ON K-CONTACT EINSTEIN MANIFOLDS

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Abstract. The object of the present paper is to characterize K-contact Einstein manifolds satisfying the curvature condition $R \cdot C = Q(S, C)$, where C is the conformal curvature tensor and R the Riemannian curvature tensor. Next we study K-contact Einstein manifolds satisfying the curvature conditions $C \cdot S = 0$ and $S \cdot C = 0$, where S is the Ricci tensor. Finally, we consider K-contact Einstein manifolds satisfying the curvature condition $Z \cdot C = 0$, where Z is the concircular curvature tensor.

1. Introduction

An emerging branch of modern mathematics is the geometry of contact manifolds. The notion of contact geometry has evolved from the mathematical formalism of classical mechanics [8]. Two important classes of contact manifolds are K-contact manifolds and Sasakian manifolds ([1], [14]). K-contact and Sasakian manifolds have been studied by several authors such as De and Biswas [5], De and De [6], Tarafdar and De [17], Tanno [16], Yildiz et al [21], Prasad et al [13] and many others. It is clear that every Sasakian manifold is a K-contact manifold but the converse is not always true, as it is shown in the three dimensional case [10]. Also a compact K-contact Einstein manifold is Sasakian [3].

The nature of a manifold mostly depends on its curvature tensor. Using the tools of conformal transformation geometers have deduced conformal curvature tensor. Apart from conformal curvature tensor, the concircular curvature tensor is another important tensor from the differential geometric point of view.

Let M be an $n(= 2m + 1)$-dimensional differentiable manifold. Suppose that (ϕ, ξ, η, g) is an almost contact metric structure on M. This means that (ϕ, ξ, η, g) is a quadruple consisting of a $(1, 1)$-tensor field ϕ, an associated vector field ξ, a 1-form η and a Riemannian metric g on M satisfying the following relations

$$(1.1) \quad \phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

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where X, Y are smooth vector fields on M. In the addition we have

(1.2) $\phi\xi = 0,\ \eta(\phi X) = 0,\ g(X, \xi) = \eta(X),\ g(\phi X, Y) = -g(X, \phi Y).$

An almost contact structure is said to be a contact structure if $g(X, \phi Y) = d\eta(X, Y)$. A contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M^n \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^n \times \mathbb{R}$. A normal contact metric manifold is called a Sasakian manifold. If ξ is a Killing vector field on a contact metric manifold (M^n, g) then the manifold is called a K-contact metric manifold or simply a K-contact manifold ([1], [14]). An almost contact manifold is Sasakian [1], if and only if

(1.3) $(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X,$

where ∇ is the Levi-Civita connection. It is well known that K-contact manifold is Sasakian if and only if

(1.4) $R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$

for any vector fields X, Y on (M^n, g), where R is the Riemannian curvature tensor of type $(1, 3)$ defined by

$$R(X, Y)W = \nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X,Y]}W.$$

A complete regular contact metric manifold M^{2m+1} carries a K-contact structure (ϕ, ξ, η, g), defined in terms of the almost Käehler structure (J, G) of the base manifold M^{2m}. Here the K-contact structure (ϕ, ξ, η, g) is Sasakian if and only if the base manifold (M^{2m}, J, G) is Kaehlerian. If (M^{2m}, J, G) is only almost Käehler, then (ϕ, ξ, η, g) is only K-contact [1]. In a Sasakian manifold the Ricci operator Q commutes with ϕ, that is, $Q\phi = \phi Q$. Recently in [11], it has been shown that there exists K-contact manifold with $Q\phi = \phi Q$ which are not Sasakian. It is seen that K-contact structure is the intermediate between contact and Sasakian structure.

