Y) := \operatorname{Re} g_{\mathcal{I}(v)}(\mathcal{I}_*(X), \mathcal{I}_*(Y)), \ \forall X, Y \in T_vV_\mathbb{R}, \ v \neq 0, \tag{9}$$

where $\mathrm{Re}(\tau) = \frac{1}{2}(\tau + \bar{\tau})$, for any given form $\tau \in \mathcal{A}^{p,q}(M)$. To obtain the form of the normal of the indicatrix $I_{\mathbb{R}} := \mathcal{I}^{-1}(I_z M)$, set $\mathcal{I}^{-1}(\eta) = (v^j)$, where (v^j) taken as $v^{2j-1} := x^j$ and $v^{2j} = y^j$ denote the coordinates of $V_{\mathbb{R}}$. Let $X(t),\ t \in (-\epsilon, \epsilon)$ be a smooth curve on $I_{\mathbb{R}}$ with $X(0) = v^j \frac{\partial}{\partial v^j}$. Then, by taking $X := X(0)$ and $Y := \dot{X}(0) = \frac{dv^i}{dt}\frac{\partial}{\partial v^j}|_{t=0}$ the tangent vector of the curve, we obtain

$$\begin{aligned}\hat{g}_{X(0)}(X(0), \dot{X}(0)) &= \mathrm{Re} g_{\mathcal{I}(X(0))}(\mathcal{I}_*(X(0)), \mathcal{I}_*(\dot{X}(0))) \\ &= \mathrm{Re}\ \frac{\partial L_z}{\partial \eta^i}\frac{d\mathcal{I}^i(X)}{dt}\bigg|_{t=0} = \frac{1}{2}\ \frac{d}{dt}\bigg|_{t=0} L_z(\mathcal{I}(X(t))) = 0,\end{aligned}$$

where we used $g_{j\bar{k}}\eta^{\bar{k}} = \frac{\partial L}{\partial \eta^j}$ (and its conjugate). Hence, $\mathbf{n}|_{(v^j)} = X(0) = v^j \frac{\partial}{\partial v^j}$ is the normal vector to the indicatrix, which expressed in terms of $I_z M$ vector fields is exactly $N = \eta^j \frac{\partial}{\partial \eta^j} + \eta^{\bar{j}} \frac{\partial}{\partial \bar{\eta}^j}$ if $\eta \in I_z M$, or equivalently, $N = l^j \frac{\partial}{\partial \eta^j} + l^{\bar{j}} \frac{\partial}{\partial \bar{\eta}^j}$ for an arbitrary $\eta \in \widetilde{T'_z M}$.

Now, considering that complex indicatrix is a CR hypersurface of the holomorphic tangent bundle $T'_z M$, we can apply the theory of submanifolds. If we denote the induced Levi-Civita connection by $\tilde{\nabla}$, we can determine the Gauss-Weingarten formulae as follows:

$$\nabla_X Y = \tilde{\nabla}_X Y + h(X, Y), \tag{10}$$

$$\nabla_X W = -A_W X + \nabla^{\perp}_X W, \tag{11}$$

for any $X, Y \in \Gamma(T_R(I_z M))$ and $W \in \Gamma(T_R(I_z M)^{\perp})$. Here $h : \Gamma(T(I_z M)) \times \Gamma(T(I_z M)) \to \Gamma(T(I_z M)^{\perp})$ is the $\mathcal{F}(T'_z M)$-bilinear *second fundamental form* and $A : \Gamma(T(I_z M)^{\perp}) \times \Gamma(T(I_z M)) \to \Gamma(T(I_z M))$, $A_W X = A(W, X)$ is the $\mathcal{F}(T'_z M)$-bilinear *shape operator* (or *Weingarten operator*) of the indicatrix hypersurface. Since $T_R(I_z M)^{\perp} = span\{N\}$, these maps are defined by the following set of coefficients $h_{\alpha\beta},\ A^{\alpha}_{\beta}$, regarded as:

$$h(X_\beta, X_\alpha) = h_{\alpha\beta} N, \qquad A_N(X_\beta) = A^{\alpha}_{\beta} X_\alpha, \qquad \forall X_\alpha, X_\beta \in T_R(I_z M).$$

The Riemannian metric on $T'_z M$ and the induced metric on the indicatrix will be denoted by the same symbol G_R, given in (8). If we denote $B^i_\alpha := \frac{\partial \eta^i}{\partial u^\alpha}$, $B^{\bar{i}}_\alpha := \frac{\partial \bar{\eta}^i}{\partial u^\alpha}$ and we use $\mathrm{Re}(\tau) = \frac{1}{2}(\tau + \bar{\tau})$, for any given form τ, by direct computation we get

