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INEXTENSIBLE FLOWS OF PARTIALLY NULL AND PSEUDO NULL CURVES IN SEMI-EUCLIDEAN 4-SPACE WITH INDEX 2

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Abstract. In this paper, we consider the inextensible flows in semi-Euclidean 4-space with index 2 ($\mathbb{E}_2^4$). We give the necessary and sufficient conditions for the flow to be inextensible and we find the evolution equations for the inextensible flows in semi-Euclidean 4-space with index 2 ($\mathbb{E}_2^4$).

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1. Introduction

The time evolution of a curve or surface is generated by its corresponding flow in $\mathbb{E}^3$. For this reason we shall also refer to curve and surface evolutions as flows throughout this article. Flow is said to be inextensible if, in the former case, its arclength is preserved, and in the latter case, if its intrinsic curvature is preserved. Physically, inextensible curve and surface flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example, or of a piece of paper carried by the wind, can be described by inextensible curve and surface flows. Such motions arise quite naturally in a wide range of physical applications. For example, both Chirikjian and Burdick [4] and Mochiyama et al. [15] study the shape control of hyper-redundant, or snake-like, robots. Inextensible curve and surface flows also arise in the context of many problems in computer vision [10] and [14] and computer animation [5], and even structural mechanics [19].

Inextensible flows are studied in Euclidean 3-space by Körpınar in [12]. In addition, many researchers have studied on inextensible flows such as [8], [11], [13], [1] and [7]. In [1] and [13], the authors studied inextensible flows in Minkowski space-time $\mathbb{E}_1^4$. By drawing inspiration from them, in this paper, we consider the inextensible flows in semi-Euclidean 4-space with index 2 ($\mathbb{E}_2^4$). We give the necessary and sufficient conditions for the flow to be inextensible and we find the evolution equations for the inextensible flows in semi-Euclidean 4-space with index 2 ($\mathbb{E}_2^4$).

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2. Preliminaries

The semi-Euclidean 4-space with index 2 ($\mathbb{E}_2^4$) is the Euclidean 4-space $\mathbb{E}^4$ equipped with indefinite flat metric given by

$$g = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}_2^4$. Recall that a vector $v \in \mathbb{E}_2^4 \backslash \{0\}$ can be spacelike if $g(v, v) > 0$, timelike if $g(v, v) < 0$ and null (lightlike) if $g(v, v) = 0$. In particular, the vector $v = 0$ is said to be lightlike. The norm of a vector v is given by $||v|| = \sqrt{|g(v, v)|}$. Two vectors v and w are said to be orthogonal, if $g(v, w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_2^4$, can locally be spacelike, timelike or null (lightlike), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null ([16]). Recall that a non-null curve in $\mathbb{E}_2^4$ is called pseudo null curve or partially null curve, if respectively its principal normal vector is null or its first binormal vector is null ([3]).

A null curve α is parameterized by pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$ ([2]). On the other hand, a non-null curve α is parametrized by the arclength parameter s if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along a curve α in $\mathbb{E}_2^4$, consisting of the tangent, the principal normal, the first binormal and the second binormal vector field respectively.

If α is a non-null curve whose Frenet frame $\{T, N, B_1, B_2\}$ contains only non-null vector fields, the Frenet equations are given by ([9])

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2\kappa_1 & 0 & 0 \\ -\epsilon_1\kappa_1 & 0 & \epsilon_3\kappa_2 & 0 \\ 0 & -\epsilon_2\kappa_2 & 0 & \epsilon_1\epsilon_2\epsilon_3\kappa_3 \\ 0 & 0 & -\epsilon_3\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.1}$$

where $g(T, T) = \epsilon_1$, $g(N, N) = \epsilon_2$, $g(B_1, B_1) = \epsilon_3$, $g(B_2, B_2) = \epsilon_4$, $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1$, $\epsilon_i \in \{-1, 1\}$, $i \in \{1, 2, 3, 4\}$. In particular, the following conditions hold:

$$g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(N, B_2) = g(B_1, B_2) = 0.$$

If α is a pseudo null curve, the Frenet formulas read ([17])

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ 0 & 0 & \kappa_2 & 0 \\ 0 & \kappa_3 & 0 & -\epsilon_2\kappa_2 \\ -\epsilon_1\kappa_1 & 0 & -\epsilon_2\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.2}$$

where the first curvature $\kappa_1(s) = 0$, if α is straight line, or $\kappa_1(s) = 1$ in all other cases. Then, the following conditions are satisfied:

$$g(T, T) = \epsilon_1\ , \ g(B_1, B_1) = \epsilon_2, \quad g(N, N) = g(B_2, B_2) = 0,$$

$$g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(B_1, B_2) = 0,$$

$$g(N, B_2) = 1, \ \epsilon_1\epsilon_2 = -1.$$

If α is a Cartan null curve, the Frenet formulas read ([6],[18])

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\epsilon_1\kappa_2 & 0 & -\epsilon_1\kappa_1 & 0 \\ 0 & \kappa_2 & 0 & \kappa_3 \\ -\epsilon_2\kappa_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.3}$$

where the first curvature $\kappa_1(s) = 0$, if α is straight line, or $\kappa_1(s) = 1$ in all other cases. Then, the following conditions are satisfied:

$$g(N, N) = \epsilon_1 \ , \ g(B_2, B_2) = \epsilon_2, \quad g(T, T) = g(B_1, B_1) = 0,$$

$$g(T, N) = g(T, B_2) = g(N, B_1) = g(N, B_2) = g(B_1, B_2) = 0,$$

$$g(T, B_1) = 1, \ \epsilon_1\epsilon_2 = -1.$$

If α is a partially null curve, the Frenet formulas read ([17])