A Riemannian manifold (M, g) is called locally symmetric if its curvature tensor R is parallel, that is, $\nabla R = 0$. The notion of semisymmetric, a proper generalization of locally symmetric manifold, is defined by $R(X, Y) \cdot R = 0$, where $R(X, Y)$ acts on R as a derivation. A complete intrinsic classification of these manifolds was given by Szabó in [15]. In contact geometry Tanno [16] showed that a semisymmetric K-contact manifold M is locally isometric to the unit sphere $S^n(1)$. In [4], Chaki and Tarafdar studied a Sasakian manifold satisfying the condition $R(X, Y) \cdot C = 0$, where C denotes the Weyl conformal curvature tensor defined by

$$
\begin{aligned}
C(X, Y)W &= R(X, Y)W - \frac{1}{n-2}\{S(Y, W)X - S(X, W)Y + g(Y, W)QX \\
&\quad -g(X, W)QY\} + \frac{r}{(n-1)(n-2)}\{g(Y, W)X - g(X, W)Y\},
\end{aligned}
$$

(1.5)

S is the Ricci tensor of type $(0,2)$, Q is the Ricci operator defined by

$$S(X,Y) = g(QX,Y)$$

and r is the scalar curvature of M. Generalizing the result of Chaki and Tarafdar [4], Guha and De [9] proved that if a K-contact manifold with characteristic vector field ξ belonging to the k-nullity distribution satisfies the condition $R(\xi,X) \cdot C = 0$, then $C(\xi,X)Y = 0$ for any vector fields X,Y. In [7], De and Ghosh studied contact manifold satisfying the condition $R(\xi,X) \cdot R = 0$, where ξ belongs to the k-nullity distribution.

For a $(0,k)$-tensor field T, $k \geq 1$, on (M^n, g) we define the tensors $R \cdot T$ and $Q(S,T)$ by

$$
\begin{aligned}
(R(X,Y) \cdot T)(X_1, X_2, ..., X_k) \;=\; & -T(R(X,Y)X_1, X_2, ..., X_k) \\
& -T(X_1, R(X,Y)X_2, ..., X_k) \\
& -\cdots - T(X_1, X_2, ..., R(X,Y)X_k)
\end{aligned}
$$

(1.6)

and

$$
\begin{aligned}
Q(S,T)(X_1, X_2, ..., X_k) \;=\; & -T((X \wedge_S Y)X_1, X_2, ..., X_k) \\
& -T(X_1, (X \wedge_S Y)X_2, ..., X_k) \\
& -\cdots - T(X_1, X_2, ..., (X \wedge_S Y)X_k),
\end{aligned}
$$

(1.7)

respectively [18].
Furthermore we define endomorphism $X \wedge_A Y$ by

$$(1.8) \qquad (X \wedge_A Y)W = A(Y,W)X - A(X,W)Y,$$

where $X, Y, W \in \chi(M)$, $\chi(M)$ is the Lie algebra of vector fields on M and A is a symmetric $(0,2)$-tensor. In the present paper our aim is to investigate under what conditions a K-contact manifold will be a Sasakian manifold. This paper is organized as follows:
After preliminaries in section 2, we consider a K-contact Einstein manifold satisfying the curvature condition $R \cdot C = Q(S,C)$ and in this case we have shown that the manifold becomes a Sasakian manifold. In section 4, we prove that in a K-contact Einstein manifold the curvature condition $C \cdot S = 0$ holds. In section 5, we discuss K-contact Einstein manifold satisfying the curvature condition $S \cdot C = 0$ and prove that under this condition a K-contact Einstein manifold is Sasakian. Finally, in section 6, we have shown that if a K-contact Einstein manifold satisfies the curvature condition $Z \cdot C = 0$, then either the manifold is Sasakian, or, the scalar curvature of the manifold is constant.

2. Preliminaries

Besides the above results in an $n(= 2m + 1)$-dimensional K-contact manifold, the following results hold ([1], [20])

$$(2.1) \qquad \nabla_X \xi = -\phi X,$$

(2.2) $g(R(\xi, X)Y, \xi) = \eta(R(\xi, X)Y) = g(X, Y) - \eta(X)\eta(Y),$

(2.3) $R(\xi, X)\xi = -X + \eta(X)\xi,$

(2.4) $S(X, \xi) = (n - 1)\eta(X),$

(2.5) $(\nabla_X \phi)Y = R(\xi, X)Y,$

for any vector fields $X, Y \in \chi(M)$.

