ON $\phi$-SYMMETRIC LP-SASAKIAN MANIFOLDS ADMITTING SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. The object of the present paper is to study locally $\phi$-symmetric LP-Sasakian manifolds admitting a semi-symmetric metric connection and obtain a necessary and sufficient condition for a locally $\phi$-symmetric LP-Sasakian manifold with respect to semi-symmetric metric connection to be locally $\phi$-symmetric LP-Sasakian manifold with respect to the Levi-Civita connection.

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1. Introduction

Analogously to the Sasakian manifolds, in 1989 Matsumoto [12] introduced the notion of LP-Sasakian manifolds. Again the same notion was studied by Mihai and Rosca [13] and they obtained many results. LP-Sasakian manifolds were also studied by De et. al. [8], Shaikh et. al. ([15], [16], [17], [19]), Taleshian and Asghari [27], Venkatesha and Bagewadi [28] and many others. The notion of a local $\phi$-symmetry on a 3-dimensional LP-Sasakian manifold was studied by Shaikh and De [20].

In 1924 Friedmann and Schouten [10] introduced the notion of a semi-symmetric linear connection on a differentiable manifold. Then in 1932 Hayden [11] introduced the idea of metric connection with torsion on a Riemannian manifold. A systematic study of the semi-symmetric metric connection on a Riemannian manifold has been given by Yano [29] in 1970. Also semi-symmetric metric connection on a Riemannian manifold has been studied by Barua and Mukhopadhyay [1], Binh [1], Chaki and Chaki [3], Chaturvedi and Pandey [10], Shaikh and Hui [22], Sharfuddin and Hussain [24] and many others. Recently Shaikh and Jana studied the quarter-symmetric metric connection on a $(k, \mu)$-contact metric manifold [23].

The study of Riemann symmetric manifolds began with the work of Cartan [4]. A Riemannian manifold $(M^n, g)$ is said to be locally symmetric due to Cartan [4] if its curvature tensor $R$ satisfies the relation $\nabla R = 0$, where $\nabla$
denotes the operator of covariant differentiation with respect to the metric tensor $g$. As a weaker version of local symmetry, the notion of a locally $\phi$-symmetric Sasakian manifold was introduced by Takahashi [26]. Shaikh and Baishya [13] studied locally $\phi$-symmetric LP-Sasakian manifolds in the sense of Takahashi. The notion of locally $\phi$-symmetric manifolds in different structures has been studied by several authors (see, [7], [13], [18], [21], [26]). An LP-Sasakian manifold is said to be $\phi$-symmetric [7] if it satisfies the condition

$$\phi^2((\nabla W R)(X, Y)Z) = 0$$

for arbitrary vector fields $X, Y, Z$ and $W$ on $M$.

In particular, if $X, Y, Z, W$ are horizontal vector fields, i.e., orthogonal to $\xi$, then it is called a locally $\phi$-symmetric LP-Sasakian manifold [26].

It is easy to check that an LP-Sasakian manifold is $\phi$-symmetric if and only if it is locally symmetric or locally $\phi$-symmetric.

Recently De and Sarkar [9] studied $\phi$-Ricci symmetric Sasakian manifolds. In this connection Shukla and Shukla [25] studied $\phi$-Ricci symmetric Kenmotsu manifolds. An LP-Sasakian manifold is said to be $\phi$-Ricci symmetric [9] if it satisfies

$$\phi^2((\nabla X Q)(Y)) = 0,$$

where $Q$ is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$ for all vector fields $X, Y$.

If $X, Y$ are horizontal vector fields then the manifold is said to be locally $\phi$-Ricci symmetric.

It is easy to check that an LP-Sasakian manifold is $\phi$-Ricci symmetric if and only if it is Ricci symmetric or locally $\phi$-Ricci symmetric.