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ \kappa_1 & 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 & 0 \\ 0 & -\epsilon_2\kappa_2 & 0 & -\kappa_3 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.4}$$

where the third curvature $\kappa_3(s) = 0$ for each s. Moreover, the following conditions hold:

$$g(T, T) = \epsilon_1 \ , \ g(N, N) = \epsilon_2, \quad g(B_1, B_1) = g(B_2, B_2) = 0,$$

$$g(T, N) = g(T, B_2) = g(T, B_1) = g(N, B_1) = g(N, B_2) = 0,$$

$$g(B_1, B_2) = 1, \ \epsilon_1\epsilon_2 = -1.$$

3. Inextensible flows of partially null and pseudo null curves in $\mathbb{E}_2^4$

In this paper, we assume that $\gamma : [0, l] \times [0, \omega] \to \mathbb{E}_2^4$ is a one parameter family of smooth partially null or pseudo null curves in the semi-Euclidean 4-space with index 2, where l is arclength of the initial curve. Let u be the curve parametrization variable, $0 \leq u \leq l$. The arclength of γ is given by

$$s(u) = \int_0^u \left\| \frac{\partial \gamma}{\partial t} \right\| du.$$

The operator $\frac{\partial}{\partial s}$ is given in terms of u by

$$\frac{\partial}{\partial s} = \frac{1}{v} \frac{\partial}{\partial u}$$

where $v = \left\| \frac{\partial \gamma}{\partial t} \right\|$. The arclength parameter is $ds = vdu$.

Definition 3.1. Let γ be a partially null or pseudo null curve with the Frenet frame $\{T, N, B_1, B_2\}$ in the semi-Euclidean space with index 2. Any flow of the partially null or pseudo null curves can be given as follows

$$\frac{\partial\gamma}{\partial t} = \beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2 \tag{3.1}$$

where β_i $(1 \leq i \leq 4)$ is a C^∞-function.

Let the arclength parameter be

$$s(u,t) = \int_0^u v du.$$

In $\mathbb{E}_2^4$, the requirement that the partially null or pseudo null curves are not subjected to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t} s(u,t) = \int_0^u \frac{\partial v}{\partial t} du = 0$$

where $u \in [0, l]$.

Definition 3.2. Let γ be a partially null or pseudo null curve in $\mathbb{E}_2^4$. A partially null or pseudo null curve evolution $\gamma(u,t)$ and its flow $\frac{\partial\gamma}{\partial t}$ are said to be inextensible if

$$\frac{\partial}{\partial t}\left\|\frac{\partial\gamma}{\partial u}\right\| = 0. \tag{3.2}$$

3.1. Inextensible flows of partially null curves in $\mathbb{E}_2^4$

In this section, we consider inextensible flows of partially null curves in $\mathbb{E}_2^4$.

Lemma 3.3. *Let $\frac{\partial\gamma}{\partial t} = \beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2$ be a smooth flow of a partially null curve γ with $\kappa_3 = 0$ in $\mathbb{E}_2^4$. If the flow is inextensible, then*

$$\frac{\partial v}{\partial t} = \varepsilon_1\left(\frac{\partial\beta_1}{\partial u} + \beta_2 v k_1\right). \tag{3.3}$$

Proof. Assume that $\frac{\partial\gamma}{\partial t}$ is a smooth flow of a partially null curve γ with $\kappa_3 = 0$ in $\mathbb{E}_2^4$. By using the definition of γ, we get

$$v^2 = g\left(\frac{\partial\gamma}{\partial u}, \frac{\partial\gamma}{\partial u}\right). \tag{3.4}$$

Differentiating (3.4) with resprect to t, we have

$$2v\frac{\partial v}{\partial t} = \frac{\partial}{\partial t} g\left(\frac{\partial\gamma}{\partial u}, \frac{\partial\gamma}{\partial u}\right) = 2g\left(\frac{\partial\gamma}{\partial u}, \frac{\partial}{\partial u}\left(\frac{\partial\gamma}{\partial t}\right)\right) \tag{3.5}$$

which leads to the following

$$v\frac{\partial v}{\partial t} = g\left(\frac{\partial\gamma}{\partial u}, \frac{\partial}{\partial u}\left(\frac{\partial\gamma}{\partial t}\right)\right). \tag{3.6}$$

Substituting (3.1) in (3.6), we find

$$v\frac{\partial v}{\partial t}=g\left(\frac{\partial \gamma}{\partial u},\frac{\partial}{\partial u}\left(\beta_1 T+\beta_2 N+\beta_3 B_1+\beta_4 B_2\right)\right) \tag{3.7}$$

which implies that

$$\begin{aligned}\frac{\partial v}{\partial t} \quad = \quad & g\left(T,\left(\frac{\partial \beta_1}{\partial u}+\beta_2 v k_1\right)T+\left(\beta_1 v k_1+\frac{\partial \beta_2}{\partial u}-\beta_4 v\varepsilon_2 k_2\right)N\right.\\ & \left.+\left(\beta_2 v k_2+\frac{\partial \beta_3}{\partial u}\right)B_1+\left(\frac{\partial \beta_4}{\partial u}\right)B_2\right).\end{aligned} \tag{3.8}$$

From (3.8) , we obtain

$$\frac{\partial v}{\partial t}=\varepsilon_1\left(\frac{\partial \beta_1}{\partial u}+\beta_2 v k_1\right) \tag{3.9}$$

which completes the proof. □

Theorem 3.4. *Let* $\frac{\partial \gamma}{\partial t}=\beta_1 T+\beta_2 N+\beta_3 B_1+\beta_4 B_2$ *be a smooth flow of a partially null curve* γ *with* $\kappa_3=0$ *in* $\mathbb{E}_2^4$*. Then the flow is inextensible if and only if*

$$\frac{\partial \beta_1}{\partial u}=-\beta_2 v k_1. \tag{3.10}$$

Proof. Let $\frac{\partial \gamma}{\partial t}$ be inextensible. From (3.2) , we have

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u\frac{\partial v}{\partial t}du=0 \tag{3.11}$$

Substituting (3.3) in (3.11), we obtain

$$\frac{\partial \beta_1}{\partial u}=-\beta_2 v k_1. \tag{3.12}$$

□

We now restrict ourselves to arc length parametrized curves. That is, $v=1$ and the local coordinate u corresponds to the curve arclength s. Then, we have the following lemma.