Proposition 2.1. *Let (M, F) be a complex Finsler manifold and $I_z M$ the complex indicatrix in an arbitrary fixed point $z \in M$. Then, with respect to the Levi-Civita connection ∇, we obtain:*

$$\nabla_{X_\alpha} N = \tfrac{1}{F} X_\alpha, \qquad \nabla_{X_\beta} X_\alpha = 2\mathrm{Re}\left[(X_\beta(B^i_\alpha) + B^k_\beta B^j_\alpha C^i_{jk})\dot{\partial}_i\right],$$
$$\textit{and} \qquad g_{\alpha\beta} := G_R(X_\alpha, X_\beta) = \mathrm{Re}(g_{i\bar{j}} B^i_\alpha B^{\bar{j}}_\beta).$$

Using this Proposition and relations $G_R(\nabla_X Y, N) = G_R(h(X, Y), N)$, respectively $G_R(\nabla_X N, X_\beta) = -G_R(A_N X, X_\beta)$, we obtain :

Proposition 2.2. *The coefficients of the second fundamental form and Weingarten operator of the complex indicatrix are given by*

$$h_{\alpha\beta} = \mathrm{Re}\left(l_i X_\beta(B^i_\alpha) + l_i B^k_\beta B^j_\alpha C^i_{jk}\right) \quad \text{and} \quad A^\alpha_\beta = -\frac{1}{F}\delta^\alpha_\beta.$$

Moreover, one can notice that

$$\nabla^\perp_{X_\alpha} N = 0, \quad \forall X_\alpha \in T\left(I_z M\right), \tag{12}$$

and thus, the Weingarten formula becomes

$$\nabla_{X_\alpha} N = -A_N X_\alpha, \qquad \forall X_\alpha \in T\left(I_z M\right). \tag{13}$$

Now, using Proposition 2.1, it follows that

$$A_N X_\alpha = -\frac{1}{F} X_\alpha. \tag{14}$$

Considering that $T'_z M$ is endowed with the Riemannian metric G_R and the linear Levi-Civita connection ∇ metric with respect to G_R, i.e. $\nabla G_R = 0$, between the second fundamental form and the shape operator of any of the indicatrix hypersurface $I_z M$ can be established the following relation

$$G_R(h(X_\alpha, X_\beta), N) = G_R(A_N X_\alpha, X_\beta), \quad \forall X_\alpha, X_\beta \in \Gamma(T_R(I_z M)), \tag{15}$$

and using (14), we get

Proposition 2.3. *The coefficients of the second fundamental form of the complex indicatrix satisfy*

$$h_{\alpha\beta} = -\frac{1}{F} g_{\alpha\beta}.$$

Corollary 2.4. *The second fundamental form relative to the second fundamental form is symmetric, i.e. its local coefficients satisfy $h_{\alpha\beta} = h_{\beta\alpha}$.*

A CR submanifold is said to be *totally geodesic* if $h(X,Y) = 0$ for any of its tangent vectors X, Y (cf. [7]). Thus, from the last Proposition it is obtained that the complex indicatrix is *not* a totally geodesic hypersurface of the holomorphic tangent bundle $\widetilde{T'_z M}$.

Also, in [7], we find that a submanifold $\tilde{M}$ is *totally umbilical* if the first and the second fundamental forms are proportional, i.e.

$$h(X,Y) = Hg(X,Y), \quad \forall X, Y \in \Gamma(T\tilde{M}),$$

where H is a normal vector field, named the *field of curvature vectors.* If we consider now in this formula the case of the tangent vectors of the complex indicatrix, X_α and X_β, and compare with the result from Proposition 2.3, we obtain the following theorem.

Theorem 2.5. *The complex indicatrix $I_z M$ in an arbitrary fixed point $z \in M$ of the complex Finsler space (M, F), is a totally umbilical manifold with constant field of curvature vectors $H = -\frac{1}{F} N$.*

Remark 2.6. This Theorem, together with (14), verifies the equivalent totally umbilical condition for the complex indicatrix, as $A_N X_\alpha = G_R(N,H)X_\alpha$, for any tangent vector X_α, normal vector N and Hermitian metric G_R.

In the Riemannian theory, a submanifold $\tilde{M}$ of a Riemannian manifold M is said to be an extrinsic sphere if it is totally umbilical and it has non-zero parallel mean curvature vector (cf. Nomizu-Yano [17]). Many of the basic results concerning extrinsic spheres in Riemannian and Kählerian geometry were obtained by B.Y.Chen ([9, 8]). Let $\tilde{M}$ be an orientable hypersurface in a Kähler manifold M. Then $\tilde{M}$ is an *extrinsic hypersphere* of M, if $h(X,Y) = Hg(X,Y)$ is satisfied for any X, Y vector fields on $\tilde{M}$. Here H denotes the mean curvature vector field of $\tilde{M}$ and its norm is a non zero constant function on the extrinsic hypersphere $\tilde{M}$. So, considering this definition and the result from Proposition 2.3 we obtain a result which can be found in ([12]) as well:

