

EXISTENCE AND NON-EXISTENCE OF POSITIVE SOLUTIONS OF FOUR-POINT BVPs FOR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS ON WHOLE LINE¹

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Abstract. This paper is concerned with four-point boundary value problems of second order singular differential equations on whole lines. The Green's function $G(t, s)$ for the problem

$$-(\rho(t)x'(t))' = 0, \lim_{t \rightarrow -\infty} \rho(t)x'(t) - kx(\xi) = \lim_{t \rightarrow +\infty} \rho(t)x'(t) + lx(\eta) = 0$$

is obtained. We proved that $G(t, s) \geq 0$ under some assumptions which actually generalizes a corresponding result in [Appl. Math. Comput. 217(2)(2010) 811-819]. Sufficient conditions to guarantee the existence and non-existence of positive solutions are established.

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1. Introduction

The investigation of nonlocal boundary value problems for ordinary differential equations was initiated by Il'in and Moiseev [14]. Since then, more general nonlocal boundary value problems (BVPs) were studied by several authors, see the text books [1, 11, 13], the paper [21], and the survey papers [16, 17] and the references cited there. However, the study of existence of positive solutions of nonlocal boundary value problems for differential equations on whole real lines does not seem to be sufficiently developed [2, 3, 4, 5, 6, 20] and the references therein.

In recent years, the existence of solutions of boundary value problems of the differential equations governed by nonlinear differential operators has been studied by many authors, see [5, 7, 8, 9, 10, 15, 18, 19] and the references therein.

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In [12], Deren and Hamal investigated the existence and multiplicity of nonnegative solutions for the following integral boundary-value problem on the whole line

$$(1.1) \quad \begin{cases} (p(t)x'(t))' + \lambda q(t)f(t, x(t), x'(t)) = 0, & a.e., t \in \mathbf{R}, \\ a_1 \lim_{t \rightarrow -\infty} x(t) - b_1 \lim_{t \rightarrow -\infty} p(t)x'(t) = \int_{-\infty}^{+\infty} g_1(s, x(s), x'(s))\psi(s)ds, \\ a_2 \lim_{t \rightarrow +\infty} x(t) + b_2 \lim_{t \rightarrow +\infty} p(t)x'(t) = \int_{-\infty}^{+\infty} g_2(s, x(s), x'(s))\psi(s)ds, \end{cases}$$

where $\lambda > 0$ is a parameter, $f, g_1, g_2 \in C(\mathbf{R}, \times [0, \infty) \times \mathbf{R}, [0, \infty))$, $q, \psi \in C(\mathbf{R}, (0, \infty))$ and $p \in C(\mathbf{R}, (0, \infty)) \cap C^1(\mathbf{R})$. Here, the values of

$$\int_{-\infty}^{+\infty} g_i(s, x(s), x'(s))ds, (i = 1, 2), \int_{-\infty}^{+\infty} \frac{ds}{p(s)} < +\infty$$

and $\sup_{s \in \mathbf{R}} \psi(s)$ are finite and $a_1 + a_2 > 0$, $b_i > 0$ ($i = 1, 2$) satisfying $D = a_2 b_1 + a_1 b_2 + a_1 a_2 \int_{-\infty}^{+\infty} \frac{ds}{p(s)} > 0$.

Motivated by the mentioned papers, we consider the following four-point boundary value problem for second order singular differential equation on the whole line

$$(1.2) \quad \begin{cases} [\rho(t)x'(t)]' + f(t, x(t), x'(t)) = 0, & a.e., t \in \mathbf{R}, \\ \lim_{t \rightarrow -\infty} \rho(t)x'(t) - kx(\xi) = 0, \\ \lim_{t \rightarrow +\infty} \rho(t)x'(t) + lx(\eta) = 0 \end{cases}$$

where

- (a) $-\infty < \xi < \eta < +\infty$, $k, l \geq 0$ are constants,
- (b) f is a nonnegative Carathéodory function, see Definition 2.3,
- (c) $\rho \in C^0(\mathbf{R}, [0, \infty))$ with $\rho(t) > 0$ for all $t \neq 0$ satisfying

$$(1.3) \quad 1 - k \int_{-\infty}^{\xi} \frac{du}{\rho(u)} \geq 0,$$

$$(1.4) \quad 1 - l \int_{\eta}^{+\infty} \frac{du}{\rho(u)} \geq 0,$$

$$(1.5) \quad \Delta = k + l + kl \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} > 0.$$

The purpose is to establish sufficient conditions for the existence and non-existence of positive solutions of BVP(1.2). Our results and methods are different from those in [2, 3, 4, 5, 7, 8, 20].

The main features of our paper are as follows. Firstly, compared with [12], we establish the existence results of solutions of second order singular differential equation on the whole line. Secondly, we investigate the existence

of positive solutions by a different method and imposing growth conditions on f . Thirdly, compared with [15], we consider the case $\int_{-\infty}^{+\infty} \frac{1}{\rho(s)} ds < +\infty$ in this paper while $\int_{-\infty}^{+\infty} \frac{1}{\rho(s)} ds = +\infty$ was considered in [15]. Finally, the Green's function $G(t, s)$ for the problem $-(\rho(t)x'(t))' = 0$, $\lim_{t \rightarrow -\infty} \rho(t)x'(t) - kx(\xi) = 0$, $\lim_{t \rightarrow +\infty} \rho(t)x'(t) - lx(\eta) = 0$ is obtained. We proved that $G(t, s) \geq 0$ under some assumptions, which generalizes a corresponding result in ([22]Appl. Math. Comput. 217(2)(2010) 811-819), see Remark 2.1.

The remainder of this paper is organized as follows: the preliminary results are given in Section 2, the existence result of positive solutions of BVP(1.2) is proved in Section 3. Finally the non-existence results on positive solutions of BVP(1.2) are presented in Section 4.

2. Preliminary Results

In this section, we present some background definitions in Banach spaces see [11] and state an important fixed point theorem see Theorem 2.2.11 in [13]. The preliminary results are given too.

Definition 2.1. Let X be a real Banach space. The nonempty convex closed subset P of X is called a cone in X if $ax \in P$ for all $x \in P$ and $a \geq 0$ and $x \in X$ and $-x \in X$ imply $x = 0$.

Definition 2.2. An operator $T : X \mapsto X$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Definition 2.3. f is called a Carathéodory function when

- (i) $t \mapsto f\left(t, x, \frac{1}{\rho(t)}y\right)$ is defined almost everywhere on \mathbf{R} and is measurable on \mathbf{R} for any $x, y \in \mathbf{R}$,
- (ii) $(x, y) \mapsto f\left(t, x, \frac{1}{\rho(t)}y\right)$ is uniformly continuous on \mathbf{R}^2 for almost every $t \in \mathbf{R}$, i.e., for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| f\left(t, x_1, \frac{1}{\rho(t)}y_1\right) - f\left(t, x_2, \frac{1}{\rho(t)}y_2\right) \right| < \epsilon$$

for almost every $t \in \mathbf{R}$, $|x_1 - x_2| < \delta$, $|y_1 - y_2| < \delta$,

- (iii) for each $r > 0$, there exists a nonnegative function $\phi_r \in L^1(\mathbf{R})$ such that $|u|, |v| \leq r$ implies

$$\left| f\left(t, x, \frac{1}{\rho(t)}y\right) \right| \leq \phi_r(t), \text{ a.e. } t \in \mathbf{R}.$$

Lemma 2.1 ([11, 13]). *Let X be a real Banach space, P a cone of X , and let Ω_1, Ω_2 be two nonempty bounded open sets of P with $0 \in \Omega_1 \subseteq \overline{\Omega_1} \subseteq \Omega_2$. Suppose that $T : \overline{\Omega_2} \rightarrow P$ is a completely continuous operator, and*

(E1) $Tx \neq \lambda x$ for all $\lambda \in [0, 1)$ and $x \in \partial\Omega_1$, $Tx \neq \lambda x$ for all $\lambda \in (1, +\infty)$ and $x \in \partial\Omega_2$;

or

(E2) $Tx \neq \lambda x$ for all $\lambda \in (1, +\infty)$ and $x \in \partial\Omega_1$, $Tx \neq \lambda x$ for all $\lambda \in [0, 1)$ and $x \in \partial\Omega_2$.

Then T has at least one fixed point $x \in \overline{\Omega_2} \setminus \Omega_1$.

Choose

$$X = \left\{ x : \mathbf{R} \mapsto \mathbf{R} : \begin{array}{l} x \in C^0(\mathbf{R}), \rho x' \in C^0(\mathbf{R}), \lim_{t \rightarrow \pm\infty} x(t) \\ \text{and } \lim_{t \rightarrow \pm\infty} \rho(t)x'(t) \text{ exist and are finite} \end{array} \right\}.$$

For $x \in X$, define

$$\|x\| = \max \left\{ \sup_{t \in \mathbf{R}} |x(t)|, \sup_{t \in \mathbf{R}} \rho(t)|x'(t)| \right\}.$$

Lemma 2.2. X is a Banach space with $\|\cdot\|$ defined above.

Proof. It is easy to see that X is a normed linear space. Let $\{x_u\}$ be a Cauchy sequence in X . Then $\|x_u - x_v\| \rightarrow 0$, $u, v \rightarrow +\infty$. It follows that

$$\lim_{t \rightarrow -\infty} x_u(t), \lim_{t \rightarrow +\infty} x_u(t), \lim_{t \rightarrow -\infty} \rho(t)x'_u(t), \lim_{t \rightarrow +\infty} \rho(t)x'_u(t) \text{ exist,}$$

$$\sup_{t \in \mathbf{R}} |x_u(t) - x_v(t)| \rightarrow 0, u, v \rightarrow +\infty,$$

$$\sup_{t \in \mathbf{R}} \rho(t)|x'_u(t) - x'_v(t)| \rightarrow 0, u, v \rightarrow +\infty.$$

Thus there exist two functions x_0, y_0 defined on \mathbf{R} such that

$$\lim_{u \rightarrow +\infty} x_u(t) = x_0(t), \lim_{u \rightarrow +\infty} \rho(t)x'_u(t) = y_0(t).$$

It follows that

$$\sup_{t \in \mathbf{R}} |x_u(t) - x_0(t)| \rightarrow 0, u \rightarrow +\infty,$$

$$\sup_{t \in \mathbf{R}} |\rho(t)x'_u(t) - y_0(t)| \rightarrow 0, u \rightarrow +\infty.$$

This means that the functions $x_0, y_0 : \mathbf{R} \mapsto \mathbf{R}$ are well defined.

Step 1. Prove that $x_0, y_0 \in C(\mathbf{R})$.