Lemma 3.5. *Let* $\frac{\partial \gamma}{\partial t}=\beta_1 T+\beta_2 N+\beta_3 B_1+\beta_4 B_2$ *be a smooth inextensible flow of a partially null curve* γ *with* $\kappa_3=0$ *in* $\mathbb{E}_2^4$*. Then we have the following*

$$\frac{\partial T}{\partial t}=\left(\beta_1 k_1+\frac{\partial \beta_2}{\partial s}-\beta_4\varepsilon_2 k_2\right)N+\left(\beta_2 k_2+\frac{\partial \beta_3}{\partial s}\right)B_1+\frac{\partial \beta_4}{\partial s}B_2,$$

$$\frac{\partial N}{\partial t}=\left(\beta_1 k_1+\frac{\partial \beta_2}{\partial s}-\varepsilon_2\beta_4 k_2\right)T+\psi_2 B_1+\psi_1 B_2,$$

$$\frac{\partial B_1}{\partial t}=-\varepsilon_1\frac{\partial \beta_4}{\partial s}T-\varepsilon_2\psi_1 N+\psi_3 B_1,$$

$$\frac{\partial B_2}{\partial t}=-\varepsilon_1\left(\beta_2 k_2+\frac{\partial \beta_3}{\partial s}\right)T-\varepsilon_2\psi_2 N-\psi_3 B_2$$

where $\psi_1=g\left(\frac{\partial N}{\partial t},B_1\right)$, $\psi_2=g\left(\frac{\partial N}{\partial t},B_2\right)$ *and* $\psi_3=g\left(\frac{\partial B_1}{\partial t},B_2\right)$.

Proof. Let $\frac{\partial \gamma}{\partial t}=\beta_1 T+\beta_2 N+\beta_3 B_1+\beta_4 B_2$ be a smooth inextensible flow of a partially null curve γ with $\kappa_3=0$ in $\mathbb{E}_2^4$. Then

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial t}\frac{\partial \gamma}{\partial s}=\frac{\partial}{\partial s}\frac{\partial \gamma}{\partial t}=\frac{\partial}{\partial s}\left(\beta_1 T+\beta_2 N+\beta_3 B_1+\beta_4 B_2\right) \tag{3.13}$$

which brings about

$$\frac{\partial T}{\partial t}=\left(\beta_1 k_1+\frac{\partial \beta_2}{\partial s}-\beta_4\varepsilon_2 k_2\right)N+\left(\beta_2 k_2+\frac{\partial \beta_3}{\partial s}\right)B_1+\frac{\partial \beta_4}{\partial s}B_2. \tag{3.14}$$

From (3.14), we obtain

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}g(T,N)=g\left(\frac{\partial T}{\partial t},N\right)+g\left(T,\frac{\partial N}{\partial t}\right)\\
&= \varepsilon_2\left(\beta_1 k_1+\frac{\partial \beta_2}{\partial s}-\beta_4\varepsilon_2 k_2\right)+g\left(T,\frac{\partial N}{\partial t}\right),\\
0 &= \frac{\partial}{\partial t}g(T,B_1)=g\left(\frac{\partial T}{\partial t},B_1\right)+g\left(T,\frac{\partial B_1}{\partial t}\right)\\
&= \left(\frac{\partial \beta_4}{\partial s}-\beta_4 k_3\right)+g\left(T,\frac{\partial B_1}{\partial t}\right),\\
0 &= \frac{\partial}{\partial t}g(T,B_2)=g\left(\frac{\partial T}{\partial t},B_2\right)+g\left(T,\frac{\partial B_2}{\partial t}\right)\\
&= \left(\beta_2 k_2+\frac{\partial \beta_3}{\partial s}\right)+g\left(T,\frac{\partial B_2}{\partial t}\right),\\
0 &= \frac{\partial}{\partial t}g(N,B_1)=g\left(\frac{\partial N}{\partial t},B_1\right)+g\left(N,\frac{\partial B_1}{\partial t}\right)\\
&= \psi_1+g\left(N,\frac{\partial B_1}{\partial t}\right),\\
0 &= \frac{\partial}{\partial t}g(N,B_2)=g\left(\frac{\partial N}{\partial t},B_2\right)+g\left(N,\frac{\partial B_2}{\partial t}\right)\\
&= \psi_2+g\left(N,\frac{\partial B_2}{\partial t}\right),\\
0 &= \frac{\partial}{\partial t}g(B_1,B_2)=g\left(\frac{\partial B_1}{\partial t},B_2\right)+g\left(B_1,\frac{\partial B_2}{\partial t}\right)\\
&= \psi_3+g\left(B_1,\frac{\partial B_2}{\partial t}\right)
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{\partial N}{\partial t} &= \left(\beta_1 k_1 + \frac{\partial \beta_2}{\partial s} - \beta_4 \varepsilon_2 k_2\right) T + \psi_2 B_1 + \psi_1 B_2, \\
\frac{\partial B_1}{\partial t} &= -\varepsilon_1 \frac{\partial \beta_4}{\partial s} T - \varepsilon_2 \psi_1 N + \psi_3 B_1, \\
\frac{\partial B_2}{\partial t} &= -\varepsilon_1 \left(\beta_2 k_2 + \frac{\partial \beta_3}{\partial s}\right) T - \varepsilon_2 \psi_2 N - \psi_3 B_2
\end{aligned}$$

where $\psi_1 = g\left(\frac{\partial N}{\partial t}, B_1\right)$, $\psi_2 = g\left(\frac{\partial N}{\partial t}, B_2\right)$ and $\psi_3 = g\left(\frac{\partial B_1}{\partial t}, B_2\right)$. This completes the proof. □