Theorem 2.7. *Let* (M,F) *be a complex Finsler manifold and* $z \in M$ *an arbitrary fixed point. Then* $\widetilde{T'_zM}$ *is a Kähler manifold and* $I_zM = \{\eta \in T'_zM : F(z,\eta)=1\}$ *is an extrinsic hypersphere of* T'_zM*.*

3. The fundamental equations of the indicatrix

In order to introduce the fundamental equations of the complex indicatrix, i.e. the Gauss, Codazzi and Ricci equations on the indicatrix hypersurface, we consider $\tilde{\nabla}$ and $\nabla^{\perp}$, the induced tangent and normal connection on I_zM of the Levi-Civita connection of T'_zM, as above. To get a link between curvatures $R(X,Y)Z$ of ∇ connection and $\tilde{R}(X,Y)Z$ of $\tilde{\nabla}$ connection, for $X,Y,Z \in \Gamma(T_R(I_zM))$ we act similarly as in [16, 18].

Firstly, the covariant derivative of the second fundamental form is being defined as $(\nabla_X h)(Y,Z) = \nabla^{\perp}_X(h(Y,Z)) - h(\tilde{\nabla}_X Y, Z) - h(Y, \tilde{\nabla}_X Z)$ and using (12), we get:

$$(\nabla_X h)(Y,Z) = -h(\tilde{\nabla}_X Y, Z) - h(Y, \tilde{\nabla}_X Z).$$

Considering the curvature and torsion definitions $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}$ and $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$, respectively, for $X,Y,Z \in \Gamma(T_R(I_zM))$ and by applying the Gauss-Weingarten formulae (10) and (11), we get:

$$\begin{aligned} R(X,Y)Z = \quad & \tilde{R}(X,Y)Z + A(h(X,Z),Y) - A(h(Y,Z),X) \\ & + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) + h(\tilde{T}(X,Y),Z), \end{aligned}$$

where $\tilde{T}(X,Y)$ is the torsion of the induced tangent connection $\tilde{\nabla}$. Since ∇ is the Levi-Civita connection, it is torsion free, and using the symmetry of the second fundamental form, from Corollary 2.4, it implies that the induced connection $\tilde{\nabla}$ is also torsion free, i.e. $\tilde{T} \equiv 0$.

Equating the tangent and the normal component with the help of the metric structure G_R from (8), we obtain

$$(16)\quad G_R(R(X,Y)Z,U) = G_R(\tilde{R}(X,Y)Z,U) + G_R\left(A_{h(X,Z)}Y - A_{h(Y,Z)}X, U\right)$$

and respectively,

$$G_R\left(R(X,Y)Z,N\right) = G_R\left(\left(\nabla_X h\right)(Y,Z) - \left(\nabla_Y h\right)(X,Z),N\right) \tag{17}$$

for any $X,Y,Z,U \in \Gamma(T_R(I_zM))$. Relations (16) and (17) are called *the Gauss equations*, respectively *H-Codazzi equations* of the complex indicatrix.

If we choose to use the curvature tensor of $(0,4)$ type given by the Riemannian curvature tensor $R(X,Y;U,Z) = G_R(R(X,Y)Z,U)$, by applying relation (15) the Gauss equation can be rewritten as

$$\begin{aligned} R(X,Y;U,Z) = \; & \tilde{R}(X,Y;U,Z) + G_R(h(X,Z),h(Y,U)) \\ & -G_R(h(Y,Z),h(X,U)), \end{aligned} \tag{18}$$

with $\tilde{R}(X,Y;U,Z)$ the induced Riemann curvature tensor on I_zM. Using in addition Theorem 2.5, the above relation becomes

$$\begin{aligned} R(X,Y;U,Z) = \; & \tilde{R}(X,Y;U,Z) + \tfrac{1}{L}G_R(X,Z)G_R(Y,U) \\ & -\tfrac{1}{L}G_R(Y,Z)G_R(X,U). \end{aligned} \tag{19}$$

Moreover, if we denote by $[R(X,Y)Z]^{\perp}$ the normal component of the curvature, the H-Codazzi equation becomes

$$[R(X,Y)Z]^{\perp} = \left(\nabla_X h\right)(Y,Z) - \left(\nabla_Y h\right)(X,Z).$$