We have for $t_0 \in \mathbf{R}$ that

$$\begin{aligned} |x_0(t) - x_0(t_0)| &\leq |x_0(t) - x_N(t)| + |x_N(t) - x_N(t_0)| + |x_N(t_0) - x_0(t_0)| \\ &\leq 2 \sup_{t \in \mathbf{R}} |x_N(t) - x_0(t)| + |x_N(t) - x_N(t_0)|. \end{aligned}$$

Since $\sup_{t \in \mathbf{R}} |x_u(t) - x_0(t)| \rightarrow 0, u \rightarrow +\infty$ and $x_u(t)$ is continuous on \mathbf{R} , then for any $\epsilon > 0$ we can choose N and $\delta > 0$ such that $\sup_{t \in \mathbf{R}} |x_N(t) - x_0(t)| < \epsilon$ and $|x_N(t) - x_N(t_0)| < \epsilon$ for all $|t - t_0| < \delta$. Thus $|x_0(t) - x_0(t_0)| < 3\epsilon$ for all $|t - t_0| < \delta$. So $x_0 \in C(\mathbf{R})$. Similarly we can prove that $y_0 \in C(\mathbf{R})$.

Step 2. Prove that the limits $\lim_{t \rightarrow -\infty} x_0(t), \lim_{t \rightarrow +\infty} x_0(t), \lim_{t \rightarrow -\infty} y_0(t), \lim_{t \rightarrow +\infty} y_0(t)$ exist.

Suppose that $\lim_{t \rightarrow -\infty} x_u(t) = A_u^-$ is a finite real number. By $\sup_{t \in \mathbf{R}} |x_u(t) - x_v(t)| \rightarrow 0, u, v \rightarrow +\infty$, we know that A_u^- is a Cauchy sequence. Then $\lim_{u \rightarrow +\infty} A_u^-$ exists. By $\sup_{t \in \mathbf{R}} |x_u(t) - x_0(t)| \rightarrow 0, u \rightarrow +\infty$, we get that

$$\lim_{t \rightarrow -\infty} x_0(t) = \lim_{t \rightarrow -\infty} \lim_{u \rightarrow +\infty} x_u(t) = \lim_{u \rightarrow +\infty} \lim_{t \rightarrow -\infty} x_u(t) = \lim_{u \rightarrow +\infty} A_u^-.$$

Hence $\lim_{t \rightarrow -\infty} x_0(t)$ exists. Similarly we can prove that $\lim_{t \rightarrow +\infty} x_0(t), \lim_{t \rightarrow -\infty} y_0(t), \lim_{t \rightarrow +\infty} y_0(t)$ exist.

Step 3. Prove that $y_0(t) = \rho(t)x'_0(t)$.

We have

$$\begin{aligned} & \left| x_u(t) - \lim_{t \rightarrow -\infty} x_u(t) - \int_{-\infty}^t \frac{y_0(s)}{\rho(s)} ds \right| = \left| \int_{-\infty}^t x'_u(s) ds - \int_{-\infty}^t \frac{y_0(s)}{\rho(s)} ds \right| \\ & \leq \int_{-\infty}^t \left| x'_u(s) - \frac{y_0(s)}{\rho(s)} \right| ds \leq \int_{-\infty}^t \frac{1}{\rho(t)} ds \sup_{t \in \mathbf{R}} |\rho(t)x'_u(t) - y_0(t)| \\ & \leq \int_{-\infty}^{+\infty} \frac{1}{\rho(t)} ds \sup_{t \in \mathbf{R}} |\rho(t)x'_u(t) - y_0(t)| \rightarrow 0 \text{ as } u \rightarrow +\infty. \end{aligned}$$

So $\lim_{u \rightarrow +\infty} \left(x_u(t) - \lim_{t \rightarrow -\infty} x_u(t) \right) = \int_{-\infty}^t \frac{y_0(s)}{\rho(s)} ds$. Then $x_0(t) - c_0 = \int_{-\infty}^t \frac{y_0(s)}{\rho(s)} ds$.

Here $c_0 := \lim_{u \rightarrow +\infty} \lim_{t \rightarrow -\infty} x_u(t) = \lim_{u \rightarrow +\infty} A_u^-$. It follows that $\frac{y_0(t)}{\rho(t)} = x'_0(t)$. So $x_0 \in X$ with $x_u \rightarrow x_0$ as $u \rightarrow +\infty$. It follows that X is a Banach space. \square

Lemma 2.3. Let M be a subset of X . Then M is relatively compact if and only if the following conditions are satisfied:

- (i) both $\{x : x \in M\}$ and $\{\rho(t)x' : x \in M\}$ are uniformly bounded,
- (ii) both $\{x : x \in M\}$ and $\{\rho(t)x' : x \in M\}$ are equicontinuous in any subinterval $[a, b]$ in \mathbf{R} ,
- (iii) both $\{x : x \in M\}$ and $\{\rho(t)x' : x \in M\}$ are equi-convergent as $t \rightarrow \pm\infty$.

Proof. " \Leftarrow ". From Lemma 2.2, we know X is a Banach space. In order to prove that the subset M is relatively compact in X , we only need to show M is totally bounded in X , that is for all $\epsilon > 0$, M has a finite ϵ -net.

For any given $\epsilon > 0$, by **(i)** and **(iii)**, there exist constants $A_x, C_x, T > 0, a > 0$, we have

$$|x(t) - A_x| \leq \frac{\epsilon}{3}, |\rho(t)x'(t) - C_x| < \frac{\epsilon}{3}, t \leq -T, x \in M,$$

$$|x(t_1) - x(t_2)| \leq \frac{\epsilon}{3}, |\rho(t_1)x'(t_1) - \rho(t_2)x'(t_2)| < \frac{\epsilon}{3}, t_1, t_2 \geq T, x \in M,$$

$$|x(t_1) - x(t_2)| \leq \frac{\epsilon}{3}, |\rho(t_1)x'(t_1) - \rho(t_2)x'(t_2)| < \frac{\epsilon}{3}, t_1, t_2 \leq -T, x \in M.$$

For $T > 0$, define $X|_{[-T, T]} = \{x : x, \rho(t)x' \in C[-T, T]\}$. For $x \in X|_{[-T, T]}$, define

$$\|x\|_T = \max \left\{ \max_{t \in [-T, T]} |x(t)|, \max_{t \in [-T, T]} \rho(t)|x'(t)| \right\}.$$

Similarly to Lemma 2.2, we can prove that $X|_{[-T, T]}$ is a Banach space.

Let $M|_{[-T, T]} = \{t \mapsto x(t), t \in [-T, T] : x \in M\}$. Then $M|_{[-T, T]}$ is a subset of $X|_{[-T, T]}$. By **(i)** and **(ii)**, and Ascoli-Arzelà theorem, we can know that $M|_{[-T, T]}$ is relatively compact. Thus, there exist $x_1, x_2, \dots, x_k \in M$ such that, for any $x \in M$, we have that there exists some $i = 1, 2, \dots, k$ such that

$$\|x - x_i\|_T = \max \left\{ \sup_{t \in [-T, T]} |x(t) - x_i(t)|, \sup_{t \in [-T, T]} \rho(t)|x'(t) - x'_i(t)| \right\} \leq \frac{\epsilon}{3}.$$

Therefore, for $x \in M$, we have that

$$\|x - x_i\|_X = \max \left\{ \sup_{t \in [-T, T]} |x(t) - x_i(t)|, \sup_{t \in [-T, T]} \rho(t)|x'(t) - x'_i(t)|, \right.$$

$$\left. \sup_{t \geq T} |x(t) - x_i(t)|, \sup_{t \geq T} \rho(t)|x'(t) - x'_i(t)|, \right.$$

$$\left. \sup_{t \leq -T} |x(t) - x_i(t)|, \sup_{t \leq -T} \rho(t)|x'(t) - x'_i(t)| \right\}$$

$$\leq \max \left\{ \frac{\epsilon}{3}, \sup_{t \geq T} |x(t) - x(T)| + |x(T) - x_i(T)| + \sup_{t \geq T} |x_i(T) - x_i(t)|, \right.$$

$$\left. \sup_{t \geq T} |\rho(t)x'(t) - \rho(T)x'(T)| + |\rho(T)x'(T) - \rho(T)x'_i(T)| \right.$$

$$\left. + \sup_{t \geq T} |\rho(T)x'_i(T) - \rho(t)x'_i(t)| \right.$$

$$\left. \sup_{t \leq -T} |x(t) - x(T)| + |x(T) - x_i(T)| + \sup_{t \leq -T} |x_i(T) - x_i(t)|, \right.$$

$$\left. \sup_{t \leq -T} |\rho(t)x'(t) - \rho(T)x'(T)| + |\rho(T)x'(T) - \rho(T)x'_i(T)| \right.$$

$$\left. + \sup_{t \leq -T} |\rho(T)x'_i(T) - \rho(t)x'_i(t)| \right\} \leq \epsilon.$$

So, for any $\epsilon > 0$, M has a finite ϵ -net $\{U_{x_1}, U_{x_2}, \dots, U_{x_k}\}$, that is, M is totally bounded in X . Hence M is relatively compact in X .