Theorem 3.6. *Let $\frac{\partial \gamma}{\partial t} = \beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2$ be a smooth inextensible flow of a partially null curve γ with $\kappa_3 = 0$ in $\mathbb{E}^4_2$. Then the following partial differential equation holds:*

$$\frac{\partial k_1}{\partial t} = \left[\frac{\partial^2 \beta_2}{\partial s^2} + \frac{\partial}{\partial s}(\beta_1 k_1) - \varepsilon_2 \frac{\partial}{\partial s}(\beta_4 k_2) - \varepsilon_2 k_2 \frac{\partial \beta_4}{\partial s}\right].$$

Proof. From lemma 3.5, we get

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial T}{\partial t} &= k_1 \left(\beta_1 k_1 + \frac{\partial \beta_2}{\partial s} - \beta_4 \varepsilon_2 k_2\right) T \\
&+ \left(\left(\frac{\partial^2 \beta_2}{\partial s^2} + \frac{\partial}{\partial s}(\beta_1 k_1) - \varepsilon_2 \frac{\partial}{\partial s}(\beta_4 k_2)\right) - \varepsilon_2 k_2 \frac{\partial \beta_4}{\partial s}\right) N \\
&+ \left(k_2 \left(\beta_1 k_1 + \frac{\partial \beta_2}{\partial s} - \beta_4 \varepsilon_2 k_2\right) + \frac{\partial^2 \beta_3}{\partial s^2} + \frac{\partial}{\partial s}(\beta_2 k_2)\right) B_1 \\
&+ \left(\frac{\partial^2 \beta_4}{\partial s^2} - k_2 \frac{\partial \beta_4}{\partial s}\right) B_2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial T}{\partial s} &= \frac{\partial}{\partial t}(k_1 N) = \frac{\partial k_1}{\partial s} N + k_1 \frac{\partial N}{\partial t} = k_1 \left(\beta_1 k_1 + \frac{\partial \beta_2}{\partial s} - \beta_4 \varepsilon_2 k_2\right) T \\
&+ \frac{\partial k_1}{\partial t} N + k_1 \psi_2 B_1 + k_1 \psi_1 B_2.
\end{aligned}$$

From equality of the coefficients of N in above equalities, we get

$$\frac{\partial k_1}{\partial t} = \left[\frac{\partial^2 \beta_2}{\partial s^2} + \frac{\partial}{\partial s}(\beta_1 k_1) - \varepsilon_2 \frac{\partial}{\partial s}(\beta_4 k_2) - \varepsilon_2 k_2 \frac{\partial \beta_4}{\partial s}\right].$$

□

Corollary 3.7. *In theorem 3.6, from equality of the coefficients of B_1 and B_2 respectively, we obtain*

$$k_1 \psi_2 = k_2 \frac{\partial \beta_2}{\partial s} + k_1 k_2 \beta_1 - \varepsilon_2 k_2^2 \beta_4 + \frac{\partial^2 \beta_3}{\partial s^2} + \frac{\partial}{\partial s}(\beta_2 k_2),$$

$$k_1 \psi_1 = \frac{\partial^2 \beta_4}{\partial s^2} - k_2 \frac{\partial \beta_4}{\partial s}.$$

Theorem 3.8. *Let $\frac{\partial\gamma}{\partial t}=\beta_1 T+\beta_2 N+\beta_3 B_1+\beta_4 B_2$ be a smooth inextensible flow of a partially null curve γ with $\kappa_3=0$ in $\mathbb{E}_2^4$. Then the following partial differential equation holds:*

$$\frac{\partial k_1}{\partial t}-\varepsilon_1 k_2\frac{\partial\beta_4}{\partial s}=\frac{\partial^2\beta_2}{\partial s^2}+\frac{\partial}{\partial s}(\beta_1 k_1)-\varepsilon_2\frac{\partial}{\partial s}(\beta_4 k_2).$$

Proof. From lemma 3.5, we get

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial N}{\partial t} &= \left(\frac{\partial^2\beta_2}{\partial s^2}+\frac{\partial}{\partial s}(\beta_1 k_1)-\varepsilon_2\frac{\partial}{\partial s}(\beta_4 k_2)\right)T\\ &\quad+\left(\left(\frac{\partial\beta_2}{\partial s}+\beta_1 k_1-\beta_4\varepsilon_2 k_2\right)k_1-\varepsilon_2 k_2\psi_1\right)N\\ &\quad+\frac{\partial\psi_2}{\partial s}B_1+\left(\frac{\partial\psi_1}{\partial s}-\psi_1 k_3\right)B_2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial N}{\partial s} &= \left(\frac{\partial k_1}{\partial t}-\varepsilon_1 k_2\frac{\partial\beta_4}{\partial s}\right)T+\left(k_1\frac{\partial\beta_2}{\partial s}+\beta_1 k_1^2-\beta_4\varepsilon_2 k_1 k_2-\varepsilon_2 k_2\psi_1\right)N\\ &\quad+\left(\frac{\partial k_2}{\partial t}+k_1\frac{\partial\beta_3}{\partial s}+k_1 k_2\beta_2+k_2\psi_3\right)B_1+\left(k_1\frac{\partial\beta_4}{\partial s}\right)B_2.\end{aligned}$$