The object of the present paper is to study the locally $\phi$-symmetric and locally $\phi$-Ricci symmetric LP-Sasakian manifolds admitting semi-symmetric metric connection. The paper is organized as follows. Section 2 is concerned with some preliminaries about LP-Sasakian manifolds and semi-symmetric metric connections. Section 3 is devoted to the study of locally $\phi$-symmetric LP-Sasakian manifolds admitting a semi-symmetric metric connection and obtained a necessary and sufficient condition for a locally $\phi$-symmetric LP-Sasakian manifold with respect to semi-symmetric metric connection to be locally $\phi$-symmetric LP-Sasakian manifold with respect to the Levi-Civita connection. Section 4 deals with the study of locally $\phi$-Ricci symmetric LP-Sasakian manifolds admitting semi-symmetric metric connection.

2. Preliminaries

An $n$-dimensional smooth manifold $M$ is said to be an LP-Sasakian manifold ([13], [16]) if it admits a $(1, 1)$ tensor field $\phi$, a unit timelike vector field $\xi$, an 1-form $\eta$ and a Lorentzian metric $g$, which satisfy

$$\eta(\xi) = -1, \quad g(X, \xi) = \eta(X), \quad \phi^2 X = X + \eta(X)\xi,$$
(2.2) \[ g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad \nabla_X \xi = \phi X, \]

(2.3) \[ (\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \]

where \( \nabla \) denotes the operator of covariant differentiation with respect to the Lorentzian metric \( g \). It can be easily seen that in an LP-Sasakian manifold, the following relations hold:

(2.4) \[ \phi \xi = 0, \quad \eta \circ \phi = 0, \quad \text{rank} \phi = n - 1. \]

Again, if we take

\[ \Omega(X, Y) = g(X, \phi Y) \]

for any vector fields \( X, Y \), then the tensor field \( \Omega(X, Y) \) is a symmetric \((0,2)\) tensor field \([12]\). Also, since the vector field \( \eta \) is closed in an LP-Sasakian manifold, we have \([8, 12]\)

(2.5) \[ (\nabla_X \eta)(Y) = \Omega(X, Y), \quad \Omega(X, \xi) = 0 \]

for any vector fields \( X \) and \( Y \).

Let \( M \) be an \( n \)-dimensional LP-Sasakian manifold with structure \((\phi, \xi, \eta, g)\). Then the following relations hold \([15, 16]\):

(2.6) \[ R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \]

(2.7) \[ \eta(R(X, Y)Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z), \]

(2.8) \[ S(X, \xi) = (n - 1)\eta(X), \]

(2.9) \[ S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y), \]

(2.10) \[ (\nabla_W R)(X, Y)\xi = \Omega(Y, W)X - \Omega(X, W)Y - R(X, Y)\phi W, \]

(2.11) \[ (\nabla_W R)(X, Y)Y = \Omega(W, Z)X - g(X, Z)\phi W - R(X, \phi W)Z \]

for any vector fields \( X, Y, Z \), where \( R \) is the curvature tensor of \( g \).

Let \( M \) be an \( n \)-dimensional LP-Sasakian manifold and \( \nabla \) be the Levi-Civita connection on \( M \). A linear connection \( \tilde{\nabla} \) on \( M \) is said to be semi-symmetric if the torsion tensor \( \tau \) of the connection \( \tilde{\nabla} \)

\[ \tau(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] \]

satisfies

(2.12) \[ \tau(X, Y) = \eta(Y)X - \eta(X)Y \]
for all $X, Y \in \chi(M)$; $\chi(M)$ being the Lie algebra of all smooth vector fields on $M$. A semi-symmetric connection $\nabla$ is called semi-symmetric metric connection if it further satisfies

$$\nabla g = 0.$$  

(2.13)

A semi-symmetric metric connection $\nabla$ in an LP-Sasakian manifold is defined by ([24], [20]):

$$\nabla_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi.$$  

(2.14)