A K-contact manifold M of dimension > 3 is said to be Einstein if its Ricci tensor S is of the form $S = \lambda g$, where λ is a constant.

In this case we have

(2.6) $S(X, Y) = \lambda g(X, Y).$

Substituting $X = Y = \xi$ in (2.6) and then using (2.4) and (1.1), we get

(2.7) $\lambda = (n - 1).$

Thus using (2.7) we obtain from (2.6)

(2.8) $S(X, Y) = (n - 1)g(X, Y).$

Again we have from (2.8)

(2.9) $QX = (n - 1)X.$

Using (2.8), (2.9) in (1.5), we get

(2.10) $C(X, Y)W = R(X, Y)W - [g(Y, W)X - g(X, W)Y].$

Further using (2.3), (2.4) and (1.1), we have from (2.10)

(2.11) $C(X, Y)\xi = R(X, Y)\xi - \eta(Y)X + \eta(X)Y,$

(2.12) $C(\xi, Y)W = R(\xi, Y)W - [g(Y, W)\xi - \eta(W)Y],$

(2.13) $C(\xi, Y)\xi = 0,$

for all vector fields X, Y and W on M.

3. K-contact Einstein manifolds satisfying $R(\xi, Y) \cdot C = Q(S, C)$

A Riemannian or a semi-Riemannian manifold $(M^n, g), n > 3$, is said to be Ricci generalized pseudosymmetric [22] if and only if the relation

(3.1) $$R \cdot R = fQ(S, R)$$

holds on the set $U_R = \{x \in M : R \neq 0 \text{ at } x\}$, where f is some function on U_R. A very important subclass of this class of manifolds realizing the condition is

(3.2) $$R \cdot R = Q(S, R).$$

Every three dimensional manifold satisfies the equation (3.2) identically. Other examples are the semi-Riemannian manifolds (M, g) admitting a non-zero 1-form ω such that the equality

$$\omega(X)R(Y, Z) + \omega(Y)R(Z, X) + \omega(Z)R(X, Y) = 0$$

holds on M. The condition (3.2) also appears in theory of plane gravitational wave. This section is devoted to study K-contact Einstein manifolds satisfying the curvature condition

$$R(\xi, Y) \cdot C = Q(S, C),$$

that is,

$$(R(\xi, Y) \cdot C)(U, V)W = Q(S, C)(U, V)W,$$

for all Y, U, V and $W \in \chi(M)$.
The above equation implies

$$
\begin{aligned}
&R(\xi, Y)C(U, V)W - C(R(\xi, Y)U, V)W \\
&-C(U, R(\xi, Y)V)W - C(U, V)R(\xi, Y)W \\
&= (\xi \wedge_S Y)C(U, V)W - C((\xi \wedge_S Y)U, V)W
\end{aligned}
$$

(3.3) $$-C(U, (\xi \wedge_S Y)V)W - C(U, V)(\xi \wedge_S Y)W.$$

Making use of (1.8) and (2.8) we obtain from (3.3)

$$
\begin{aligned}
&R(\xi, Y)C(U, V)W - C(R(\xi, Y)U, V)W \\
&-C(U, R(\xi, Y)V)W - C(U, V)R(\xi, Y)W \\
&= S(Y, C(U, V)W)\xi - (n-1)\eta(C(U, V)W)Y \\
&-S(Y, U)C(\xi, V)W + (n-1)\eta(U)C(Y, V)W \\
&-S(Y, V)C(U, \xi)W + (n-1)\eta(V)C(U, Y)W
\end{aligned}
$$