From equality of the coefficients of T in above equalities, we get

$$\frac{\partial k_1}{\partial t}-\varepsilon_1 k_2\frac{\partial\beta_4}{\partial s}=\frac{\partial^2\beta_2}{\partial s^2}+\frac{\partial}{\partial s}(\beta_1 k_1)-\varepsilon_2\frac{\partial}{\partial s}(\beta_4 k_2).$$

□

Corollary 3.9. *In Theorem 3.8, from the equality of the coefficients of B_1 and B_2 respectively, we obtain*

$$\begin{aligned}\frac{\partial\psi_2}{\partial s} &= \frac{\partial k_2}{\partial t}+k_1\frac{\partial\beta_3}{\partial s}+k_1 k_2\beta_2+k_2\psi_3,\\ k_1\frac{\partial\beta_4}{\partial s} &= \frac{\partial\psi_1}{\partial s}-k_3\psi_1.\end{aligned}$$

Theorem 3.10. *Let $\frac{\partial\gamma}{\partial t}=\beta_1 T+\beta_2 N+\beta_3 B_1+\beta_4 B_2$ be a smooth inextensible flow of a partially null curve γ with $\kappa_3=0$ in $\mathbb{E}_2^4$. Then the following differential equation holds:*

$$\frac{\partial}{\partial s}\left(\frac{1}{k_1}\frac{\partial^2\beta_4}{\partial s^2}\right)=k_1\frac{\partial\beta_4}{\partial s}.$$

Proof. From lemma 3.5, we get

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial B_1}{\partial t} &= \left(-\varepsilon_1\frac{\partial^2\beta_4}{\partial s^2}-\varepsilon_2\psi_1 k_1\right)T-\left(\varepsilon_2\frac{\partial\psi_1}{\partial s}+\varepsilon_1 k_1\frac{\partial\beta_4}{\partial s}\right)N\\ &\quad+\left(-\varepsilon_2\psi_1 k_2+\frac{\partial\psi_3}{\partial s}\right)B_1.\end{aligned}$$

On the other hand, $\frac{\partial}{\partial t}\frac{\partial B_1}{\partial s} = 0$. Thus, we have

$$\begin{aligned}\psi_1 &= \frac{1}{k_1}\frac{\partial^2 \beta_4}{\partial s^2},\\ \frac{\partial \psi_1}{\partial s} &= k_1 \frac{\partial \beta_4}{\partial s},\\ \frac{\partial \psi_3}{\partial s} &= \varepsilon_2 k_2 \psi_1\end{aligned}$$

which implies that

$$\frac{\partial}{\partial s}\left(\frac{1}{k_1}\frac{\partial^2 \beta_4}{\partial s^2}\right) = k_1 \frac{\partial \beta_4}{\partial s}.$$

□

Theorem 3.11. *Let $\frac{\partial \gamma}{\partial t} = \beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2$ be a smooth inextensible flow of a partially null curve γ with $\kappa_3 = 0$ in $\mathbb{E}^4_2$. Then the following differential equation holds:*

$$-\varepsilon_2 k_2 \frac{\partial \beta_2}{\partial s} - \varepsilon_2 k_1 k_2 \beta_1 + k_2^2 \beta_4 = -\varepsilon_1 \frac{\partial^2 \beta_3}{\partial s^2} - \varepsilon_1 \frac{\partial}{\partial s}(\beta_1 k_2) - \varepsilon_2 k_1 \psi_2.$$

Proof. From lemma 3.5, we get

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial B_2}{\partial t} &= \left(-\varepsilon_1 \frac{\partial^2 \beta_3}{\partial s^2} - \varepsilon_1 \frac{\partial}{\partial s}(\beta_1 k_2) - \varepsilon_2 k_1 \psi_2\right) T\\ &\quad + \left(-\varepsilon_1 k_1 \left(\frac{\partial \beta_3}{\partial s} + k_2 \beta_2\right) - \varepsilon_2 \frac{\partial \psi_2}{\partial s} + \psi_3 \varepsilon_2 k_2\right) N\\ &\quad -\varepsilon_2 k_2 \psi_2 B_1 - \left(\frac{\partial \psi_3}{\partial s} - k_3 \psi_3\right) B_2\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial B_2}{\partial s} &= \left(-\varepsilon_2 k_2 \frac{\partial \beta_2}{\partial s} - \varepsilon_2 k_1 k_2 \beta_1 + k_2^2 \beta_4\right) T - \varepsilon_2 \frac{\partial k_2}{\partial t} N\\ &\quad -\varepsilon_2 k_2 \psi_2 B_1 - \varepsilon_2 k_2 \psi_1 B_2.\end{aligned}$$

From the equality of the coefficients of T in above equalities, we get

$$-\varepsilon_2 k_2 \frac{\partial \beta_2}{\partial s} - \varepsilon_2 k_1 k_2 \beta_1 + k_2^2 \beta_4 = -\varepsilon_1 \frac{\partial^2 \beta_3}{\partial s^2} - \varepsilon_1 \frac{\partial}{\partial s}(\beta_1 k_2) - \varepsilon_2 k_1 \psi_2.$$

□

Corollary 3.12. *In Theorem 3.11, from the equality of the coefficients of N and B_2 respectively, we obtain*

$$\begin{aligned}-\frac{\partial k_2}{\partial t} &= k_1 \frac{\partial \beta_3}{\partial s} + k_1 k_2 \beta_2 - \frac{\partial \psi_2}{\partial s} + k_2 \psi_3,\\ -\varepsilon_2 k_2 \psi_1 &= -\frac{\partial \psi_3}{\partial s} + k_3 \psi_3.\end{aligned}$$

3.2. Inextensible flows of pseudo null curves in $\mathbb{E}_2^4$

In this section, we consider inextensible flows of pseudo null curves in $\mathbb{E}_2^4$. Since the proofs of the following theorems are similiar to previous proofs, we omit some of those proofs.