If $R$ and $\tilde{R}$ are respectively the curvature tensor of the Levi-Civita connection $\nabla$ and the semi-symmetric metric connection $\nabla$ in an LP-Sasakian manifold, then we have [14]

$$\tilde{R}(X,Y)Z = R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y - g(Y,Z)LX + g(X,Z)LY,$$

(2.15)

where $\alpha$ is a symmetric (0,2) tensor field given by

$$\alpha(X,Y) = (\nabla_X \eta)(Y) + \frac{1}{2}g(X,Y),$$  

(2.16)

$$LX = \nabla_X \xi + \frac{1}{2}X = \phi X - \frac{1}{2}X - \eta(X)\xi$$  

(2.17)

and

$$g(LX,Y) = \alpha(X,Y).$$  

(2.18)

**Lemma 2.1.** [14] In an LP-Sasakian manifold with semi-symmetric metric connection $\nabla$, we have

$$\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0,$$

(2.19)

$$g(\tilde{R}(X,Y)Z, U) = -g(\tilde{R}(Y,X)Z, U),$$  

(2.20)

$$g(\tilde{R}(X,Y)Z, U) = -g(\tilde{R}(X,Y)U, Z),$$  

(2.21)

$$g(\tilde{R}(X,Y)Z, U) = g(\tilde{R}(Z,U)X, Y).$$  

(2.22)

**Lemma 2.2.** [14] In an $n$-dimensional LP-Sasakian manifold the Ricci tensor $\tilde{S}$ and scalar curvature $\tilde{r}$ with respect to the semi-symmetric metric connection $\nabla$ are given by

$$\tilde{S}(X,Y) = S(X,Y) - (n-2)\alpha(X,Y) - ag(X,Y)$$  

(2.23)

and

$$\tilde{r} = r - 2(n-1)a,$$

(2.24)

where $a = \text{tr.} \alpha$, $S$ and $r$ denote the Ricci tensor and scalar curvature of the Levi-Civita connection $\nabla$ respectively.
Lemma 2.3. [13] Let $M$ be an $n$-dimensional LP-Sasakian manifold with the semi-symmetric metric connection $\tilde{\nabla}$. Then we have

\begin{align}
(2.25) \quad & g(\tilde{R}(X,Y)Z,\xi) = \eta(\tilde{R}(X,Y)Z) = (\tilde{\nabla}_X\eta)(Z)\eta(Y) - (\tilde{\nabla}_Y\eta)(Z)\eta(X), \\
(2.26) \quad & \tilde{R}(\xi,X)\xi = -\tilde{\nabla}_X\xi = X + \eta(X)\xi - \phi X, \\
(2.27) \quad & \tilde{R}(X,Y)\xi = \eta(X)\tilde{\nabla}_Y\xi - \eta(Y)\tilde{\nabla}_X\xi, \\
(2.28) \quad & \tilde{R}(\xi,X)Y = \eta(Y)\tilde{\nabla}_X\xi - g(Y,\tilde{\nabla}_X\xi)\xi, \\
(2.29) \quad & \tilde{S}(X,\xi) = \left(\frac{n}{2} - a\right)\eta(X), \\
(2.30) \quad & \tilde{S}(\phi X,\phi Y) = S(X,Y) + \left(\frac{n}{2} - a\right)\eta(X)\eta(Y) - (n - 2)\alpha(X,Y) - ag(X,Y)
\end{align}

for arbitrary vector fields $X$, $Y$ and $Z$.

From (2.22), (2.23), (2.24), (2.25) and (2.27), we get

\begin{align}
(2.31) \quad & (\tilde{\nabla}_W\tilde{R})(X,Y)\xi = R(X,Y)W - R(X,Y)\phi W + \alpha(X,W)Y \\
& - \alpha(Y,W)X + g(X,W)LX - g(Y,W)LY + \alpha(Y,\phi W)X + \Omega(Y,W)LX \\
& - \Omega(X,W)LY + g(X,W)Y - g(Y,W)X + \Omega(X,\phi W)Y + \Omega(Y,W)\phi X \\
& - \Omega(\phi W,W)Y + \Omega(X,\phi W)Y - \Omega(Y,W)\phi X \\
& - \eta(X)[g(Y,W) - \Omega(Y,W)]\xi \\
& - \eta(Y)[g(X,W) - \Omega(X,W)]\xi
\end{align}