Now, if we follow analogously steps for the curvatures $R\,(X,Y)\,N$ and $\tilde{R}\,(X,Y)\,N$, we define the covariant derivative of the shape operator as

$$\left(\nabla_X A\right)(N,Y) = \tilde{\nabla}_X\left(A_N Y\right) - A(\nabla^{\perp}_X N, Y) - A(N, \tilde{\nabla}_X Y)$$

and considering the curvature form $R^{\perp}$ of the normal Finsler connection, we can apply (12) and we obtain for each $X,Y \in \Gamma(T_R(I_zM))$

$$\left(\nabla_X A\right)(N,Y) = \tilde{\nabla}_X\left(A_N Y\right) - A(N, \tilde{\nabla}_X Y) \text{ and } R^{\perp}(X,Y)N = 0.$$

Thus, using the Gauss-Weingarten equations we obtained that:

$$\begin{aligned} R(X,Y)N = \; & h\left(Y, A_N X\right) - h\left(X, A_N Y\right) + \left(\nabla_Y A\right)(N,X) \\ & - \left(\nabla_X A\right)(N,Y) - A_N(\tilde{T}\,(X,Y)). \end{aligned}$$

If we separate the tangent and the normal components using G_R, and keeping in mind that $\tilde{\nabla}$ is torsion free, we obtain

$$G_R\left(R(X,Y)N,Z\right) = G_R\left(\left(\nabla_Y A\right)(N,X) - \left(\nabla_X A\right)(N,Y), Z\right), \tag{20}$$

and

$$G_R\left(R(X,Y)N,N\right) = G_R\left(h\left(Y,A_N X\right) - h\left(X,A_N Y\right),N\right) \tag{21}$$

for any $X,Y,Z \in \Gamma(T_R(I_zM))$. These relations represent the *A-Codazzi equations*, respectively *Ricci equations* of the complex indicatrix.

Using (15) for the Ricci equation, we obtain that $G_R\left(R(X,Y)N,N\right)=0$, and thus, $R(X,Y)N$ has only tangent component to the complex indicatrix. Then, the A-Codazzi equation becomes

$$R(X,Y)N=\left(\nabla_Y A\right)(N,X)-\left(\nabla_X A\right)(N,Y).$$

To express the Gauss-Codazzi equations in terms of tangent vectors X_α to the complex indicatrix, we use

$$\begin{aligned}[X_\alpha,X_\beta] &= 2\mathrm{Re}\{[X_\alpha(B^j_\beta)-X_\beta(B^i_\alpha)]\dot{\partial}_j\},\\ R(X_\alpha,X_\beta)X_\gamma &= 2\mathrm{Re}\{B^j_\gamma[B^k_\beta X_\alpha(C^i_{jk})-B^k_\alpha X_\beta(C^i_{jk})]\dot{\partial}_i\},\end{aligned}$$

and thus, from the Gauss equation (18) and Proposition 2.3 we obtain

Proposition 3.1. *The Riemannian curvature tensor of the induced tangent Levi-Civita connection associated to the complex indicatrix is given by*

$$\begin{aligned}\tilde{R}(X_\alpha,X_\beta;X_\gamma,X_\delta)= &\ 2\mathrm{Re}\{B^j_\gamma B^{\bar{l}}_{\bar{\delta}}[B^k_\beta X_\alpha(C^i_{jk})-B^k_\alpha X_\beta(C^i_{jk})]g_{i\bar{l}}\}\\ &-\tfrac{1}{L}g_{\gamma\alpha}g_{\delta\beta}+\tfrac{1}{L}g_{\gamma\beta}g_{\delta\alpha}.\end{aligned}$$

Using that by direct calculus $R(X_\alpha,X_\beta)N=0$ and (20) we find that $\left(\nabla_Y A\right)(N,X)=\left(\nabla_X A\right)(N,Y)$, i.e. $\left(\nabla_X A\right)(N,Y)$ is symmetric in the tangent vectors X and Y of the complex indicatrix.

4. The integrability of CR-distributions from the complex indicatrix

Since each real orientable hypersurface considered on a Kähler manifold is a CR-submanifold with $J\mathcal{D}^\perp_x=T^\perp_x$, we get that the anti-invariant distribution $\mathcal{D}^\perp$ of the indicatrix of a complex Finsler space in a fixed point must satisfy $J\mathcal{D}^\perp=span\{N\}$, where N is the unit normal vector field to I_zM given in (7) and J is the complex structure from (5). Then

$$\xi=JN=\mathrm{i}\left(l^k\dot{\partial}_k-l^{\bar{k}}\dot{\partial}_{\bar{k}}\right),\quad \mathrm{i}:=\sqrt{-1}, \tag{22}$$

is a real tangent unit vector on I_zM, $\xi=\bar{\xi}$, such that $N=-J\xi$. Let then $\mathcal{D}$ be the maximal J-invariant subspace of the tangent space of I_zM, orthogonal to $\mathcal{D}^\perp=span\{\xi\}$, such that $T_R(I_zM)=\mathcal{D}\oplus\mathcal{D}^\perp$. Thus, $\dim_R I_zM=2n-1$ and $\dim_R\mathcal{D}=2n-2$, since M is an n dimensional complex manifold.