(\Rightarrow) Assume that M is relatively compact, then for any $\epsilon > 0$, there exists a finite ϵ -net of M . Let the finite ϵ -net be $\{U_{x_1}, U_{x_2}, \dots, U_{x_k}\}$ with $x_i \subset M$. Then for any $x \in M$, there exists U_{x_i} such that $x \in U_{x_i}$ and

$$|x(t)| \leq |x(t) - x_i(t)| + |x_i(t)| \leq \epsilon + \max \left\{ \sup_{t \in \mathbf{R}} |x_i(t)| : i = 1, 2, \dots, k \right\},$$

$$\rho(t)|x'(t)| \leq \epsilon + \max \left\{ \sup_{t \in \mathbf{R}} \rho(t)|x'_i(t)| : i = 1, 2, \dots, k \right\}.$$

It follows that both M and $\{\rho(t)x' : x \in M\}$ are uniformly bounded. Then (i) holds. Furthermore, there exists $T > 0$ such that $|x_i(t_1) - x_i(t_2)| < \epsilon$ for all $t_1, t_2 \geq T$ and all $t_1, t_2 \leq -T$ and $i = 1, 2, \dots, k$. Then we have for $t_1, t_2 \in R$ that

$$\begin{aligned} |x(t_1) - x(t_2)| &\leq |x(t_1) - x_i(t_1)| + |x_i(t_1) - x_i(t_2)| + |x_i(t_2) - x(t_2)| \\ &\leq 3\epsilon \text{ for all } t_1, t_2 \geq T, t_1, t_2 \leq -T, x \in M. \end{aligned}$$

Similarly we have that

$$|\rho(t_1)x'(t_1) - \rho(t_2)x'(t_2)| \leq 3\epsilon \text{ for all } t_1, t_2 \geq T, t_1, t_2 \leq -T, x \in M.$$

Thus (iii) is valid. Similarly we can prove that (ii) holds. Consequently, Lemma 2.3 is proved. \square

Denote

$$G(t, s) = \frac{1}{\Delta} \left\{ \begin{array}{l} 1 + k \int_{\xi}^t \frac{du}{\rho(u)} + k \int_u^{\xi} \frac{du}{\rho(u)} + kl \int_u^{\xi} \frac{du}{\rho(u)} \int_t^{\eta} \frac{du}{\rho(u)} \\ + l \int_u^{\eta} \frac{du}{\rho(u)} + kl \int_u^{\eta} \frac{du}{\rho(u)} \int_{\xi}^t \frac{du}{\rho(u)} - l \int_u^t \frac{du}{\rho(u)} - k \int_u^t \frac{du}{\rho(u)} \\ - kl \int_u^t \frac{du}{\rho(u)} \int_{\xi}^{\eta} \frac{du}{\rho(u)}, s \leq \min\{\xi, t\}, \\ \\ 1 + k \int_{\xi}^t \frac{du}{\rho(u)}, s \geq \max\{\eta, t\}, \\ \\ 1 + k \int_{\xi}^t \frac{du}{\rho(u)} + k \int_u^{\xi} \frac{du}{\rho(u)} + kl \int_u^{\xi} \frac{du}{\rho(u)} \int_t^{\eta} \frac{du}{\rho(u)} \\ + l \int_u^{\eta} \frac{du}{\rho(u)} + kl \int_u^{\eta} \frac{du}{\rho(u)} \int_{\xi}^t \frac{du}{\rho(u)}, t < s \leq \xi < \eta, \\ \\ 1 + k \int_{\xi}^t \frac{du}{\rho(u)} + l \int_u^{\eta} \frac{du}{\rho(u)} \\ + kl \int_u^{\eta} \frac{du}{\rho(u)} \int_{\xi}^t \frac{du}{\rho(u)}, \min\{t, \xi\} < s \leq \eta, \\ \\ 1 + k \int_{\xi}^t \frac{du}{\rho(u)} + l \int_u^{\eta} \frac{du}{\rho(u)} + kl \int_u^{\eta} \frac{du}{\rho(u)} \int_{\xi}^t \frac{du}{\rho(u)} - l \int_u^t \frac{du}{\rho(u)} \\ - k \int_u^t \frac{du}{\rho(u)} - kl \int_u^t \frac{du}{\rho(u)} \int_{\xi}^{\eta} \frac{du}{\rho(u)}, \xi < s \leq \max\{t, \eta\}, \\ \\ 1 + k \int_{\xi}^t \frac{du}{\rho(u)} - l \int_u^t \frac{du}{\rho(u)} - k \int_u^t \frac{du}{\rho(u)} \\ - kl \int_u^t \frac{du}{\rho(u)} \int_{\xi}^{\eta} \frac{du}{\rho(u)}, \xi < \eta < s \leq t. \end{array} \right.$$

Lemma 2.4. $x \in X$ is a solution of BVP(1.2) if and only if

$$(2.1) \quad x(t) = \int_{-\infty}^{+\infty} G(t, s) f(s, x(s), x'(s)) ds.$$

Proof. Since $x \in X$, f is a Caratheodory function, then

$$\|x\| = \max \left\{ \sup_{t \in \mathbf{R}} |x(t)|, \sup_{t \in \mathbf{R}} \rho(t) |x'(t)| \right\} = r < +\infty,$$

and $\int_{-\infty}^{+\infty} f(r, x(r), x'(r)) dr$ converges. If x is a solution of BVP(1.2), we get from $[\rho(t)x'(t)]' + f(t, x(t), x'(t)) = 0$ that there exist constants $A, B \in \mathbf{R}$ such that

$$(2.2) \quad \begin{aligned} \rho(t)x'(t) &= A - \int_{-\infty}^t f(s, x(s), x'(s)) ds, \\ x(t) &= B + A \int_{-\infty}^t \frac{du}{\rho(u)} - \int_{-\infty}^t \left(\int_s^t \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds. \end{aligned}$$

From $\lim_{t \rightarrow -\infty} \rho(t)x'(t) - kx(\xi) = 0$ and $\lim_{t \rightarrow +\infty} \rho(t)x'(t) + lx(\eta) = 0$, we have

$$\begin{aligned} \left(1 - k \int_{-\infty}^{\xi} \frac{du}{\rho(u)} \right) A - kB &= -k \int_{-\infty}^{\xi} \left(\int_s^{\xi} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds, \\ \left(1 + l \int_{-\infty}^{\eta} \frac{du}{\rho(u)} \right) A + lB &= \int_{-\infty}^{+\infty} f(s, x(s), x'(s)) ds \\ + l \int_{-\infty}^{\eta} \left(\int_s^{\eta} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds. \end{aligned}$$

It follows that

$$\begin{aligned} A &= \frac{1}{\Delta} \left[k \int_{-\infty}^{+\infty} f(s, x(s), x'(s)) ds + kl \int_{-\infty}^{\eta} \left(\int_s^{\eta} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \right. \\ &\quad \left. - kl \int_{-\infty}^{\xi} \left(\int_s^{\xi} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \right], \\ B &= \frac{1}{\Delta} \left[\left(1 - k \int_{-\infty}^{\xi} \frac{du}{\rho(u)} \right) \int_{-\infty}^{+\infty} f(s, x(s), x'(s)) ds \right. \\ &\quad + k \left(1 + l \int_{-\infty}^{\eta} \frac{du}{\rho(u)} \right) \int_{-\infty}^{\xi} \left(\int_s^{\xi} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \\ &\quad \left. + l \left(1 - k \int_{-\infty}^{\xi} \frac{du}{\rho(u)} \right) \int_{-\infty}^{\eta} \left(\int_s^{\eta} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \right]. \end{aligned}$$

Substituting A, B into (2.2), we get that

$$\begin{aligned}
x(t) &= - \int_{-\infty}^t \left(\int_s^t \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \\
&+ \frac{1}{\Delta} \left[\left(1 - k \int_{-\infty}^{\xi} \frac{du}{\rho(u)} \right) \int_{-\infty}^{+\infty} f(s, x(s), x'(s)) ds \right. \\
&+ k \left(1 + l \int_{-\infty}^{\eta} \frac{du}{\rho(u)} \right) \int_{-\infty}^{\xi} \left(\int_s^{\xi} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \\
&\left. + l \left(1 - k \int_{-\infty}^{\xi} \frac{du}{\rho(u)} \right) \int_{-\infty}^{\eta} \left(\int_s^{\eta} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \right] \\
&+ \frac{1}{\Delta} \left[k \int_{-\infty}^{+\infty} f(s, x(s), x'(s)) ds + kl \int_{-\infty}^{\eta} \left(\int_s^{\eta} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \right. \\
&\left. - kl \int_{-\infty}^{\xi} \left(\int_s^{\xi} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \right] \int_{-\infty}^t \frac{du}{\rho(u)} \\
&= \frac{1}{\Delta} \left[- \int_{-\infty}^t \left(k + l + kl \int_{\xi}^{\eta} \frac{du}{\rho(u)} \right) \left(\int_s^t \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \right. \\
&+ \int_{-\infty}^{+\infty} \left(1 + k \int_{\xi}^t \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \\
&+ \int_{-\infty}^{\xi} \left(k \int_u^{\xi} \frac{du}{\rho(u)} + kl \int_u^{\xi} \frac{du}{\rho(u)} \int_t^{\eta} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \\
&\left. + \int_{-\infty}^{\eta} \left(l \int_u^{\eta} \frac{du}{\rho(u)} + kl \int_u^{\eta} \frac{du}{\rho(u)} \int_{\xi}^t \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \right] \\
&= \int_{-\infty}^{+\infty} G(t, s) f(s, x(s), x'(s)) ds.
\end{aligned}$$

This is just (2.1). On the other hand, if $x \in X$ satisfies (2.1), it is easy to show that x is a solution of BVP(1.2). The proof is completed. \square

Fix $c > 0$ such that $\int_{-\infty}^{-c} \frac{du}{\rho(u)} < 1$. For ease expression, denote $\mu = \int_{-\infty}^{-c} \frac{du}{\rho(u)} \left(2 \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right)^{-1}$. Let

$$P = \left\{ x \in X : x(t) \geq 0 \text{ for all } t \in \mathbf{R}, \min_{t \in [-c, c]} x(t) \geq \mu \sup_{t \in \mathbf{R}} x(t) \right\}.$$

Define the operator T on P by $(Tx)(t) = \int_{-\infty}^{+\infty} G(t, s) f(s, x(s), x'(s)) ds$ for $x \in X$.

Lemma 2.5. *Suppose that (a)-(c) hold. Then*

- (i) $T : P \mapsto X$ is well defined,

(ii) it holds that

$$(2.3) \quad \begin{cases} [\rho(t)(Tx)'(t)]' + f(t, x(t), x'(t)) = 0, a.e., t \in \mathbf{R}, \\ \lim_{t \rightarrow -\infty} (Tx)(t) - k(Tx)(\xi) = 0, \\ \lim_{t \rightarrow +\infty} (Tx)(t) - l(Tx)(\eta) = 0, \end{cases}$$

(iii) $T : P \mapsto P$ is completely continuous;

(iv) $x \in X$ is a positive solution of BVP(1.2) if and only if x is a fixed point of T in P .