for arbitrary vector fields $X$, $Y$ and $W$. Also from (2.22), (2.25) and (2.27), we have

\begin{align}
(2.32) \quad & g((\tilde{\nabla}_W\tilde{R})(X,Y)Z,U) = -g((\tilde{\nabla}_W\tilde{R})(X,Y)U,Z).
\end{align}

From (2.17) we have

\begin{align}
(2.33) \quad & \alpha(X,\xi) = \frac{1}{2}\eta(X), \\
(2.34) \quad & (\nabla_W\alpha)(X,\xi) = \frac{1}{2}\Omega(W,X) - \alpha(X,\phi W),
\end{align}
\[
(\nabla_W L)(X) = \left[ g(W, X) - \Omega(W, X) \right] \xi + \eta(X)[W - \phi W] + 2\eta(X)\eta(W)\xi.
\]

Again by the virtue of (2.33) - (2.35) we have from (2.14) and (2.15) that

\[
(\nabla_X S)(Y, Z) = (\nabla_X S)(Y, Z) - [S(X, Y) + \alpha(X, Y)]\eta(Z) + [\frac{3}{2}g(X, Z) + (n - 2)\Omega(X, Z)]\eta(Y) - (n - 2)[g(X, Y) - \Omega(X, Y)]\eta(Z) - da(X)g(Y, Z).
\]

Also from (2.38) we have

\[
(\nabla_X S)(Y, \xi) = (n - 1)\Omega(X, Y) - S(Y, \phi X).
\]

3. Locally \(\phi\)-symmetric LP-Sasakian manifolds admitting semi-symmetric metric connection

Definition 3.1. An LP-Sasakian manifold \(M\) is said to be locally \(\phi\)-symmetric with respect to a semi-symmetric metric connection if its curvature tensor \(\tilde{R}\) satisfies the condition

\[
\phi^2((\nabla_W \tilde{R})(X, Y) Z) = 0
\]

for all horizontal vector fields \(X, Y, Z\) and \(W\).

We now consider a locally \(\phi\)-symmetric LP-Sasakian manifold with respect to a semi-symmetric metric connection. Then by the virtue of (2.31) it follows from (3.1) that

\[
(\nabla_W \tilde{R})(X, Y) Z + \eta((\nabla_W \tilde{R})(X, Y) Z) \xi = 0.
\]

Using (2.32) in (3.2), we get

\[
(\nabla_W \tilde{R})(X, Y) Z = g((\nabla_W \tilde{R})(X, Y) \xi, Z) \xi.
\]
In view of (2.31) it follows from (3.3) that
\[
\begin{align*}
(e_\nabla W e R)(X;Y)Z &= [g(R(X,Y)W,Z) - g(R(X,Y)\phi W,Z) + \alpha(X,W)g(Y,Z) \\
- \alpha(Y,W)g(X,Z) + g(X,W)\alpha(Y,Z) - g(Y,W)\alpha(X,Z) \\
+ \alpha(Y,\phi W)g(X,Z) - \alpha(X,\phi W)g(Y,Z) + \Omega(Y,W)\alpha(X,Z) \\
- \Omega(X,W)\alpha(Y,Z) + g(X,W)\Omega(Y,Z) + \Omega(Y,W)g(X,Z) \\
- \Omega(X,W)g(Y,Z) + \Omega(X,W)\Omega(Y,Z) - \Omega(Y,W)\Omega(X,Z)]
\end{align*}
\]
for all horizontal vector fields \(X, Y, Z\) and \(W\). Next, let us assume that in an LP-Sasakian manifold, the relation (3.4) holds for all horizontal vector fields \(X, Y, Z\) and \(W\). Then it follows from (2.36) that (3.4) holds and consequently the manifold is locally \(\phi\)-symmetric with respect to a semi-symmetric metric connection. This leads to the following:

**Theorem 3.1.** An LP-Sasakian manifold is locally \(\phi\)-symmetric with respect to semi-symmetric metric connection if and only if the relation (3.4) holds for all horizontal vector fields \(X, Y, Z\) and \(W\).