Considering now that

$$T_R(T'_zM)=T_R(I_zM)\oplus span\{N\}\quad\text{and}\quad T_R(I_zM)=\mathcal{D}\oplus span\{\xi\},$$

with N and $\xi=JN$ given in (7), respectively (22), we can take $\mathcal{D}=T_R(\tilde{M})$, where $\tilde{M}$ is a complex hypersurface of T'_zM, with $\dim_C\tilde{M}=n-1$ and complex unit normal vector

$$N'=\frac{1}{2}(N-\mathrm{i}JN)=\frac{1}{2}(N-\mathrm{i}\xi)=l^j\dot{\partial}_j.$$

Thus, we have

$$\mathcal{D} = \mathrm{Re}\{T'\tilde{M} \oplus T''\tilde{M}\}$$

and since $T'(T'_zM) = span\{\dot{\partial}_j\}$, there exist the complex projection factors P^i_a such that

$$T'\tilde{M} = span\{Y'_a := P^j_a\dot{\partial}_j\}, \quad a \in \{1,\ldots,n-1\}.$$

Further, we denote by $\mathcal{D}' := T'\tilde{M}$, $\mathcal{D}'' := T''\tilde{M}$, and so $\mathcal{D}\otimes\mathbb{C} = \mathcal{D}'\oplus\mathcal{D}''$.

Having in mind that Y'_a and N' are complex tangent vector, respectively the complex normal vector of the complex hypersurface $\tilde{M}$ with respect to the Hermitian metric $\mathcal{G}$ on $\widetilde{T'_zM}$, the projection factors P^i_a fulfil the following conditions

$$P^j_a l_j = 0, \qquad P^{\bar{j}}_{\bar{a}} l_{\bar{j}} = 0 \qquad \text{and} \qquad l^j l_j = 1. \tag{23}$$

Then, we conclude that

$$\mathcal{D} = span\{Y_a := Y'_a + \overline{Y'_a},\ JY_a = \mathrm{i}(Y'_a - \overline{Y'_a})\}, \quad \mathrm{i} = \sqrt{-1}, \tag{24}$$

which leads to the explicit form of the real tangent vectors

$$Y_a = P^j_a\dot{\partial}_j + P^{\bar{j}}_{\bar{a}}\dot{\partial}_{\bar{j}} \quad \text{and} \quad JY_a = \mathrm{i}(P^j_a\dot{\partial}_j - P^{\bar{j}}_{\bar{a}}\dot{\partial}_{\bar{j}}), \quad a \in \{1,\ldots,n-1\},$$

Their components on the holomorphic and anti-holomorphic bundles are $Y'_a = P^j_a\dot{\partial}_j$, $\overline{Y'_a} = P^{\bar{j}}_{\bar{a}}\dot{\partial}_{\bar{j}}$, $JY'_a = \mathrm{i}P^i_a\dot{\partial}_i$ and $\overline{JY'_a} = -\mathrm{i}P^{\bar{j}}_{\bar{a}}\dot{\partial}_{\bar{j}}$.

Applying G_R to the real tangent vectors generating $\mathcal{D}$, by direct calculations involving (8) and the above observations we get

$$\begin{aligned} G_R(Y_a,Y_b) = &\quad G_R(JY_a,JY_b) = \mathrm{Re}(g_{j\bar{k}}P^j_aP^{\bar{k}}_{\bar{b}}) =: \mathrm{Re}(g_{a\bar{b}}) \quad \text{and} \\ G_R(Y_a,JY_b) = &\quad G_R(JY_a,Y_b) = -\mathrm{Re}(\mathrm{i}g_{j\bar{k}}P^j_aP^{\bar{k}}_{\bar{b}}) =: -\mathrm{Re}(\mathrm{i}g_{a\bar{b}}), \end{aligned}$$

with $a,b \in \{1,\ldots,n-1\}$. Also, using conditions (23) we can easily verify

$$G_R(\xi,Y_a) = G_R(\xi,JY_a) = G_R(N,Y_a) = G_R(N,JY_a) = 0,$$

where we used $N' = l^j\dot{\partial}_j$, $\overline{N'} = l^{\bar{j}}\dot{\partial}_{\bar{j}}$, $\xi' = \mathrm{i}l^j\dot{\partial}_j$, $\overline{\xi'} = -\mathrm{i}l^{\bar{j}}\dot{\partial}_{\bar{j}}$, with $\mathrm{i} = \sqrt{-1}$.