Proof. (i) Since $x \in X$, f is a Carathéodory function, we know that

$$\int_{-\infty}^{+\infty} f(s, x(s), x'(s)) ds$$

is convergent. By the definition of T , we have

$$\begin{aligned} (Tx)(t) &= - \int_{-\infty}^t \left(\int_s^t \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \\ &+ \frac{1}{\Delta} \left[\left(1 - k \int_{-\infty}^{\xi} \frac{du}{\rho(u)} \right) \int_{-\infty}^{+\infty} f(s, x(s), x'(s)) ds \right. \\ &+ k \left(1 + l \int_{-\infty}^{\eta} \frac{du}{\rho(u)} \right) \int_{-\infty}^{\xi} \left(\int_s^{\xi} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \\ &+ l \left(1 - k \int_{-\infty}^{\xi} \frac{du}{\rho(u)} \right) \int_{-\infty}^{\eta} \left(\int_s^{\eta} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \left. \right] \\ &+ \frac{1}{\Delta} \left[k \int_{-\infty}^{+\infty} f(s, x(s), x'(s)) ds + kl \int_{-\infty}^{\eta} \left(\int_s^{\eta} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \right. \\ &\left. - kl \int_{-\infty}^{\xi} \left(\int_s^{\xi} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \right] \int_{-\infty}^t \frac{du}{\rho(u)}. \end{aligned}$$

We know

$$\begin{aligned} \rho(t)(Tx)'(t) &= - \int_{-\infty}^t f(s, x(s), x'(s)) ds \\ &+ \frac{1}{\Delta} \left[k \int_{-\infty}^{+\infty} f(s, x(s), x'(s)) ds + kl \int_{-\infty}^{\eta} \left(\int_s^{\eta} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \right. \\ &\left. - kl \int_{-\infty}^{\xi} \left(\int_s^{\xi} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \right]. \end{aligned}$$

Thus $Tx : t \mapsto (Tx)(t)$ and $\rho(Tx)' : t \mapsto \rho(t)(Tx)'(t)$ are continuous on \mathbf{R} . Furthermore, $\lim_{t \rightarrow \pm\infty} (Tx)(t)$ and $\lim_{t \rightarrow \pm\infty} \rho(t)(Tx)'(t)$ exist and are finite. Thus

$T : P \mapsto X$ is well defined.

(ii) By (i) and direct computation we get (2.3).

(iii) First, we prove that $T : P \mapsto P$ is well defined.

Step 1. Prove that $(Tx)(t) \geq 0$ for all $t \in \mathbf{R}$. By the definition of T , it suffices to prove that $G(t, s) \geq 0$. For reader's convenience, denote $\int_a^b \frac{du}{\rho(u)} = \int_a^b$.

Case 1. For $s \geq \max\{\eta, t\}$, we see from (1.3) that

$$\Delta G(t, s) = 1 + k \int_{\xi}^t \geq \begin{cases} 0, & \xi \leq t, \\ 1 - k \int_{-\infty}^{\xi} \geq 0, & \xi > t. \end{cases}$$

Case 2. For $s \leq \min\{\xi, t\}$, we have from (1.4) that

$$\begin{aligned} \Delta G(t, s) &= 1 + k \int_{\xi}^t + k \int_u^{\xi} + kl \int_u^{\xi} \int_t^{\eta} + l \int_u^{\eta} + kl \int_u^{\eta} \int_{\xi}^t - l \int_u^t - k \int_u^t - kl \int_u^t \int_{\xi}^{\eta} \\ &= 1 + kl \int_u^{\xi} \int_t^{\eta} + l \int_t^{\eta} + kl \int_u^{\eta} \int_{\xi}^t - kl \left[\int_u^t \int_{\xi}^t + \int_u^t \int_t^{\eta} \right] \\ &= 1 + l \int_t^{\eta} \geq \begin{cases} 0, & t \leq \eta, \\ 1 - l \int_{\eta}^{+\infty} \geq 0, & t > \eta. \end{cases} \end{aligned}$$

Case 3. For $t < s \leq \xi < \eta$, note $u \leq \xi$, we have from (1.3) and (1.4) that

$$\begin{aligned} \Delta G(t, s) &= 1 + k \int_{\xi}^t + k \int_u^{\xi} + kl \int_u^{\xi} \int_t^{\eta} + l \int_u^{\eta} + kl \int_u^{\eta} \int_{\xi}^t \\ &= 1 + k \int_u^t + kl \int_u^{\xi} \int_t^{\eta} + l \int_u^{\eta} + kl \int_u^{\eta} \int_{\xi}^t \geq 1 + k \int_{\xi}^t + l \int_u^{\eta} + kl \int_u^{\eta} \int_{\xi}^t \\ &= \left(1 + k \int_{\xi}^t\right) \left(1 + l \int_u^{\eta}\right) \\ &\geq \begin{cases} 0, & \xi \leq t, u \leq \eta, \\ \left(1 - k \int_{-\infty}^{\xi}\right) \left(1 + l \int_u^{\eta}\right) \geq 0, & \xi > t, u \leq \eta, \\ \left(1 + k \int_{\xi}^t\right) \left(1 - l \int_{\eta}^{+\infty}\right), & \xi \leq t, \eta < u, \\ \left(1 - k \int_{-\infty}^{\xi}\right) \left(1 - l \int_{\eta}^{+\infty}\right) \geq 0, & \xi > t, \eta < u. \end{cases} \end{aligned}$$

Case 4. For $\min\{t, \xi\} < s \leq \eta$, we have from (1.3) and (1.4) that

$$\begin{aligned} \Delta G(t, s) &= 1 + k \int_{\xi}^t + l \int_u^{\eta} + kl \int_u^{\eta} \int_{\xi}^t \\ &= \left(1 + k \int_{\xi}^t\right) \left(1 + l \int_u^{\eta}\right) \\ &\geq \begin{cases} 0, \xi \leq t, u \leq \eta, \\ \left(1 - k \int_{-\infty}^{\xi}\right) \left(1 + l \int_u^{\eta}\right) \geq 0, \xi > t, u \leq \eta, \\ \left(1 + k \int_{\xi}^t\right) \left(1 - l \int_{\eta}^{+\infty}\right), \xi \leq t, \eta < u, \\ \left(1 - k \int_{-\infty}^{\xi}\right) \left(1 - l \int_{\eta}^{+\infty}\right) \geq 0, \xi > t, \eta < u. \end{cases} \end{aligned}$$

Case 5. For $\xi < s \leq \max\{t, \eta\}$, note $\xi \leq u \leq \min\{t, \eta\}$, we consider two cases:

Subcase 5.1. $u \geq t$. From (1.3) and (1.4), we have

$$\begin{aligned} \Delta G(t, s) &= 1 + k \int_{\xi}^t + l \int_u^{\eta} + kl \int_u^{\eta} \int_{\xi}^t - l \int_u^t - k \int_u^t - kl \int_u^t \int_{\xi}^{\eta} \\ &= 1 + k \int_{\xi}^u + l \int_t^{\eta} + kl \int_u^{\eta} \int_{\xi}^t - kl \int_u^t \int_{\xi}^{\eta} \\ &\geq 1 + k \int_{\xi}^t + l \int_u^{\eta} + kl \int_u^{\eta} \int_{\xi}^t = \left(1 + k \int_{\xi}^t\right) \left(1 + l \int_u^{\eta}\right) \\ &\geq \begin{cases} 0, \xi \leq t, u \leq \eta, \\ \left(1 - k \int_{-\infty}^{\xi}\right) \left(1 + l \int_u^{\eta}\right) \geq 0, \xi > t, u \leq \eta, \\ \left(1 + k \int_{\xi}^t\right) \left(1 - l \int_{\eta}^{+\infty}\right), \xi \leq t, \eta < u, \\ \left(1 - k \int_{-\infty}^{\xi}\right) \left(1 - l \int_{\eta}^{+\infty}\right) \geq 0, \xi > t, \eta < u. \end{cases} \end{aligned}$$

Subcase 5.2. $u < t$. Let

$$F(t) = 1 + k \int_{\xi}^u + l \int_t^{\eta} + kl \int_u^{\eta} \int_{\xi}^t - kl \int_u^t \int_{\xi}^{\eta}.$$

Using (1.3), we have

$$F'(t) = -\frac{l}{\rho(t)} \left(1 + k \int_u^{\eta} - k \int_{\xi}^{\eta}\right) = -\frac{l}{\rho(t)} \left(1 + k \int_u^{\xi} \frac{du}{\rho(u)}\right) \leq 0.$$

On the other hand, we have

$$\begin{aligned}
 F(+\infty) &= 1 + k \int_{\xi}^u -l \int_{\eta}^{+\infty} + kl \int_u^{\eta} \int_{\xi}^{+\infty} -kl \int_u^{+\infty} \int_{\xi}^{\eta} \\
 &\geq k \left(\int_{\xi}^u + l \int_u^{\eta} \int_{\xi}^{+\infty} -l \int_u^{+\infty} \int_{\xi}^{\eta} \right) \\
 &= k \left(\int_{\xi}^u + l \int_u^{\eta} \int_{\xi}^{\eta} + l \int_u^{\eta} \int_{\eta}^{+\infty} -l \int_u^{+\infty} \int_{\xi}^{\eta} \right) \\
 &= k \left(\int_{\xi}^u -l \int_{\eta}^{+\infty} \int_{\xi}^{\eta} + l \int_u^{\eta} \int_{\eta}^{+\infty} \right) \\
 &= k \int_{\xi}^u \left(1 - l \int_{\eta}^{+\infty} \right) \geq 0.
 \end{aligned}$$

Since F is decreasing and $F(+\infty) \geq 0$, we get $F(t) \geq F(+\infty) \geq 0$ for all $t < +\infty$.

Case 6. For $\xi < \eta < s \leq t$, note $u \geq \eta$, we have from (1.3) and (1.4) that

$$\begin{aligned}
 \Delta G(t, s) &= 1 + k \int_{\xi}^t -l \int_u^t -k \int_u^t -kl \int_u^t \int_{\xi}^{\eta} \\
 &\geq 1 + k \int_{\xi}^u -l \int_u^t -kl \int_u^t \int_{\xi}^u = \left(1 + k \int_{\xi}^u \right) \left(1 - l \int_u^t \right) \\
 &\geq \begin{cases} 0, \xi \leq u, u \geq t, \\ \left(1 - k \int_{-\infty}^{\xi} \right) \left(1 - l \int_u^t \right) \geq 0, \xi > u, u \geq t, \\ \left(1 + k \int_{\xi}^u \right) \left(1 - l \int_{\eta}^{+\infty} \right) \geq 0, \xi \leq u, u < t, \\ \left(1 - k \int_{-\infty}^{\xi} \right) \left(1 - l \int_{\eta}^{+\infty} \right) \geq 0, \xi > u, u < t. \end{cases}
 \end{aligned}$$

From cases 1-6, together with (1.5), this step is proved.

Step 2. Prove that $\min_{t \in [-c, c]} (Tx)(t) \geq \mu \sup_{t \in \mathbf{R}} (Tx)(t)$.