In view of (2.32), it follows from (3.2) that
\[
(\nabla_W \hat{R})(X,Y)\xi = 0.
\]
From (2.31) and (3.5) it follows that
\[
R(X,Y)W - R(X,Y)\phi W = g(Y,W)X - g(X,W)Y + g(X,W)\phi Y - g(Y,W)\phi X \\
+ \Omega(X,W)Y - \Omega(Y,W)X + \Omega(Y,W)\phi X - \Omega(X,W)\phi Y \\
+ \alpha(Y,W)X - \alpha(X,W)Y + g(Y,W)LX - g(X,W)LX \\
+ \alpha(X,\phi W)Y - \alpha(Y,\phi W)X + \Omega(X,W)LX - \Omega(Y,W)LX
\]
for horizontal vector fields \(X, Y\) and \(W\). Contracting (3.6), we get
\[
S(Y,W) - S(Y,\phi W) = (n - 1 + a - \psi)[g(Y,W) - \Omega(Y,W)] \\
+ (n - 2)[\alpha(Y,W) - \alpha(Y,\phi W)],
\]
where \(\psi = \text{tr.} \Omega\) and \(a = \text{tr.} \alpha\). Hence we can state the following:

**Theorem 3.2.** In a locally \(\phi\)-symmetric LP-Sasakian manifold with a semi-symmetric metric connection, the curvature tensor and the Ricci tensor are respectively given by (3.6) and (3.7).

We now consider a locally \(\phi\)-symmetric LP-Sasakian manifold with the Levi-Civita connection. Then in [15], Shaikh and Baishya proved that
Theorem 3.3. An LP-Sasakian manifold \((M^n, g)\) is locally \(\phi\)-symmetric with respect to the Levi-Civita connection if and only if the following relation holds for arbitrary vector fields \(X, Y, Z, W \in \chi(M)\).

\[
\begin{align*}
(\nabla_W R)(X, Y)Z &= \\
&= [2\{\Omega(Y, W)g(X, Z) - \Omega(X, W)g(Y, Z)\} \\
&+ \Omega(Y, Z)g(X, W) - \Omega(X, Z)g(Y, W) \\
&+ 2\{\Omega(Y, Z)\eta(X)\eta(W) - \Omega(X, Z)\eta(Y)\eta(W)\} - g(\phi R(X, Y)W, Z)]\xi \\
&+ \eta(X)[\Omega(W, Z)Y - g(Y, Z)\phi W - R(Y, \phi W)Z] \\
&- \eta(Y)[\Omega(W, Z)X - g(X, Z)\phi W - R(X, \phi W)Z] \\
&- \eta(Z)[2\{\Omega(Y, W)X - \Omega(X, W)Y\} - \phi R(X, Y)W - g(Y, W)\phi X] \\
&+ g(X, W)\phi Y] + 2\{\eta(Y)\phi X - \eta(X)\phi Y\}\eta(Z)\eta(W).
\end{align*}
\]

(3.8)

holds for arbitrary vector fields \(X, Y, Z, W \in \chi(M)\).

Now we take a locally \(\phi\)-symmetric LP-Sasakian manifold with respect to a semi-symmetric metric connection. Then the relation (3.8) holds for any horizontal vector fields \(X, Y, Z, W\).