Lemma 3.13. *Let $\frac{\partial \gamma}{\partial t} = \beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2$ be a smooth flow of a pseudo null curve γ with $\kappa_1 = 1$ in $\mathbb{E}_2^4$. If the flow is inextensible, then*

$$\frac{\partial v}{\partial t} = \frac{\partial \beta_1}{\partial u} - \varepsilon_1 v \beta_4. \tag{3.15}$$

Theorem 3.14. *Let $\frac{\partial \gamma}{\partial t} = \beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2$ be a smooth flow of a pseudo null curve γ with $\kappa_1 = 1$ in $\mathbb{E}_2^4$. Then the flow is inextensible if and only if*

$$\frac{\partial \beta_1}{\partial u} = \beta_4 v \varepsilon_1.$$

Proof. Let $\frac{\partial \gamma}{\partial t}$ be inextensible. From (3.2), we have

$$\frac{\partial}{\partial t} s(u,t) = \int_0^u \frac{\partial v}{\partial t} du = 0. \tag{3.16}$$

Substituting (3.15) in (3.16), we obtain

$$\frac{\partial \beta_1}{\partial u} = \beta_4 v \varepsilon_1.$$

□

We now restrict ourselves to arc length parametrized curves. That is, $v = 1$ and the local coordinate u corresponds to the curve arclength s. In this case we have the following lemma.

Lemma 3.15. *Let $\frac{\partial \gamma}{\partial t} = \beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2$ be a smooth inextensible flow of a pseudo null curve γ with $\kappa_1 = 1$ in $\mathbb{E}_2^4$. Then we have the following*

$$\frac{\partial T}{\partial t} = \left(\frac{\partial \beta_2}{\partial s} + \beta_3 k_3 + \beta_1\right) N + \left(\beta_2 k_2 + \frac{\partial \beta_3}{\partial s} - \varepsilon_2 k_3 \beta_4\right) B_1 + \left(\frac{\partial \beta_4}{\partial s} - \varepsilon_2 k_2 \beta_3\right) B_2,$$

$$\frac{\partial N}{\partial t} = -\left(\beta_3 k_2 + \varepsilon_1 \frac{\partial \beta_4}{\partial s}\right) T + \psi_2 N + \varepsilon_2 \psi_1 B_1,$$

$$\frac{\partial B_1}{\partial t} = \left(\beta_2 k_2 + \frac{\partial \beta_3}{\partial s} - \beta_4 \varepsilon_2 k_3\right) T + \psi_3 N - \psi_1 B_2,$$

$$\frac{\partial B_2}{\partial t} = -\varepsilon_1 \left(\frac{\partial \beta_2}{\partial s} + \beta_3 k_3 + \beta_1\right) T - \varepsilon_2 \psi_3 B_1 - \psi_2 B_2$$

where $\psi_1 = g(\frac{\partial N}{\partial t}, B_1)$, $\psi_2 = g(\frac{\partial N}{\partial t}, B_2)$ and $\psi_3 = g(\frac{\partial B_1}{\partial t}, B_2)$.

Theorem 3.16. *Let* $\frac{\partial \gamma}{\partial t} = \beta_1 T + \beta_2 N + \beta_3 B_1 + \beta_4 B_2$ *be a smooth inextensible flow of a pseudo null curve* γ *with* $\kappa_1 = 1$ *in* $\mathbb{E}^4_2$. *Then the following partial differential equation holds:*

$$\left(\beta_3 k_2 + \varepsilon_1 \frac{\partial \beta_4}{\partial s}\right) = \varepsilon_1 \left(\frac{\partial \beta_4}{\partial s} - \beta_3 \varepsilon_2 k_2\right). \tag{3.17}$$

Proof. From lemma 3.15, we get

$$\frac{\partial}{\partial s}\frac{\partial T}{\partial t} = -\varepsilon_1 \left(\frac{\partial \beta_4}{\partial s} - \beta_3 \varepsilon_2 k_2\right) T$$
$$+ \left(\frac{\partial^2 \beta_2}{\partial s^2} + \frac{\partial \beta_1}{\partial s} + \frac{\partial}{\partial s}(\beta_3 k_3) + k_3 \left(\frac{\partial \beta_3}{\partial s} + \beta_2 k_2 - \beta_4 \varepsilon_2 k_3\right)\right) N$$
$$+ \left(\left(\beta_1 + \frac{\partial \beta_2}{\partial s} + \beta_3 k_3\right) k_2 + \frac{\partial^2 \beta_3}{\partial s^2} + \frac{\partial}{\partial s}(\beta_2 k_2)\right) B_1$$
$$-\varepsilon_2 \left(\frac{\partial}{\partial s}(\beta_4 k_3) + k_3 \left(\frac{\partial \beta_4}{\partial s} - \beta_3 \varepsilon_2 k_2\right)\right) B_1$$
$$+ \left(-\varepsilon_2 k_2 \left(\frac{\partial \beta_3}{\partial s} + \beta_2 k_2 - \beta_4 \varepsilon_2 k_3\right) + \frac{\partial^2 \beta_4}{\partial s^2} - \varepsilon_2 \frac{\partial}{\partial s}(\beta_3 k_2)\right) B_2.$$

On the other hand,

$$\frac{\partial}{\partial t}\frac{\partial T}{\partial s} = \frac{\partial}{\partial t} N = -\left(\beta_3 k_2 + \varepsilon_1 \frac{\partial \beta_4}{\partial s}\right) T + \psi_2 N + \varepsilon_2 \psi_1 B_1.$$