Firstly, we prove that $(Tx)(t)$ is concave with respect to $\tau = \tau(t) = \int_{-\infty}^t \frac{du}{\rho(u)}$. It is easy to see that $\tau \in C \left(\mathbf{R}, \left(0, \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right) \right)$ and $\frac{d\tau}{dt} = \frac{1}{\rho(t)} > 0$. Thus

$$(2.4) \quad \frac{d(Tx)}{dt} = \frac{d(Tx)}{d\tau} \frac{d\tau}{dt} = \frac{d(Tx)}{d\tau} \frac{1}{\rho(t)}.$$

It follows that $\rho(t) \frac{d(Tx)}{dt} = \frac{d(Tx)}{d\tau}$. Since $\frac{d[\rho(t)(Tx)'(t)]}{dt} = -f(t, x(t), x'(t)) \leq 0$ and $\frac{d\tau}{dt} > 0$, we get that $\frac{d[\rho(t)(Tx)'(t)]}{d\tau} \leq 0$ on $\left(0, \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right)$. Then $\frac{d(Tx)}{d\tau}$ is decreasing with respect to $\tau \in \left(0, \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right)$. Hence $(Tx)(t)$ is concave with respect to $\tau \in \left(0, \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right)$.

Since $\frac{d\tau}{dt} > 0$ for all $t \in \mathbf{R}$, there exists the inverse function of $\tau = \tau(t)$. Denote the inverse function of $\tau = \tau(t)$ by $t = t(\tau)$.

Case 1. there exists $\tau_0 \in \mathbf{R}$ such that $\sup_{t \in \mathbf{R}} (Tx)(t) = (Tx)(\tau_0)$. One sees

$$\min_{t \in [-c, c]} (Tx)(t) = \min \{(Tx)(-c), (Tx)(c)\}.$$

Denote $\tau(+\infty) = \int_{-\infty}^{+\infty} \frac{du}{\rho(u)}$. If

$$\min\{(Tx)(-c), (Tx)(c)\} = (Tx)(c) = (Tx)(t(\tau(c))),$$

noting $\tau(-c) < 1$ (by definition of c), one has for $t \in [-c, c]$ that

$$\begin{aligned} (Tx)(t) &\geq (Tx)(t(\tau(c))) \\ &= (Tx) \left(t \left(\frac{\tau(+\infty) - \tau(c) + \tau(\tau_0)}{\tau(+\infty) + \tau(\tau_0)} \frac{\tau(c)\tau(+\infty)}{\tau(+\infty) - \tau(c) + \tau(\tau_0)} + \frac{\tau(c)}{\tau(+\infty) + \tau(\tau_0)} \tau(\tau_0) \right) \right). \end{aligned}$$

Noting that $\tau(+\infty) > \tau(c)$ and $(Tx)(t)$ is concave with respect to τ , then, for $t \in [-c, c]$,

$$\begin{aligned} (Tx)(t) &\geq \frac{\tau(+\infty) - \tau(c) + \tau(\tau_0)}{\tau(+\infty) + \tau(\tau_0)} (Tx) \left(t \left(\frac{\tau(c)\tau(+\infty)}{\tau(+\infty) - \tau(c) + \tau(\tau_0)} \right) \right) \\ &\quad + \frac{\tau(c)}{\tau(+\infty) + \tau(\tau_0)} (Tx) (t(\tau(\tau_0))) \\ &\geq \int_{-\infty}^c \frac{1}{\rho(s)} ds \frac{1}{2 \int_{-\infty}^{+\infty} \frac{1}{\rho(s)} ds} (Tx)(\tau_0) \\ &\geq \int_{-\infty}^{-c} \frac{1}{\rho(s)} ds \frac{1}{2 \int_{-\infty}^{+\infty} \frac{1}{\rho(s)} ds} (Tx)(\tau_0) \\ &= \mu \sup_{t \in \mathbf{R}} (Tx)(t). \end{aligned}$$

Similarly, if $\min\{(Tx)(-c), (Tx)(c)\} = (Tx)(-c) = (Tx)(t(\tau(-c)))$, noting $\tau(-c) < 1$ (by definition of c), one has for $t \in [-c, c]$ that

$$\begin{aligned} (Tx)(t) &\geq (Tx)(t(\tau(-c))) \\ &= (Tx) \left(t \left(\frac{\tau(+\infty) + \tau(\tau_0) - \tau(-c)}{\tau(+\infty) + \tau(\tau_0) - \tau(-c)} \frac{\tau(+\infty)\tau(-c)}{\tau(+\infty) + \tau(\tau_0)} + \frac{\tau(-c)}{\tau(+\infty) + \tau(\tau_0)} \tau(\tau_0) \right) \right) \\ &\geq \frac{\tau(+\infty) + \tau(\tau_0) - \tau(-c)}{\tau(+\infty) + \tau(\tau_0)} (Tx) \left(t \left(\frac{\tau(-c)\tau(+\infty)}{\tau(+\infty) + \tau(\tau_0) - \tau(-c)} \right) \right) \\ &\quad + \frac{\tau(-c)}{\tau(+\infty) + \tau(\tau_0)} (Tx) (t(\tau(\tau_0))) \\ &\geq \int_{-\infty}^{-c} \frac{1}{\rho(s)} ds \frac{1}{2 \int_{-\infty}^{+\infty} \frac{1}{\rho(s)} ds} (Tx)(\tau_0) > \mu \sup_{t \in \mathbf{R}} (Tx)(t). \end{aligned}$$

Case 2. $\sup_{t \in \mathbf{R}} (Tx)(t) = \lim_{t \rightarrow -\infty} (Tx)(t)$ or $\lim_{t \rightarrow +\infty} (Tx)(t)$. Choose $\tau_0 \in \mathbf{R}$. By the same methods used in Case 1, we get

$$\min_{t \in [-c, c]} (Tx)(t) \geq \mu(Tx)(\tau_0).$$

Let $\tau_0 \rightarrow -\infty$. We get

$$\min_{t \in [-c, c]} (Tx)(t) \geq \mu \lim_{t \rightarrow -\infty} (Tx)(t) = \mu \sup_{t \in \mathbf{R}} (Tx)(t) \text{ or } \mu \lim_{t \rightarrow +\infty} (Tx)(t).$$

So $\min_{t \in [-c, c]} (Tx)(t) \geq \mu \sup_{t \in \mathbf{R}} (Tx)(t)$ is proved. We see from Step 1 and Step 2 that $Tx \in P$. Hence $T : P \rightarrow P$ is well defined.

Step 3. We prove that T is continuous on P . Since f is a Carathéodory function, then the result follows.

Step 4. We show that T maps bounded subsets into bounded sets.

Given a bounded set $D \subseteq X$. Then, there exists $M > 0$ such that $D \subseteq \{x \in X : \|x\| \leq M\}$. Then there exists $\phi_M \in L^1(\mathbf{R})$ such that $|f(t, x(t), x'(t))| \leq \phi_M(t)$ for all $t \in \mathbf{R}$. By the definition of $G(t, s)$, the expressions of $(Tx)(t)$ and $\rho(t)(Tx)'(t)$, we get that

$$G(t, s) \leq \frac{1+3k \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} + 2l \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} + 3kl \left(\int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right)^2}{\Delta}.$$

Then

$$|(Tx)(t)| \leq \frac{1+3k \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} + 2l \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} + 3kl \left(\int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right)^2}{\Delta} \int_{-\infty}^{+\infty} \phi_M(s) ds.$$

On the other hand, we have

$$\rho(t)|(Tx)'(t)| \leq \left[1 + \frac{k+2kl \int_{-\infty}^{+\infty} \frac{du}{\rho(u)}}{\Delta} \right] \int_{-\infty}^{+\infty} \phi_M(s) ds.$$

Then

$$\|(Tx)\| = \max \left\{ \sup_{t \in \mathbf{R}} |(Tx)(t)|, \sup_{t \in \mathbf{R}} \rho(t)|(Tx)'(t)| \right\} < \infty.$$

So, $\{TD\}$ is bounded.

Step 5. we prove that both $\{Tx : x \in D\}$ and $\{t \mapsto \rho(t)(Tx)'(t) : x \in D\}$ are equi-continuous on each finite subinterval on \mathbf{R} .

The proof is standard and is omitted. One may see [22].

Step 6. we show that both $\{Tx : x \in D\}$ and $\{t \mapsto \rho(t)(Tx)'(t) : x \in D\}$ are equi-convergent at both $+\infty$ and $-\infty$ respectively.

We have that

$$\begin{aligned}
& \left| (Tx)(t) - \frac{1}{\Delta} \left[\left(1 - k \int_{-\infty}^{\frac{du}{\rho(u)}} \right) \int_{-\infty}^{+\infty} f(s, x(s), x'(s)) ds \right. \right. \\
& \quad + k \left(1 + l \int_{-\infty}^{\eta} \frac{du}{\rho(u)} \right) \int_{-\infty}^{\xi} \left(\int_s^{\xi} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \\
& \quad \left. \left. + l \left(1 - k \int_{-\infty}^{\frac{du}{\rho(u)}} \right) \int_{-\infty}^{\eta} \left(\int_s^{\eta} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \right] \right| \\
& \leq \int_{-\infty}^t \left(\int_s^t \frac{du}{\rho(u)} \right) |f(s, x(s), x'(s))| ds \\
& \quad + \frac{1}{\Delta} \left[k \int_{-\infty}^{+\infty} |f(s, x(s), x'(s))| ds + kl \int_{-\infty}^{\eta} \left(\int_s^{\eta} \frac{du}{\rho(u)} \right) |f(s, x(s), x'(s))| ds \right. \\
& \quad \left. + kl \int_{-\infty}^{\xi} \left(\int_s^{\xi} \frac{du}{\rho(u)} \right) |f(s, x(s), x'(s))| ds \right] \int_{-\infty}^t \frac{du}{\rho(u)} \\
& \leq \frac{1}{\Delta} \left[\Delta + k + kl \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} + kl \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right] \int_{-\infty}^{+\infty} \Phi_M(s) ds \int_{-\infty}^t \frac{du}{\rho(u)} \\
& \rightarrow 0 \text{ uniformly as } t \rightarrow -\infty.
\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}
& \left| \rho(t)(Tx)'(t) - \frac{1}{\Delta} \left[k \int_{-\infty}^{+\infty} f(s, x(s), x'(s)) ds \right. \right. \\
& \quad \left. \left. + kl \int_{-\infty}^{\eta} \left(\int_s^{\eta} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds - kl \int_{-\infty}^{\xi} \left(\int_s^{\xi} \frac{du}{\rho(u)} \right) f(s, x(s), x'(s)) ds \right] \right| \\
& \leq - \int_{-\infty}^t |f(s, x(s), x'(s))| ds \\
& \leq \int_{-\infty}^t \phi_M(s) ds \rightarrow 0 \text{ uniformly as } t \rightarrow -\infty.
\end{aligned}$$

Hence $\{t \mapsto \rho(t)(Tx)'(t) : x \in D\}$ and $\{Tx : x \in D\}$ are equiconvergent at $-\infty$.