(3.4) $$-S(Y, W)C(U, V)\xi + (n-1)\eta(W)C(U, V)Y.$$

Substituting $U = W = \xi$ in (3.4) and using (2.3) and (2.13) yields

(3.5) $$C(Y, V)\xi + C(\xi, V)Y = (n-1)C(Y, V)\xi + (n-1)C(\xi, V)Y.$$

Hence for $n > 3$, it follows from (3.5) that

$$(3.6) \qquad C(Y, V)\xi + C(\xi, V)Y = 0.$$

With the help of (2.11) and (2.12) we obtain from (3.6)

$$(3.7) \qquad R(Y, V)\xi + R(\xi, V)Y - g(V, Y)\xi + 2\eta(Y)V - \eta(V)Y = 0.$$

Interchanging Y and V in (3.7) yields

$$(3.8) \qquad R(V, Y)\xi + R(\xi, Y)V - g(Y, V)\xi + 2\eta(V)Y - \eta(Y)V = 0.$$

Subtracting (3.8) from (3.7) and using the Bianchi's 1st identity we have

$$R(Y, V)\xi = \eta(V)Y - \eta(Y)V,$$

which implies that the manifold is Sasakian.
This leads to the following:

Theorem 3.1. *Let (M, g) be an $n(> 3)$-dimensional K-contact Einstein manifold satisfying the curvature condition $R(\xi, Y) \cdot C = Q(S, C)$. Then the manifold is a Sasakian manifold.*

4. K-contact Einstein manifolds

In this section we consider a K-contact Einstein manifold.
From (2.10) we have

$$(4.1) \qquad C(X, Y)U = R(X, Y)U - [g(Y, U)X - g(X, U)Y].$$

Taking inner product of (4.1) with V yields

$$(4.2) \quad g(C(X, Y)U, V) = g(R(X, Y)U, V) - [g(Y, U)g(X, V) - g(X, U)g(Y, V)].$$

Interchanging U and V in (4.2) we have

$$(4.3) \quad g(C(X, Y)V, U) = g(R(X, Y)V, U) - [g(Y, V)g(X, U) - g(X, V)g(Y, U)].$$

Adding (4.2) and (4.3) we get

$$(4.4) \qquad g(C(X, Y)U, V) + g(C(X, Y)V, U) = 0.$$

Now we have

$$(4.5) \qquad (C(X, Y) \cdot S)(U, V) = -S(C(X, Y)U, V) - S(U, C(X, Y)V).$$

Using (2.8) we obtain from (4.5)

$$(4.6) \quad (C(X, Y) \cdot S)(U, V) = -(n - 1)[g(C(X, Y)U, V) + g(C(X, Y)V, U)]$$

From (4.4) and (4.6) it follows that

$$C(X, Y) \cdot S = 0.$$

By the above discussions we have the following:

Theorem 4.1. *Let (M, g) be an $n(= 2m + 1)$-dimensional K-contact Einstein manifold. Then the curvature condition $C(X, Y) \cdot S = 0$ holds on M.*

5. *K*-contact Einstein manifolds satisfying $S \cdot C = 0$

In this section we investigate K-contact Einstein manifolds satisfying the curvature condition $S \cdot C = 0$, where S is the Ricci tensor of type $(0, 2)$. Let the manifold (M, g) satisfies the condition

$$(S(X, Y) \cdot C)(U, V)W = 0,$$

where X, Y, U, V and $W \in \chi(M)$. The above equation implies

$$(X \wedge_S Y)C(U, V)W + C((X \wedge_S Y)U, V)W$$
(5.1)
$$+C(U, (X \wedge_S Y)V)W + C(U, V)(X \wedge_S Y)W = 0.$$

Using (1.8) and (5.1) we obtain

$$S(Y, C(U, V)W)X - S(X, C(U, V)W)Y + S(Y, U)C(X, V)W$$
$$-S(X, U)C(Y, V)W + S(Y, V)C(U, X)W - S(X, V)C(U, Y)W$$
(5.2)
$$+S(Y, W)C(U, V)X - S(X, W)C(U, V)Y = 0.$$