From the equality of the coefficients of T in above equalities, we get

$$\left(\beta_3 k_2 + \varepsilon_1 \frac{\partial \beta_4}{\partial s}\right) = \varepsilon_1 \left(\frac{\partial \beta_4}{\partial s} - \beta_3 \varepsilon_2 k_2\right).$$

□

Corollary 3.17. *In Theorem 3.16, from the equality of the coefficients of* N, B_1 *and* B_2 *respectively, we obtain*

$$\begin{aligned}
\psi_2 &= \frac{\partial^2 \beta_2}{\partial s^2} + \frac{\partial \beta_1}{\partial s} + \frac{\partial}{\partial s}(\beta_3 k_3) + k_3 \frac{\partial \beta_3}{\partial s} \\
&\quad + \beta_2 k_3 k_2 - \varepsilon_2 k_3^2 \beta_4 \\
\varepsilon_2 \psi_1 &= k_2 \frac{\partial \beta_2}{\partial s} + k_2 \beta_1 + k_2 k_3 \beta_3 + \frac{\partial^2 \beta_3}{\partial s^2} + \frac{\partial}{\partial s}(\beta_2 k_2) \\
&\quad - \varepsilon_2 \frac{\partial}{\partial s}(\beta_4 k_3) - \varepsilon_2 k_3 \left(\frac{\partial \beta_4}{\partial s} - \varepsilon_2 \beta_3 k_2\right) \\
0 &= -\varepsilon_2 k_2 \left(\frac{\partial \beta_3}{\partial s} + \beta_2 k_2 - \varepsilon_2 k_3 \beta_4\right) + \frac{\partial^2 \beta_4}{\partial s^2} - \varepsilon_2 \frac{\partial}{\partial s}(\beta_3 k_2)
\end{aligned}$$

Theorem 3.18. *Let $\frac{\partial\gamma}{\partial t}=\beta_1 T+\beta_2 N+\beta_3 B_1+\beta_4 B_2$ be a smooth inextensible flow of a pseudo null curve γ with $\kappa_1=1$ in $\mathbb{E}_2^4$. Then the following partial differential equation holds:*

$$k_2\frac{\partial\beta_3}{\partial t}+k_2^2\beta_2-\varepsilon_2 k_2 k_3\beta_4=-\frac{\partial}{\partial s}\left(\beta_3 k_2\right)-\varepsilon_1\frac{\partial^2\beta_2}{\partial s^2}. \tag{3.18}$$

Proof. From lemma 3.15, we get

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial N}{\partial t} &= -\left(\frac{\partial}{\partial s}\left(\beta_3 k_2\right)+\varepsilon_1\frac{\partial^2\beta_2}{\partial s^2}\right)T\\ &\quad+\left(\varepsilon_2 k_3\psi_1-\varepsilon_1\frac{\partial\beta_4}{\partial s}-\beta_3 k_2+\frac{\partial\psi_2}{\partial s}\right)N\\ &\quad+\left(\psi_2 k_2+\varepsilon_2\frac{\partial\psi_1}{\partial s}\right)B_1-\psi_1 k_2 B_2\end{aligned}$$

On the other hand,

$$\frac{\partial}{\partial t}\frac{\partial N}{\partial s}=k_2\left(\frac{\partial\beta_3}{\partial t}+k_2\beta_2-\varepsilon_2 k_3\beta_4\right)T+k_2\psi_3 N+\frac{\partial k_2}{\partial t}B_1-k_2\psi_1 B_2.$$

From the equality of the coefficients of T in above equalities, we get

$$k_2\frac{\partial\beta_3}{\partial t}+k_2^2\beta_2-\varepsilon_2 k_2 k_3\beta_4=-\frac{\partial}{\partial s}\left(\beta_3 k_2\right)-\varepsilon_1\frac{\partial^2\beta_2}{\partial s^2}.$$

□

Corollary 3.19. *In Theorem 3.18, from the equality of the coefficients of N and B_1 respectively, we obtain*

$$\begin{aligned}k_2\psi_3 &= -\varepsilon_1\frac{\partial\beta_4}{\partial s}-\beta_3 k_2+\frac{\partial\psi_2}{\partial s}+\varepsilon_2 k_3\psi_1,\\ \frac{\partial k_2}{\partial t} &= \varepsilon_2\frac{\partial\psi_1}{\partial s}+k_2\psi_2.\end{aligned}$$

Theorem 3.20. *Let $\frac{\partial\gamma}{\partial t}=\beta_1 T+\beta_2 N+\beta_3 B_1+\beta_4 B_2$ be a smooth inextensible flow of a pseudo null curve γ with $\kappa_1=1$ in $\mathbb{E}_2^4$. Then the following differential equation holds:*

$$-2k_2k_3\beta_3-\varepsilon_1 k_3\frac{\partial\beta_4}{\partial s}-k_2\frac{\partial\beta_2}{\partial s}-k_2\beta_1=\frac{\partial^2\beta_3}{\partial s^2}+\frac{\partial}{\partial s}\left(k_2\beta_2\right)-\varepsilon_2\frac{\partial}{\partial s}\left(k_3\beta_4\right)+\varepsilon_1\psi_1.$$