Similarly we can prove that both $\{Tx : x \in D\}$ and $\{t \mapsto \rho(t)(Tx)'(t) : x \in D\}$ are equiconvergent at $+\infty$. The details are omitted.

From Steps 3-6, we see that T maps bounded sets into relatively compact sets.

Therefore, the operator $T : P \rightarrow P$ is completely continuous. The proof of (iii) is complete.

(iv) It is easy to see that $x \in P$ is a positive solution of BVP(1.2) if and only if x is a fixed point of T in P . The proof of (iv) is complete. Thus the proof of Lemma 2.5 is ended. \square

Remark 2.1. From Cases 1-6 in Step 1 in the proof of Lemma 2.5, we know $G(t, s) \geq 0$ for all $s, t \in \mathbf{R}$. This result generalizes a corresponding one in [22].

3. Existence of positive solutions

In this section we establish existence result on positive solutions of BVP(1.2).

Fix c such that $\int_{-\infty}^{-c} \frac{du}{\rho(u)} < 1$. Let $\mu = \int_{-\infty}^{-c} \frac{du}{\rho(u)} \left(\int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right)^{-1}$. Δ is defined by (1.5). To simplify notation, for nonnegative function $\phi \in L^1(\mathbf{R})$ and nonnegative constants L_1, L_2 and a, b , denote

$$M_0 = \max \left\{ \frac{a}{\int_{-c}^0 \phi(r) dr \int_{-\infty}^{-c} \frac{du}{\rho(u)}}, \frac{a}{\int_0^c \phi(r) dr \int_c^{+\infty} \frac{du}{\rho(u)}} \right\},$$

$$W_0 = \min \left\{ \frac{L_1}{\int_{-\infty}^0 \phi(r) dr}, \frac{L_1}{\int_0^{+\infty} \phi(r) dr} \right\},$$

$$E_0 = \max \left\{ \frac{\Delta b}{\left[1 + 3k \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} + 2l \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} + 3kl \left(\int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right)^2 \right] \int_{-\infty}^{+\infty} \phi(s) ds}, \frac{\Delta L_2}{\left[\Delta + k + 2kl \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right] \int_{-\infty}^{+\infty} \phi(s) ds} \right\}.$$

Theorem 3.1. *Suppose that (a)-(c) hold and there exists nonnegative function $\phi \in L^1(\mathbf{R})$ and nonnegative constants L_1, L_2 and a, b such that $L_1 < L_2$ and $a < b$ and*

$$f \left(t, u, \frac{v}{\rho(t)} \right) \geq M_0 \phi(t), \quad t \in [-c, c], u \in [\mu a, a], v \in [-L_1, L_1];$$

$$f \left(t, u, \frac{v}{\rho(t)} \right) \geq W_0 \phi(t), \quad t \in \mathbf{R}, u \in [0, a], v \in [-L_1, L_1],$$

$$f \left(t, u, \frac{v}{\rho(t)} \right) \leq E_0 \phi(t), \quad t \in \mathbf{R}, u \in [0, b], v \in [-L_2, L_2].$$

If $E_0 > \max\{M_0, W_0\}$, then BVP(1.2) has at least one positive solution x satisfying

$$(3.1) \quad a \leq \sup_{t \in \mathbf{R}} x(t) \leq b, \quad 0 < \sup_{t \in \mathbf{R}} \rho(t) |x'(t)| \leq L_2$$

or

$$(3.2) \quad 0 < \sup_{t \in \mathbf{R}} x(t) \leq b, \quad L_1 \leq \sup_{t \in \mathbf{R}} \rho(t) |x'(t)| \leq L_2.$$

Proof. Let X, P and the operator T be defined in section 2. By the definition of T , Lemma 2.5, we know that $T : P \mapsto P$ is completely continuous, x is a positive solution of BVP(1.2) if and only if x is a fixed point the T in P .

Define

$$\bar{\xi}(x) = \sup_{t \in \mathbf{R}} x(t), \quad x \in P, \quad \bar{\eta}(x) = \sup_{t \in \mathbf{R}} \rho(t) |x'(t)|, \quad x \in P,$$

$$\Omega_1 = \{x \in P : \bar{\xi}(x) < a, \bar{\eta}(x) < L_1\}, \quad \Omega_2 = \{x \in P : \bar{\xi}(x) < b, \bar{\eta}(x) < L_2\}.$$

It is easy to see that $\bar{\xi}$ and $\bar{\eta}$ are continuous functionals and Ω_1 and Ω_2 are bounded nonempty open subsets of P and $\bar{\xi}(x), \bar{\eta}(x) \leq \|x\| = \max\{\bar{\xi}(x), \bar{\eta}(x)\}$. To apply (E1) in Lemma 2.1, we do the following two steps:

Step 1. We prove that

$$(3.3) \quad Tx \neq \lambda x \text{ for all } \lambda \in [0, 1) \text{ and } x \in \partial\Omega_1.$$

Let

$$C_1 = \{x \in P : \bar{\xi}(x) = a, \bar{\eta}(x) \leq L_1\}, \quad D_1 = \{x \in P : \bar{\xi}(x) \leq a, \bar{\eta}(x) = L_1\},$$

$$C_2 = \{x \in P : \bar{\xi}(x) = b, \bar{\eta}(x) \leq L_2\}, \quad D_2 = \{x \in P : \bar{\xi}(x) \leq b, \bar{\eta}(x) = L_2\}.$$

Sub-step 1.1. For $x \in C_1$, we prove that $\bar{\xi}(Tx) \geq a$.

In fact, $x \in C_1$ implies that

$$\mu a \leq x(t) \leq a, \quad t \in [-c, c], \quad -L_1 \leq \rho(t)x'(t) \leq L_1.$$

Then we get

$$f(t, x(t), x'(t)) = f\left(t, x(t), \frac{\rho(t)x'(t)}{\rho(t)}\right) \geq M_0\phi(t), \quad t \in [-c, c].$$

We consider three cases:

Case 1. $\rho(t)(Tx)'(t) > 0$ for all $t \in \mathbf{R}$. In this case, we see that Tx is increasing on \mathbf{R} and $(Tx)(t) \geq 0$ for all $t \in \mathbf{R}$. Then

$$\begin{aligned} \bar{\xi}(Tx) &\geq (Tx)(-c) = \int_{-\infty}^{-c} \frac{1}{\rho(s)} \left(\int_s^{+\infty} f(r, x(r), x'(r)) dr \right) ds \\ &= \int_{-c}^{+\infty} f(r, x(r), x'(r)) dr \int_{-\infty}^{-c} \frac{du}{\rho(u)} + \int_{-\infty}^{-c} \int_{-\infty}^r \frac{du}{\rho(u)} f(r, x(r), x'(r)) dr \\ &\geq \int_{-c}^{+\infty} f(r, x(r), x'(r)) dr \int_{-\infty}^{-c} \frac{du}{\rho(u)} \\ &\geq \int_{-c}^c f(r, x(r), x'(r)) dr \int_{-\infty}^{-c} \frac{du}{\rho(u)} \\ &\geq M_0 \int_{-c}^c \phi(r) dr \int_{-\infty}^{-c} \frac{du}{\rho(u)} \geq a. \end{aligned}$$

Case 2. $\rho(t)(Tx)'(t) < 0$ for all $t \in \mathbf{R}$. In this case, we see that Tx is decreasing on \mathbf{R} and $(Tx)(t) \geq 0$ for all $t \in \mathbf{R}$. Then

$$\begin{aligned} \bar{\xi}(Tx) &\geq (Tx)(c) \geq \int_c^{+\infty} \frac{1}{\rho(s)} \int_{-\infty}^s f(r, x(r), x'(r)) dr ds \\ &= \int_{-\infty}^c f(r, x(r), x'(r)) ds \int_c^{+\infty} \frac{du}{\rho(u)} + \int_c^{+\infty} \int_r^{+\infty} \frac{du}{\rho(u)} f(r, x(r), x'(r)) dr \\ &\geq \int_{-c}^c f(r, x(r), x'(r)) ds \int_c^{+\infty} \frac{du}{\rho(u)} \\ &\geq M_0 \int_{-c}^c \phi(r) dr \int_c^{+\infty} \frac{du}{\rho(u)} \geq a. \end{aligned}$$

Case 3. there exists $\tau_0 \in \mathbf{R}$ such that $\rho(\tau_0)(Tx)'(\tau_0) = 0$. From $(Tx)(t) \geq 0$, we have

$$\rho(t)(Tx)'(t) = \begin{cases} -\int_{\tau_0}^t f(r, x(r), x'(r))dr, & t \geq \tau_0, \\ \int_t^{\tau_0} f(r, x(r), x'(r))dr, & t \leq \tau_0. \end{cases}$$

So

$$\begin{aligned} (Tx)(t) &\geq \begin{cases} \int_t^{+\infty} \frac{1}{\rho(s)} \int_{\tau_0}^s f(r, x(r), x'(r))drds, & t \geq \tau_0, \\ \int_{-\infty}^t \frac{1}{\rho(s)} \int_s^{\tau_0} f(r, x(r), x'(r))drds, & t \leq \tau_0 \end{cases} \\ &\geq \begin{cases} \int_{\tau_0}^t f(r, x(r), x'(r))dr \int_t^{+\infty} \frac{du}{\rho(u)}, & t \geq \tau_0, \\ \int_t^{\tau_0} f(r, x(r), x'(r))dr \int_{-\infty}^t \frac{du}{\rho(u)}, & t \leq \tau_0. \end{cases} \end{aligned}$$

If $\tau_0 \geq 0$, then

$$\bar{\xi}(Tx) \geq \int_{-c}^0 f(r, x(r), x'(r))dr \int_{-\infty}^{-c} \frac{du}{\rho(u)} \geq M_0 \int_{-c}^0 \phi(r)dr \int_{-\infty}^{-c} \frac{du}{\rho(u)} \geq a.$$

If $\tau_0 \leq 0$, then

$$\bar{\xi}(Tx) \geq \int_0^c f(r, x(r), x'(r))dr \int_c^{+\infty} \frac{du}{\rho(u)} \geq M_0 \int_0^c \phi(r)dr \int_c^{+\infty} \frac{du}{\rho(u)} \geq a.$$

Sub-step 1.2. For $x \in D_1$, we prove that $\bar{\eta}(Tx) \geq L_1$.