In order to study if the real or complex integrability conditions of the complex indicatrix distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are fulfilled, we will need the following results regarding the Lie brackets of these real vectors

$$\begin{aligned} [\xi,Y_a] = &\quad 2\mathrm{Re}\,(\xi(P^j_a) - \mathrm{i}Y_a(l_j))\dot{\partial}_j; \\ [\xi,JY_a] = &\quad 2\mathrm{Re}\{\mathrm{i}(\xi(P^j_a) - JY_a(l^j))\dot{\partial}_j\}; \\ [Y_a,Y_b] = &\quad 2\mathrm{Re}\,(Y_a(P^j_b) - Y_b(P^j_a))\dot{\partial}_j; \\ [Y_a,JY_b] = &\quad 2\mathrm{Re}\,(\mathrm{i}Y_a(P^j_b) - JY_b(P^j_a))\dot{\partial}_j; \\ [JY_a,JY_b] = &\quad 2\mathrm{Re}\{\mathrm{i}(JY_a(P^j_b) - JY_b(P^j_a))\dot{\partial}_j\}. \end{aligned} \tag{25}$$

Firstly, we start by studying if the complex involutivity condition of $\mathcal{D}'$ is fulfilled, i.e. if we have $[\Gamma(\mathcal{D}'),\Gamma(\mathcal{D}')] \subset \Gamma(\mathcal{D}')$. In [13] we find that this complex integrability condition is identical with

$$[JX,Y]+[X,JY]\in\Gamma(\mathcal{D}) \quad \text{and} \quad [JX,JY]-[X,Y]=J([JX,Y]+[X,JY]),$$

for any $X,Y\in\Gamma(\mathcal{D})$. The second condition is equivalent to the complex integrability of J, i.e. its Nijenhuis tensor satisfies $N_J=0$. Since I_zM is a real hypersurface of the Kähler manifold T'_zM, it results that this condition is accomplished on the complex indicatrix. Having in mind (24), it takes that to verify the first condition it is enough to prove that $[JY_a,Y_b]+[Y_a,JY_b]$ and $[JY_a,JY_b]-[Y_a,Y_b]\in\Gamma(\mathcal{D})$. This is confirmed by

$$G_R([JY_a,Y_b]+[Y_a,JY_b],\xi)=G_R([JY_a,JY_b]-[Y_a,Y_b],\xi)=0,$$

where we applied (25), the Riemannian metric G_R from (8) and the following identities obtained from the (23) relations by derivation

$$\begin{aligned}
\dot\partial_k(P^m_a)l_m &= -\tfrac{1}{F}P^m_a g_{mk}, & \dot\partial_{\bar k}(P^m_a)l_m &= -\tfrac{1}{F}P^m_a g_{m\bar k},\\
Y_a(P^m_b)l_m &= -\tfrac{1}{F}(g_{ab}+g_{b\bar a}), & JY_a(P^m_b)l_m &= -\tfrac{\mathrm{i}}{F}(g_{ab}-g_{b\bar a}).
\end{aligned} \tag{26}$$

Here we used the notations $g_{mk}=\frac{\partial^2 L}{\partial\eta^m\partial\eta^k}=\frac{\partial g_{m\bar j}}{\partial\eta^k}\bar\eta^j$, $g_{ab}=P^m_aP^k_bg_{mk}$, $g_{b\bar a}=P^m_bP^{\bar k}_{\bar a}g_{m\bar k}$ and the observation that $g_{mk}=g_{km}$ involves $g_{ab}=g_{ba}$.

Thus, we have the complex involutivity of $\mathcal{D}'$, which determines its complex integrability, according to Frobenius theorem. At its turn, the involutivity of $\mathcal{D}'$ involves the involutivity of $\mathcal{D}''$, but it not assures the involutivity or the integrability of $\mathcal{D}$.

In order to study now the real integrability of the complex indicatrix invariant distribution $\mathcal{D}$, we firstly verify if the involutivity property is fulfilled, i.e. $[X,Y]\in\Gamma(\mathcal{D})$ for any $X,Y\in\mathcal{D}$. But, by direct calculus, using again (25), (26) and the Riemannian metric G_R, we can easily obtain that $\mathcal{D}$ is not involutive since

$$\begin{aligned}
G_R([Y_a,Y_b],\xi) = {}& G_R([JY_a,JY_b],\xi)=\tfrac{\mathrm{i}}{F}(g_{b\bar a}-g_{a\bar b})\neq 0,\\
G_R([Y_a,JY_b],\xi) = {}& -\tfrac{1}{F}(g_{b\bar a}+g_{a\bar b})\neq 0.
\end{aligned}$$