Proof. From lemma 3.15, we get

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial B_1}{\partial t} &= \left(\frac{\partial^2\beta_3}{\partial s^2}+\frac{\partial}{\partial s}\left(k_2\beta_2\right)-\varepsilon_2\frac{\partial}{\partial s}\left(k_3\beta_4\right)+\varepsilon_1\psi_1\right)T\\ &\quad+\left(\frac{\partial\beta_3}{\partial s}+k_2\beta_2-\varepsilon_2 k_3\beta_4+\frac{\partial\psi_3}{\partial s}\right)N\\ &\quad+\left(\psi_3 k_2+\varepsilon_2 k_3\psi_1\right)B_1-\frac{\partial\psi_1}{\partial s}B_2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial B_1}{\partial s} &= \left(-2k_2k_3\beta_3-\varepsilon_1k_3\frac{\partial\beta_4}{\partial s}-k_2\frac{\partial\beta_2}{\partial s}-k_2\beta_1\right)T\\&+\left(\frac{\partial k_3}{\partial t}+k_3\psi_2\right)N+\left(\varepsilon_2k_3\psi_1+k_2\psi_3\right)B_1\\&+\left(-\varepsilon_2\frac{\partial k_2}{\partial t}+\varepsilon_2k_2\psi_2\right)B_2.\end{aligned}$$

From the equality of the coefficients of T in above equalities, we get

$$-2k_2k_3\beta_3-\varepsilon_1k_3\frac{\partial\beta_4}{\partial s}-k_2\frac{\partial\beta_2}{\partial s}-k_2\beta_1=\frac{\partial^2\beta_3}{\partial s^2}+\frac{\partial}{\partial s}\left(k_2\beta_2\right)-\varepsilon_2\frac{\partial}{\partial s}\left(k_3\beta_4\right)+\varepsilon_1\psi_1.$$

□

Corollary 3.21. *In Theorem 3.20, from the equality of the coefficients of N and B_2, respectively, we obtain*

$$\begin{aligned}\frac{\partial k_3}{\partial t}+k_3\psi_2 &= \frac{\partial\beta_3}{\partial s}+k_2\beta_2-\varepsilon_2k_3\beta_4+\frac{\partial\psi_3}{\partial s},\\ \frac{\partial\psi_1}{\partial s} &= \varepsilon_2\frac{\partial k_2}{\partial t}-\varepsilon_2k_2\psi_2.\end{aligned}$$

Theorem 3.22. *Let $\frac{\partial\gamma}{\partial t}=\beta_1T+\beta_2N+\beta_3B_1+\beta_4B_2$ be a smooth inextensible flow of a pseudo null curve γ with $\kappa_1=1$ in $\mathbb{E}^4_2$. Then the following differential equation holds:*

$$-\varepsilon_2k_3\left(\frac{\partial\beta_3}{\partial s}+k_2\beta_2-\varepsilon_2k_3\beta_4\right)=-\varepsilon_1\frac{\partial^2\beta_2}{\partial s^2}-\varepsilon_1\frac{\partial\beta_1}{\partial s}-\varepsilon_1\frac{\partial}{\partial s}\left(k_3\beta_3\right)+\varepsilon_1\psi_2.$$

Proof. From lemma 3.15, we get

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial B_2}{\partial t} &= \left(-\varepsilon_1\frac{\partial^2\beta_2}{\partial s^2}-\varepsilon_1\frac{\partial\beta_1}{\partial s}-\varepsilon_1\frac{\partial}{\partial s}\left(k_3\beta_3\right)+\varepsilon_1\psi_2\right)T\\&-\left(\varepsilon_1\left(\frac{\partial\beta_2}{\partial s}+\beta_1+k_3\beta_3\right)+\varepsilon_2k_3\psi_3\right)N\\&-\left(\varepsilon_2\frac{\partial\psi_3}{\partial s}-\varepsilon_2k_3\psi_2\right)B_1-\left(-k_2\psi_3+\frac{\partial\psi_2}{\partial s}\right)B_2\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial B_2}{\partial s} &= -\varepsilon_2k_3\left(\frac{\partial\beta_3}{\partial s}+k_2\beta_2-\varepsilon_2k_3\beta_4\right)T\\&+\left(-\varepsilon_1\frac{\partial\beta_2}{\partial s}-\varepsilon_1\beta_1-\varepsilon_1k_3\beta_3-\varepsilon_2k_3\psi_3\right)N\\&+\left(-\varepsilon_2\frac{\partial k_3}{\partial t}-\varepsilon_1\frac{\partial\beta_3}{\partial s}-\varepsilon_1k_2\beta_2-k_3\beta_4\right)B_1\\&+\left(-\varepsilon_1\frac{\partial\beta_4}{\partial s}-k_2\beta_3+\varepsilon_2k_3\psi_1\right)B_2.\end{aligned}$$

From the equality of the coefficients of T in above equalities, we get

$$-\varepsilon_2 k_3 \left(\frac{\partial \beta_3}{\partial s} + k_2\beta_2 - \varepsilon_2 k_3 \beta_4 \right) = -\varepsilon_1 \frac{\partial^2 \beta_2}{\partial s^2} - \varepsilon_1 \frac{\partial \beta_1}{\partial s} - \varepsilon_1 \frac{\partial}{\partial s}(k_3\beta_3) + \varepsilon_1 \psi_2.$$

□

Corollary 3.23. *In Theorem 3.22, from the equality of the coefficients of B_1 and B_2, respectively, we obtain*

$$\begin{aligned} -\varepsilon_2 \frac{\partial k_3}{\partial t} - \varepsilon_1 \frac{\partial \beta_3}{\partial s} - \varepsilon_1 k_2 \beta_2 - k_3\beta_4 &= -\varepsilon_2 \frac{\partial \psi_3}{\partial s} + \varepsilon_2 k_3 \psi_2, \\ -\varepsilon_1 \frac{\partial \beta_4}{\partial s} - k_2\beta_3 + \varepsilon_2 k_3 \psi_1 &= k_2\psi_3 - \frac{\partial \psi_2}{\partial s}. \end{aligned}$$

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