In fact, $x \in D_1$ implies that

$$0 \leq x(t) \leq a, \quad -L_1 \leq \rho(t)x'(t) \leq L_1, \quad t \in \mathbf{R}.$$

Then we get

$$f(t, x(t), x'(t)) = f\left(t, x(t), \frac{\rho(t)x'(t)}{\rho(t)}\right) \geq W_0\phi(t), \quad t \in \mathbf{R}.$$

We consider three cases:

Case 1. $\rho(t)(Tx)'(t) > 0$ for all $t \in \mathbf{R}$. Then

$$\rho(t)(Tx)'(t) = \lim_{t \rightarrow +\infty} \rho(t)(Tx)'(t) + \int_t^{+\infty} f(r, x(r), x'(r))dr.$$

So

$$\begin{aligned} \bar{\eta}(Tx) &= \sup_{t \in \mathbf{R}} \rho(t)|(Tx)'(t)| \geq \sup_{t \in \mathbf{R}} \int_t^{+\infty} f(r, x(r), x'(r))dr \\ &= \int_{-\infty}^{+\infty} f(r, x(r), x'(r))dr \geq \int_{-\infty}^{+\infty} \phi(r)W_0dr \geq L_1. \end{aligned}$$

Case 2. $\rho(t)(Tx)'(t) < 0$ for all $t \in \mathbf{R}$. Then

$$\rho(t)(Tx)'(t) = \lim_{t \rightarrow -\infty} \Phi(\rho(t)(Tx)'(t)) - \int_{-\infty}^t f(r, x(r), x'(r))dr.$$

So

$$\begin{aligned}
\bar{\eta}(Tx) &= \sup_{t \in \mathbf{R}} \rho(t) |(Tx)'(t)| \\
&= \sup_{t \in \mathbf{R}} \left| \lim_{t \rightarrow -\infty} \rho(t) (Tx)'(t) - \int_{-\infty}^t f(r, x(r), x'(r)) dr \right| \\
&\geq \int_{-\infty}^{+\infty} f(r, x(r), x'(r)) dr \\
&\geq \int_{-\infty}^{+\infty} \phi(r) W_0 dr \geq L_1.
\end{aligned}$$

Case 3. there exists $\tau_0 \in \mathbf{R}$ such that $\rho(\tau_0)(Tx)'(\tau_0) = 0$. Then

$$\rho(t)(Tx)'(t) = \begin{cases} -\int_{\tau_0}^t f(r, x(r), x'(r)) dr, & t \geq \tau_0, \\ \int_t^{\tau_0} f(r, x(r), x'(r)) dr, & t \leq \tau_0. \end{cases}$$

If $\tau_0 \geq 0$, then

$$\begin{aligned}
\bar{\eta}(Tx) &= \sup_{t \in \mathbf{R}} \rho(t) |(Tx)'(t)| \geq \sup_{t \in \mathbf{R}} \int_t^{\tau_0} f(r, x(r), x'(r)) dr \\
&\geq \int_{-\infty}^0 f(r, x(r), x'(r)) dr \geq \int_{-\infty}^0 \phi(r) W_0 dr \geq L_1,
\end{aligned}$$

If $\tau_0 \leq 0$, then

$$\begin{aligned}
\bar{\eta}(Tx) &= \sup_{t \in \mathbf{R}} \rho(t) |(Tx)'(t)| \geq \sup_{t \in \mathbf{R}} \left| \int_{\tau_0}^t f(r, x(r), x'(r)) dr \right| \\
&\geq \int_0^{+\infty} f(r, x(r), x'(r)) dr \geq \int_0^{+\infty} \phi(r) W_0 dr \geq L_1,
\end{aligned}$$

Now we prove (3.3). It is easy to see that

$$\partial\Omega_1 \subseteq C_1 \cup D_1, \quad \partial\Omega_2 \subseteq C_2 \cup D_2.$$

If $Tx = \lambda x$ for some $\lambda \in [0, 1)$ and $x \in \partial\Omega_1$, then either $x \in C_1$ or $x \in D_1$.

If $x \in C_1$, we get from Sub-step 1.1 that $\xi(Tx) \geq a$. On the other hand, we have $\xi(Tx) = \lambda\xi(x) < \xi(x) = a$, a contradiction.

If $x \in D_1$, from Sub-step 1.2, we have $\eta(Tx) \geq L_1$. On the other hand, we have $\eta(Tx) = \lambda\eta(x) < \eta(x) = L_1$, a contradiction too.

From above discussion, (3.3) holds.

Step 2. We prove that

$$(3.4) \quad Tx \neq \lambda x \text{ for all } \lambda \in (1, +\infty) \text{ and } x \in \partial\Omega_2.$$

For $x \in C_2$, one has $0 \leq x(t) \leq b$, $-L_2 \leq \rho(t)x'(t) \leq L_2$, $t \in \mathbf{R}$. Then we get

$$f(t, x(t), x'(t)) = f\left(t, x(t), \frac{\rho(t)x'(t)}{\rho(t)}\right) \leq E_0\phi(t), t \in \mathbf{R}.$$

Then, we have

$$\begin{aligned} \bar{\xi}(Tx) &= \sup_{t \in \mathbf{R}} \int_{-\infty}^{+\infty} G(t, s) f(s, x(s), x'(s)) ds \\ &\leq \frac{1+3k \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} + 2l \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} + 3kl \left(\int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right)^2}{\Delta} \int_{-\infty}^{+\infty} \phi(s) E_0 ds \leq b. \end{aligned}$$

For $x \in D_2$, we have

$$\bar{\eta}(Tx) = \sup_{t \in \mathbf{R}} \rho(t) |(Tx)'(t)| \leq \frac{1}{\Delta} \left[\Delta + k + 2kl \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right] \int_{-\infty}^{+\infty} E_0 \phi(s) ds \leq L_2.$$

In fact, if $Tx = \lambda x$ for some $\lambda \in (1, +\infty)$ and $x \in \partial\Omega_2$, then either $x \in C_2$ or $x \in D_2$.

If $x \in C_2$, we get from above discussion that $\xi(Tx) \leq b$. On the other hand, we have $\xi(Tx) = \lambda \xi(x) > \xi(x) = b$, a contradiction.

If $x \in D_2$, from above discussion, we have $\eta(Tx) \leq L_2$. On the other hand, we have $\eta(Tx) = \lambda \eta(x) > \eta(x) = L_2$, a contradiction too.

From above discussion, (3.4) holds.

It follows from (3.3), (3.4) and (E1) in Lemma 2.1 that T has at least one fixed point $x \in \bar{\Omega}_2 \setminus \Omega_1$. So BVP(1.2) has at least one positive solution x such that $x \in \bar{\Omega}_2 \setminus \Omega_1$ and that x satisfies (3.1) or (3.2). The proof of Theorem 3.1 is complete. \square

To simplify notation, for nonnegative function $\phi \in L^1(\mathbf{R})$ and nonnegative constants L_1, L_2 and a, b , denote

$$M_1 = \frac{\Delta a}{\left[1+3k \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} + 2l \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} + 3kl \left(\int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right)^2 \right] \int_{-\infty}^{+\infty} \phi(s) ds},$$

$$W_1 = \frac{\Delta L_1}{\left[\Delta + k + kl \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right] \int_{-\infty}^{+\infty} E_0 \phi(s) ds},$$

$$E_1 = \min \left\{ \frac{L_2}{\int_{-c}^0 \phi(r) dr \int_{-\infty}^{-c} \frac{du}{\rho(u)}}, \frac{L_2}{\int_0^c \phi(r) dr \int_c^{+\infty} \frac{du}{\rho(u)}}, \frac{L_2}{\int_{-\infty}^0 \phi(r) dr}, \frac{L_2}{\int_0^{+\infty} \phi(r) dr} \right\}.$$

Theorem 3.2. *Suppose that (a)-(c) hold and there exists a nonnegative function $\phi \in L^1(\mathbf{R})$ and $L_1 > L_2 > 0$ and $a > b > 0$ such that*

$$f \left(t, u, \frac{v}{\rho(t)} \right) \leq M_1 \phi(t), \quad t \in [-c, c], u \in [\mu a, a], v \in [-L_1, L_1];$$

$$f \left(t, u, \frac{v}{\rho(t)} \right) \leq W_1 \phi(t), \quad t \in \mathbf{R}, u \in [0, a], v \in [-L_1, L_1],$$

$$f \left(t, u, \frac{v}{\rho(t)} \right) \geq E_1 \phi(t), \quad t \in \mathbf{R}, u \in [0, b], v \in [-L_2, L_2].$$

If $E_1 > \max\{M_1, W_1\}$, then BVP(1.2) has at least one positive solution x satisfying

$$(3.5) \quad b \leq \sup_{t \in \mathbf{R}} x(t) \leq a, \quad 0 < \sup_{t \in \mathbf{R}} \rho(t) |x'(t)| \leq L_1$$

or

$$(3.6) \quad 0 < \sup_{t \in \mathbf{R}} x(t) \leq a, \quad L_2 \leq \sup_{t \in \mathbf{R}} \rho(t)|x'(t)| \leq L_1.$$

Proof. Let X , P and the nonlinear operator T be defined in Section 2. The proof is similar to the proof of Theorem 3.1 by using (E2) in Lemma 2.1 and is omitted. \square

4. Non-existence of positive solutions

Now we establish non-existence results on positive solutions of BVP(1.2).

Theorem 4.1. *Let $\Omega = \mathbf{R} \times [0, +\infty) \times \mathbf{R}$. Suppose that (a)-(c) hold and there exists a function $\phi \in L^1(\mathbf{R})$ such that*

$$(4.1) \quad \sup_{(t,u,v) \in \Omega} \frac{f(t,u,\frac{v}{\rho(t)})}{\phi(t)v} \leq 1.$$

If

$$(4.2) \quad \frac{1}{\Delta} \left[\Delta + k + kl \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right] \int_{-\infty}^{+\infty} \phi(s)ds < 1,$$

then BVP(1.2) does not admit positive solutions.