Let \(X, Y, Z, W\) be arbitrary vector fields of \(\chi(M)\). We now compute \((\tilde{\nabla}_{\tilde{\phi}W}\tilde{R})(\phi^2 X, \phi^2 Y)\phi^2 Z\) in two different ways. Firstly, by the virtue of (3.8), it follows from (3.8) that

\[
\begin{align*}
(\tilde{\nabla}_{\tilde{\phi}W}\tilde{R})(\phi^2 X, \phi^2 Y)\phi^2 Z &= \\
&= [g(R(\phi^2 X, \phi^2 Y)\phi^2 W, \phi^2 Z) - g(R(\phi^2 X, \phi^2 Y)\phi^3 W, \phi^2 Z) \\
&+ \alpha(\phi^2 X, \phi^2 W)\{g(Y, Z) + \eta(Y)\eta(Z)\} \\
&- \alpha(\phi^2 Y, \phi^2 W)\{g(X, Z) + \eta(X)\eta(Z)\} \\
&+ \alpha(\phi^2 Y, \phi^2 Z)\{g(X, W) + \eta(X)\eta(W)\} \\
&- \alpha(\phi^2 X, \phi^2 Z)\{g(Y, W) + \eta(Y)\eta(W)\} \\
&+ \alpha(\phi^2 Y, \phi^3 W)\{g(X, Z) + \eta(X)\eta(Z)\} \\
&- \alpha(\phi^2 X, \phi^3 W)\{g(Y, Z) + \eta(Y)\eta(Z)\} \\
&+ \Omega(Y, W)\alpha(\phi^2 X, \phi^2 Z) - \Omega(X, W)\alpha(\phi^2 Y, \phi^2 Z) \\
&+ \{g(X, W) + \eta(X)\eta(W)\}\{g(Y, Z) + \eta(Y)\eta(Z)\} \\
&- \{g(Y, W) + \eta(Y)\eta(W)\}\{g(X, Z) + \eta(X)\eta(Z)\} \\
&+ \{g(Y, W) + \eta(Y)\eta(W)\}\Omega(Y, Z) - \{g(X, W) + \eta(X)\eta(W)\}\Omega(Y, Z) \\
&+ \{g(X, Z) + \eta(X)\eta(Z)\}\Omega(Y, W) - \{g(Y, Z) + \eta(Y)\eta(Z)\}\Omega(X, W) \\
&+ \Omega(X, W)\Omega(Y, Z) - \Omega(Y, W)\Omega(X, Z)\}\xi.
\end{align*}
\]

(3.9)

From (3.8) we have

\[
(3.10) \quad g(\phi^2 X, \xi) = g(\phi^2 Y, \xi) = g(\phi^2 Z, \xi) = 0
\]

and hence \(\phi^2 X, \phi^2 Y, \phi^2 Z\) are horizontal vector fields of \(\chi(M)\). Then by the virtue of (3.8) it follows that

\[
\begin{align*}
R(\phi^2 X, \phi^2 Y)\phi^2 W &= R(X, Y)W + \{\eta(Y)X - \eta(X)Y\}\eta(W) \\
&+ \{g(Y, W)\eta(X) - g(X, W)\eta(Y)\}\xi,
\end{align*}
\]

(3.11)
(3.12) \( R(\phi^2 X, \phi^2 Y)\phi^3 W = R(X, Y)\phi W + \{\Omega(Y, W)\eta(X) - \Omega(X, W)\eta(Y)\}\xi, \)

(3.13) \( \alpha(\phi^2 X, \phi^2 W) = \alpha(X, W) + \frac{1}{2} \eta(X)\eta(W). \)