Since I_zM is a CR-hypersurface of the Kähler manifold T'_zM, $\mathcal{D}$ is integrable if and only if $h(X,JY)=h(JX,Y)$, $\forall X,Y\in\Gamma(\mathcal{D})$ (cf. [7]). This condition is equivalent to $G_R(h(X,JY),N)=G_R(h(JX,Y),N)$, and using that $G_R(h(X,Y),N)=G_R(\nabla_XY,N)$ and (24), we need to verify the equality

$$G_R(\nabla_XJY,N)=G_R(\nabla_{JX}Y,N)$$

for $X=Y_a, Y=Y_b$ and $X=JY_a, Y=Y_b$. By direct calculus, using

$$\begin{aligned}
\nabla_{Y_a}Y_b &= 2\mathrm{Re}\{(P^k_aP^j_bC^m_{jk}+Y_a(P^m_b))\dot\partial_m\},\\
\nabla_{Y_a}JY_b &= 2\mathrm{Re}\{\mathrm{i}(P^k_aP^j_bC^m_{jk}+Y_a(P^m_b))\dot\partial_m\},\\
\nabla_{JY_a}Y_b &= 2\mathrm{Re}\{(\mathrm{i}P^k_aP^j_bC^m_{jk}+JY_a(P^m_b))\dot\partial_m\},\\
\nabla_{JY_a}JY_b &= 2\mathrm{Re}\{(-P^k_aP^j_bC^m_{jk}+\mathrm{i}JY_a(P^m_b))\dot\partial_m\},
\end{aligned}$$

the observation that $C^m_{jk}l_m = \frac{1}{F}g_{jk}$ and (26), we get that the integrability equality can not be accomplished since the left and right factors differ by a minus, as

$$\begin{aligned} G_R(\nabla_{Y_a}JY_b, N) = & \ -G_R(\nabla_{JY_a}Y_b, N) = \mathrm{Re}\tfrac{\mathrm{i}}{F}g_{a\bar{b}} \quad \text{and} \\ G_R(\nabla_{JY_a}JY_b, N) = & \ -G_R(-\nabla_{Y_a}Y_b, N) = \mathrm{Re}\tfrac{1}{F}g_{b\bar{a}}. \end{aligned}$$

On the other hand, according to [7] the anti-invariant distribution $\mathcal{D}^\perp$ is integrable if

$$G_R((\nabla_U J)V, X) = 0, \quad \forall U, V \in \Gamma(\mathcal{D}^\perp),\ X \in \Gamma(\mathcal{D}).$$

This condition is fulfilled for any CR-submanifold of a Kählerian manifold, hence for the complex indicatrix too, given the Kählerian connection ∇. Thus, we can state

Theorem 4.1. *Let (M, F) be a complex Finsler manifold, $z \in M$ an arbitrary fixed point and I_zM the complex indicatrix. Then the following affirmations take place with respect to the distributions of the CR-hypersurface I_zM of T'_zM:*

(a) *the anti-invariant distribution $\mathcal{D}^\perp$ is integrable;*

(b) *even though the complex CR-structure $\mathcal{D}'$ is integrable, the real invariant distribution $\mathcal{D}$ is not involutive, nor integrable.*

Thereby, we can introduce the *Levi form* which measures the deviation from the real involutivity of $\mathcal{D}$. Since I_zM is a real hypersurface with $dim_R I_zM = 2n-1$, it is endowed with a CR-structure of type $(n-1, 1)$. Further we will particularize the results from [13] to the complex indicatrix case, regarding the Levi form determined on it. Let

$$E_\upsilon = \{\omega \in T^*_\upsilon(I_zM) : \ker(\omega) \supseteq \mathcal{D}_\upsilon\}, \quad \forall \upsilon \in I_zM.$$

Then $E \to I_zM$ is a line subbundle of the cotangent bundle $T^*(I_zM) \to I_zM$ and $E \simeq T_R(I_zM)/\mathcal{D}$ is a vector isomorphism. There exists a globally defined nowhere vanishing section $\zeta \in \Gamma^\infty(E)$, which is referred to as a *pseudo-Hermitian structure* on I_zM. Now, given a pseudo-Hermitian structure ζ on I_zM, *the Levi form* L_ζ is defined by

$$L_\zeta(X', \overline{Y'}) = -\mathrm{i}(d\zeta)(X', \overline{Y'}), \qquad \text{with } \mathrm{i} = \sqrt{-1}, \tag{27}$$

for any $X, Y \in T_R(I_zM)$, $X' = \frac{1}{2}(X - \mathrm{i}JX)$ and $\overline{Y'} = \frac{1}{2}(Y + \mathrm{i}JY)$.