Proof. Let X , P and T be defined in Section 2. From (4.1), we have

$$f\left(t, u, \frac{v}{\rho(t)}\right) \leq \phi(t)|v|, \quad (t, u, v) \in \mathbf{R} \times [0, +\infty) \times \mathbf{R}.$$

Assume that x is a positive fixed point of T . We have

$$f(t, x(t), x'(t)) \leq \phi(t)\rho(t)x'(t) \leq \phi(t) \sup_{t \in \mathbf{R}} \rho(t)|x'(t)|, \quad t \in \mathbf{R}.$$

It follows from (4.2) that

$$\begin{aligned} \sup_{t \in \mathbf{R}} \rho(t)|x'(t)| &\leq \frac{1}{\Delta} \left[\Delta + k + kl \int_{-\infty}^{+\infty} \frac{du}{\rho(u)} \right] \int_{-\infty}^{+\infty} \phi(s)ds \sup_{t \in \mathbf{R}} \rho(t)|x'(t)| \\ &< \sup_{t \in \mathbf{R}} \rho(t)|x'(t)| \end{aligned}$$

which is a contradiction. The proof is complete. \square

Theorem 4.2. *Let $\Omega = \mathbf{R} \times [0, +\infty) \times \mathbf{R}$. Suppose that (a)-(c) hold and there exists a function $\phi \in L^1(\mathbf{R})$ and a constant $c > 0$ such that*

$$(4.3) \quad \inf_{(t,u,v) \in \Omega} \frac{f(t,u,\frac{v}{\rho(t)})}{\phi(t)u} \geq 1.$$

If

$$(4.4) \quad \mu \min \left\{ \int_{-c}^0 \phi(r)dr \int_{-\infty}^{-c} \frac{du}{\rho(u)}, \int_0^c \phi(r)dr \int_c^{+\infty} \frac{du}{\rho(u)} \right\} > 1,$$

then BVP(1.2) does not admit positive solutions.

Proof. Let X , P and T be defined in Section 2. From (4.3), we have

$$f\left(t, u, \frac{v}{\rho(t)}\right) \geq \phi(t)u, \quad (t, u, v) \in \mathbf{R} \times [0, +\infty) \times \mathbf{R}.$$

Suppose that x is a positive solution of BVP(1.2). Then $x(t) = (Tx)(t)$ for all $t \in \mathbf{R}$. We have

$$f(t, x(t), x'(t)) \geq \phi(t)x(t) \geq \mu \phi(t) \sup_{t \in \mathbf{R}} |x(t)|, \quad t \in \mathbf{R}.$$

We consider three cases:

Case 1. $\rho(t)x'(t) > 0$ for all $t \in \mathbf{R}$. So

$$\begin{aligned} \sup_{t \in \mathbf{R}} |x(t)| &\geq x(-c) = \int_{-\infty}^{-c} \frac{1}{\rho(s)} \left(\int_s^{+\infty} f(r, x(r), x'(r)) dr \right) ds \\ &= \int_{-c}^{+\infty} f(r, x(r), x'(r)) dr \int_{-\infty}^{-c} \frac{du}{\rho(u)} + \int_{-\infty}^{-c} \int_{-\infty}^r \frac{du}{\rho(u)} f(r, x(r), x'(r)) dr \\ &\geq \int_{-c}^{+\infty} f(r, x(r), x'(r)) dr \int_{-\infty}^{-c} \frac{du}{\rho(u)} \\ &\geq \int_{-c}^c f(r, x(r), x'(r)) dr \int_{-\infty}^{-c} \frac{du}{\rho(u)} \\ &\geq \mu \sup_{t \in \mathbf{R}} |x(t)| \int_{-c}^c \phi(r) dr \int_{-\infty}^{-c} \frac{du}{\rho(u)} > \sup_{t \in \mathbf{R}} |x(t)|, \end{aligned}$$

which is a contradiction.

Case 2. $\rho(t)x'(t) < 0$ for all $t \in \mathbf{R}$. Then

$$\begin{aligned} \sup_{t \in \mathbf{R}} |x(t)| &\geq (Tx)(c) \geq \int_c^{+\infty} \frac{1}{\rho(s)} \int_{-\infty}^s f(r, x(r), x'(r)) dr ds \\ &= \int_{-\infty}^c f(r, x(r), x'(r)) ds \int_c^{+\infty} \frac{du}{\rho(u)} + \int_c^{+\infty} \int_r^{+\infty} \frac{du}{\rho(u)} f(r, x(r), x'(r)) dr \\ &\geq \int_{-c}^c f(r, x(r), x'(r)) ds \int_c^{+\infty} \frac{du}{\rho(u)} \\ &\geq \mu \sup_{t \in \mathbf{R}} |x(t)| \int_{-c}^c \phi(r) dr \int_c^{+\infty} \frac{du}{\rho(u)} > \sup_{t \in \mathbf{R}} |x(t)|, \end{aligned}$$

which is a contradiction.

Case 3. there exists $\tau_0 \in \mathbf{R}$ such that $\rho(\tau_0)(Tx)'(\tau_0) = 0$. From $(Tx)(t) \geq 0$, we have

$$\rho(t)x'(t) = \begin{cases} - \int_{\tau_0}^t f(r, x(r), x'(r)) dr, & t \geq \tau_0, \\ \int_t^{\tau_0} f(r, x(r), x'(r)) dr, & t \leq \tau_0. \end{cases}$$

So

$$\begin{aligned}
 x(t) &\geq \begin{cases} \int_t^{+\infty} \frac{1}{\rho(s)} \int_{\tau_0}^s f(r, x(r), x'(r)) dr ds, & t \geq \tau_0, \\ \int_{-\infty}^t \frac{1}{\rho(s)} \int_s^{\tau_0} f(r, x(r), x'(r)) dr ds, & t \leq \tau_0 \end{cases} \\
 &\geq \begin{cases} \int_{\tau_0}^t f(r, x(r), x'(r)) dr \int_t^{+\infty} \frac{du}{\rho(u)}, & t \geq \tau_0, \\ \int_t^{\tau_0} f(r, x(r), x'(r)) dr \int_{-\infty}^t \frac{du}{\rho(u)}, & t \leq \tau_0. \end{cases}
 \end{aligned}$$

If $\tau_0 \geq 0$, then

$$\begin{aligned}
 \sup_{t \in \mathbf{R}} |x(t)| &\geq \int_{-c}^0 f(r, x(r), x'(r)) dr \int_{-\infty}^{-c} \frac{du}{\rho(u)} \\
 &\geq \mu \sup_{t \in \mathbf{R}} |x(t)| \int_{-c}^0 \phi(r) dr \int_{-\infty}^{-c} \frac{du}{\rho(u)} > \sup_{t \in \mathbf{R}} |x(t)|,
 \end{aligned}$$

which is a contradiction.

If $\tau_0 \leq 0$, then

$$\begin{aligned}
 \sup_{t \in \mathbf{R}} |x(t)| &\geq \int_0^c f(r, x(r), x'(r)) dr \int_c^{+\infty} \frac{du}{\rho(u)} \\
 &\geq \mu \sup_{t \in \mathbf{R}} |x(t)| \int_0^c \phi(r) dr \int_c^{+\infty} \frac{du}{\rho(u)} > \sup_{t \in \mathbf{R}} |x(t)|,
 \end{aligned}$$

which is a contradiction.

From above discussion, we know that BVP(1.2) has no positive solution. The proof is completed. \square

References

- [1] Agarwal, R.P., Boundary value problems for higher order differential equations. Singapore: World Scientific, 1986.
- [2] Avramescu, C., Vladimirescu, C., Existence of Homoclinic solutions to a nonlinear second order ODE. Dynamics of continuous, discrete and impulsive systems, Ser. A, Math Anal. 15 (2008), 481-491.
- [3] Avramescu, C., Vladimirescu, C., Existence of solutions to second order ordinary differential equations having finite limits at $\pm\infty$. Electronic Journal of differential equations 18 (2004), 1-12.
- [4] Avramescu, C., Vladimirescu, C., Limits of solutions of a perturbed linear differential equation. E. J. Qualitative Theory of Diff. Equ. 3 (2002), 1-11.
- [5] Bianconi, B., Papalini, F., Non-autonomous boundary value problems on the real line. Discrete and continuous dynamical systems 15 (2006), 759-776.
- [6] Boichuk, A. A., Pokutnyi, O. A., Dichotomy and boundary value problems on the whole line. Chaotic Modeling and Simulation (CMSIM) 2 (2013), 247-255.

- [7] Cabada, A., Cid, J.A., Heteroclinic solutions for non-autonomous boundary value problems with singular Φ -Laplacian operators. *Discrete Contin. Dyn. Syst.* 2009, *Dynamical Systems, Differential Equations and Applications. 7th AIMS Conference, suppl.*, 118–122.
- [8] Calamai, A., Heteroclinic solutions of boundary value problems on the real line involving singular Φ -Laplacian operators. *Journal of Mathematical Analysis and Applications* 378 (2011), 667-679.
- [9] Cupini, G., Marcelli, C., Papalini, F., Heteroclinic solutions of boundary value problems on the real line involving general nonlinear differential operators. *Differ. Integral Equ.* 24 (7-8) (2011), 619-644.
- [10] Cupini, G., Marcelli, C., Papalini, F., On the solvability of a boundary value problem on the real line. *Bound. Value Probl.* 26, 2011.
- [11] Deimling, K., *Nonlinear Functional Analysis*. Berlin: Springer, 1985.
- [12] Deren, F. Y., Hamal, N. A., Second-order boundary-value problems with integral boundary conditions on the real line. *Electronic Journal of Differential Equations* 19 (2014), 1-13.
- [13] Ge, W., *Boundary Value Problems for Ordinary Differential Equations*. Beijing: Science Press, 2007. (in Chinese)
- [14] Il'in, V. A., Moiseev, E. I., Nonlocal boundary-value problem of the second kind for a Sturm-Liouville operator. *Differential Equations* 23 (1987), 979-987.
- [15] Liu, Y., Multiple positive solutions to mixed boundary value problems for singular ordinary differential equations on the whole line. *Nonlinear Analysis: Modelling and Control* 17 (4) (2012), 460-480.
- [16] Liu, Y., Existence of multiple positive solutions of p -Laplacian boundary value problems, *Math. Slovak.* 57(2007), 225-242.
- [17] Liu, Y., Solutions to second order non-homogeneous multi-point BVPs using a fixed-point theorem. *Electronic Journal of Differential Equations* 96 (2008), 1-52.
- [18] Marcelli, C., Existence of solutions to some boundary-value problems governed by general nonlinear non-autonomous differential operators. *Electron. J. Differ. Equ.* 171, 2012.
- [19] Marcelli, C., The role of boundary data on the solvability of some equations involving non-autonomous nonlinear differential operators. *Boundary Value Problems* 252, 2013.
- [20] Philos, C. G., Purnaras, I. K., A boundary value problem on the whole line to second order nonlinear differential equations. *Georgian Mathematical Journal* 17 (2010), 241-252.
- [21] Prasad, K. R., Rao, A. K., Rao, S. N., Existence of positive solutions for the system of higher order two-point boundary value problems, *Proceedings of the Indian Academy of Sciences Mathematical Sciences* 122 (2012), 139-152.
- [22] Zhao, J., Geng, F., Zhao, J., Ge, W., Positive solutions to a new kind Sturm-Liouville-like four-point boundary value problem. *Appl. Math. Comput.* 217(2) (2010), 811-819.