In view of (3.11) - (3.14), (3.11) yields

\[
(3.14) \quad (\tilde{\nabla}_{\phi^2 W} \tilde{R})(\phi^2 X, \phi^2 Y)\phi^2 Z
= [g(R(X, Y)W, Z) - g(R(X, Y)\phi W, Z) + \alpha(X, W)\{g(Y, Z) + \eta(Y)\eta(Z)\}
- \alpha(Y, W)\{g(X, Z) + \eta(X)\eta(Z)\} + \frac{1}{2}\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\}\eta(W)
+ \alpha(Y, Z)g(X, W) - \alpha(X, Z)g(Y, W) + \frac{1}{2}\{\eta(Y)g(X, W) - \eta(X)g(Y, W)\}\eta(Z)
+ \{\eta(Y)\alpha(Y, Z) - \eta(Y)\alpha(X, Z)\}\eta(W) + \alpha(Y, \phi W)g(X, Z) - \alpha(X, \phi W)g(Y, Z)
+ \{\eta(X)\alpha(Y, \phi W) - \eta(Y)\alpha(X, \phi W)\}\eta(Z) + \Omega(Y, W)\alpha(X, Z) - \Omega(X, W)\alpha(Y, Z)
+ \frac{1}{2}\{\eta(X)\Omega(Y, W) - \eta(Y)\Omega(X, W)\}\eta(Z) + g(X, W)g(Y, Z) - g(Y, W)g(X, Z)
+ g(Y, W)\Omega(X, Z) - g(X, W)\Omega(Y, Z) + \{\eta(Y)\Omega(X, Z) - \eta(X)\Omega(Y, Z)\}\eta(W)
+ \Omega(Y, W)g(X, Z) - \Omega(X, W)g(Y, Z) + \{\eta(X)\Omega(Y, W) - \eta(Y)\Omega(X, W)\}\eta(Z)
+ \Omega(X, W)\Omega(Y, Z) - \Omega(Y, W)\Omega(X, Z)\] \xi.

By the virtue of (2.11) we have

\[
(3.15) \quad (\tilde{\nabla}_{\phi^2 W} \tilde{R})(\phi^2 X, \phi^2 Y)\phi^2 Z = (\tilde{\nabla}_W \tilde{R})(\phi^2 X, \phi^2 Y)\phi^2 Z
+ \eta(W)(\tilde{\nabla}_\xi \tilde{R})(\phi^2 X, \phi^2 Y)\phi^2 Z.
\]

Now, for any horizontal vector fields \( X, Y \) and \( Z \), we have from (3.11) that

\[
(3.16) \quad (\tilde{\nabla}_\xi \tilde{R})(X, Y)Z = 0,
\]

which implies that

\[
(3.17) \quad (\tilde{\nabla}_\xi \tilde{R})(\phi^2 X, \phi^2 Y)\phi^2 Z = 0.
\]

Using (3.15) in (3.16) we obtain

\[
(3.18) \quad (\tilde{\nabla}_{\phi^2 W} \tilde{R})(\phi^2 X, \phi^2 Y)\phi^2 Z = (\tilde{\nabla}_W \tilde{R})(\phi^2 X, \phi^2 Y)\phi^2 Z.
\]

In view of (2.11), we have

\[
(3.19) \quad (\tilde{\nabla}_W \tilde{R})(\phi^2 X, \phi^2 Y)\phi^2 Z
= (\tilde{\nabla}_W \tilde{R})(X, Y)Z + \eta(Z)(\tilde{\nabla}_W \tilde{R})(X, Y)\xi
+ \eta(Y)(\tilde{\nabla}_W \tilde{R})(X, \xi)Z + \eta(Y)\eta(Z)(\tilde{\nabla}_W \tilde{R})(X, \xi)\xi
+ \eta(X)(\tilde{\nabla}_W \tilde{R})(\xi, Y)Z + \eta(X)\eta(Z)(\tilde{\nabla}_W \tilde{R})(\xi, Y)\xi.
\]
Using (2.30) in (3.19) we get