Considering that $T_R(I_zM) = \mathcal{D} \oplus span\{\xi\}$, with ξ given in (22), we call ξ the *characteristic direction* of I_zM. Thereby, the pseudo-Hermitian structure ζ must satisfy the properties $\zeta(\xi) = 1$ and $\iota_\xi d\zeta = 0$, where ι_ξ represents the interior product with ξ. Thus, on I_zM we take the 1-form

$$\zeta = \frac{\mathrm{i}}{2}(l_{\bar{k}}d\bar{\eta}^k - l_k d\eta^k),$$

for which it is easy to check that $\zeta(\xi)=1$. Its differential is

$$d\zeta=\frac{\mathrm{i}}{F}\left(-\frac{1}{2}l_m l_{\bar{k}}+g_{m\bar{k}}\right)d\eta^k\wedge d\bar{\eta}^k-\frac{1}{F}\mathrm{Re}\left[i\left(-\frac{1}{2}l_m l_k+g_{mk}\right)d\eta^m\wedge d\eta^k\right]$$

and it verifies $d\zeta(\xi,Y_a)=d\zeta(\xi,JY_a)=d\zeta(\xi,\xi)=0$, so that ζ is a pseudo-Hermitian structure on I_zM. Since $d\zeta$ shows that the CR structure obtained above on I_zM is nondegenerate, its Levi form is nondegenerate as well and is given by

$$L_\zeta=\frac{1}{F}\left(-\frac{1}{2}l_m l_{\bar{k}}+g_{m\bar{k}}\right)d\eta^m\wedge d\bar{\eta}^k.$$

By applying it to the tangent vectors of $\mathcal{D}'=span\{Y'_a=P^j_a\dot{\partial}_j\}$, we get that

$$L_\zeta(Y'_a,\overline{Y'_b})=\frac{1}{F}g_{a\bar{b}},$$

which is not zero, and thus it verifies once more that the real distribution $\mathcal{D}$ is not involutive. Also, the Levi form is positive definite and it justifies the strictly pseudoconvex property of the complex indicatrix I_zM as a CR-hypersurface of the punctured holomorphic tangent bundle T'_zM.

References

[1] Abate, M.,Patrizio, G., Finsler metrics—a global approach. Berlin: Springer-Verlag, 1994

[2] Aldea, N., On Chern complex linear connection. Bull. Math. Soc. Sc. Math. Roumanie Vol. 45 (93) No. 3-4 (2002), 119-131.

[3] Anastasiei, M., Gîrţu, M., Indicatrix of a Finsler vector bundle. Sci. Stud. Res. Ser. Math. Inform. 20(2) (2010), 21–28.

[4] Bao, D., Chern, S.-S., Shen, Z., An introduction to Riemann-Finsler geometry. New York: Springer-Verlag, 2000.

[5] Bejancu, A., CR submanifolds of a Kaehler manifold. I. Proc. Amer. Math. Soc. 69(1) (1978), 135–142.

[6] Bejancu, A., CR submanifolds of a Kaehler manifold. II. Trans. Amer. Math. Soc. 250 (1979), 333–345.

[7] Bejancu, A., Geometria CR-subvarietăţilor (in romanian). Mem. Secţ. Ştiinţ. Acad. Repub. Soc. România Ser. IV 5 (1982), 53–101.

[8] Chen, B.Y., Geometry of submanifolds and its applications. Science University of Tokyo, Tokyo, 1981.

[9] Chen, B.Y., CR-submanifolds of a Kaehler manifold. II. J. Differential Geom. 16 (1981), 493–509.

[10] Crâşmăreanu, M., Lagrange spaces with indicatrices as constant mean curvature surfaces or minimal surfaces. An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat. 10(1) (2002), 63–72.

[11] Dragomir, S., Cauchy-Riemann submanifolds of Kaehlerian Finsler spaces. Collect. Math. 40 (1989), 225–240.

[12] Dragomir, S., Grimaldi, R., On Rund's connection. Note Mat. 15 (1995), 85–98.

[13] Dragomir, S., Tomassini, G., Differential geometry and analysis on CR manifolds. Progress in Mathematics, 246. Boston, MA: Birkhäuser Boston, Inc., 2006.

[14] Matsumoto, M., Foundations of Finsler geometry and special Finsler spaces. Shigaken: Kaiseisha Press, 1986.

[15] Munteanu, G., Complex spaces in Finsler, Lagrange and Hamilton geometries. Dordrecht: Kluwer Academic Publishers, 2004.

[16] Munteanu, G., The equations of a holomorphic subspace in a complex Finsler space. Period. Math. Hungar. 55 (2007), 97–112.

[17] Nomizu, K., Yano, K., On circles and spheres in Riemannian geometry. Math. Ann. 210 (1974), 163-170.

[18] Popovici, E., The equations of the indicatrix of a complex Finsler space. Bull. Transilv. Univ. Braşov Ser. III, 6(55) (2013), 63–76.

[19] Rund, H., The differential geometry of Finsler spaces. Berlin-Göttingen-Heidelberg: Springer-Verlag, 1959.

[20] Zhao, W., A Gauss-Bonnet-Chern theorem for complex Finsler manifold. Available at: `http://arxiv.org/abs/1409.2657` (accessed 3 October 2014)

Received by the editors August 31, 2015