Putting $X = \xi$ in (5.2) yields

$$S(Y, C(U, V)W)\xi - S(\xi, C(U, V)W)Y + S(Y, U)C(\xi, V)W$$
$$-S(\xi, U)C(Y, V)W + S(Y, V)C(U, \xi)W - S(\xi, V)C(U, Y)W$$
(5.3)
$$+S(Y, W)C(U, V)\xi - S(\xi, W)C(U, V)Y = 0.$$

Next substituting $U = W = \xi$ in (5.3) and then using (2.4) and (2.13) we have

(5.4)
$$(n - 1)[C(Y, V)\xi + C(\xi, V)Y] = 0.$$

For $n > 3$, the above equation implies

(5.5)
$$C(Y, V)\xi + C(\xi, V)Y = 0.$$

Now computing in the same way as in Theorem 3.1 we obtain

$$R(Y, V)\xi = \eta(V)Y - \eta(Y)V,$$

which implies the manifold is a Sasakian manifold.
Thus we have the following:

Theorem 5.1. *If a K-contact Einstein manifold $(M^n, g), (n = 2m + 1 > 3)$ satisfies the curvature condition $S \cdot C = 0$, then the manifold is a Sasakian manifold.*

6. *K*-contact Einstein manifolds satisfying $Z \cdot C = 0$

A transformation which transform every geodesic circle of an n-dimensional Riemannian manifold M into a geodesic circle is called a concircular transformation ([19],[12]). A concircular transformation is always a conformal transformation [12]. Here, a geodesic circle means a curve in M whose first curvature

is constant and whose second curvature is identically zero. Thus the geometry of concircular transformation is the generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism [2]. An important invariant of a concircular transformation is the concircular curvature tensor Z, defined by [19]

$$(6.1) \qquad Z(X,Y)W = R(X,Y)W - \frac{r}{n(n-1)}[g(Y,W)X - g(X,W)Y],$$

where $X, Y, W \in \chi(M)$. A Riemannian manifold with vanishing concircular curvature tensor is of constant curvature. Thus, the concircular curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature. Now in this stage we consider K-contact Einstein manifold satisfying the curvature condition

$$(6.2) \qquad\qquad\qquad Z(X,Y) \cdot C = 0,$$

where Z and C defined in (6.1) and (2.10) respectively.
Substituting $X = \xi$ in (6.2) we have

$$(6.3) \qquad\qquad\qquad (Z(\xi,Y) \cdot C)(U,V)W = 0.$$

This implies

$$
\begin{aligned}
&Z(\xi,Y)C(U,V)W - C(Z(\xi,Y)U,V)W \\
(6.4)\qquad &-C(U,Z(\xi,Y)V)W - C(U,V)Z(\xi,Y)W = 0.
\end{aligned}
$$

Again putting $U = W = \xi$ in (6.4) and using (2.13) yields

$$(6.5) \qquad -C(Z(\xi,Y)\xi,V)\xi - C(\xi,Z(\xi,Y)V)\xi - C(\xi,V)Z(\xi,Y)\xi = 0.$$

With the help of (6.1), (2.3) and (1.2) we obtain from (6.5)

$$(6.6)\ \ (1-\frac{r}{n(n-1)})[C(Y,V)\xi - \eta(Y)C(\xi,V)\xi + C(\xi,V)Y - \eta(Y)C(\xi,V)\xi] = 0.$$

Applying (2.13) in (6.6) gives

$$(1 - \frac{r}{n(n-1)})[C(Y,V)\xi + C(\xi,V)Y] = 0.$$

Then either $r = n(n-1)$, or,

$$C(Y,V)\xi + C(\xi,V)Y = 0.$$

Now computing in the same way as in Theorem 3.1 we have

$$R(Y,V)\xi = \eta(V)Y - \eta(Y)V,$$

which implies that the manifold is a Sasakian manifold.
By the above discussions we have the following:

Theorem 6.1. *Let M be an n-dimensional K-contact Einstein manifold satisfying the curvature condition $Z(\xi,Y) \cdot C = 0$. Then either the manifold is Sasakian, or, the scalar curvature of the manifold is constant.*

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