\[(3.20) \ (\tilde{\nabla}_W \tilde{R})(\phi^2 X, \phi^2 Y)\phi^2 Z \]
\[= (\tilde{\nabla}_W \tilde{R})(X, Y)Z - \eta(Z)R(X, Y)\phi W - \eta(Y)R(X, \phi W)Z + \eta(X)R(Y, \phi W)Z \]
\[+ \frac{1}{2} \left[ \eta(Z)\{\Omega(Y, W)X - \Omega(X, W)Y\} + \eta(Y)\Omega(W, Z)X - \eta(X)\Omega(W, Z)Y \right] \]
\[- \eta(Z)\{\alpha(Y, \phi W)X - \alpha(X, \phi W)Y\} + \eta(Y)\alpha(Z, \phi W)X - \eta(X)\alpha(Z, \phi W)Y \]
\[- \eta(X)\{\alpha(Y, Z)W - \eta(W)\alpha(Y, Z)\xi\} + \eta(Y)\eta(Z)\{\alpha(X, W)\xi - \alpha(X, \phi W)\xi\} \]
\[- \eta(X)\eta(Z)\{\alpha(Y, W)\xi - \alpha(Y, \phi W)\xi\} - \frac{1}{2}\eta(X)\{g(Y, Z)W - \eta(W)g(Y, Z)\xi\} \]
\[+ \frac{1}{2}\eta(Y)\{g(X, Z)W - \eta(W)g(X, Z)\xi\}. \]

From (3.18) and (3.20) we get

\[(3.21) \ (\tilde{\nabla}_{\phi^2 W} \tilde{R})(\phi^2 X, \phi^2 Y)\phi^2 Z \]
\[= (\tilde{\nabla}_W \tilde{R})(X, Y)Z - \eta(Z)R(X, Y)\phi W - \eta(Y)R(X, \phi W)Z + \eta(X)R(Y, \phi W)Z \]
\[+ \frac{1}{2} \left[ \eta(Z)\{\Omega(Y, W)X - \Omega(X, W)Y\} + \eta(Y)\Omega(W, Z)X - \eta(X)\Omega(W, Z)Y \right] \]
\[- \eta(Z)\{\alpha(Y, \phi W)X - \alpha(X, \phi W)Y\} + \eta(Y)\alpha(Z, \phi W)X - \eta(X)\alpha(Z, \phi W)Y \]
\[- \eta(X)\{\alpha(Y, Z)W - \eta(W)\alpha(Y, Z)\xi\} + \eta(Y)\eta(Z)\{\alpha(X, W)\xi - \alpha(X, \phi W)\xi\} \]
\[- \eta(X)\eta(Z)\{\alpha(Y, W)\xi - \alpha(Y, \phi W)\xi\} - \frac{1}{2}\eta(X)\{g(Y, Z)W - \eta(W)g(Y, Z)\xi\} \]
\[+ \frac{1}{2}\eta(Y)\{g(X, Z)W - \eta(W)g(X, Z)\xi\}. \]

From (3.14) and (3.21) we obtain

\[(3.22) \ (\tilde{\nabla}_W \tilde{R})(X, Y)Z \]
\[= \left[ g(R(X, Y)W, Z) - g(R(X, Y)\phi W, Z) \right] \]
\[+ \alpha(Y, Z)g(X, W) - \alpha(X, Z)g(Y, W) \]
\[+ \frac{1}{2}\{\eta(Y)g(X, W) - \eta(X)g(Y, W)\} \eta(Z) \]
\[- \eta(Y)\eta(W)\alpha(X, Z) + \alpha(Y, \phi W)g(X, Z) - \alpha(X, \phi W)g(Y, Z) \]
\[+ \Omega(Y, W)\alpha(X, Z) - \Omega(X, W)\alpha(Y, Z) \]
\[+ \frac{1}{2}\{\eta(X)\Omega(Y, W) - \eta(Y)\Omega(X, W)\} \eta(Z) \]
\[+ g(X, W)g(Y, Z) - g(Y, W)g(X, Z) + g(Y, W)\Omega(X, Z) - g(X, W)\Omega(Y, Z) \]
\[+ \{\eta(Y)\Omega(X, Z) - \eta(X)\Omega(Y, Z)\} \eta(W) \]
\[+ \Omega(Y, W)g(X, Z) - \Omega(X, W)g(Y, Z) \]
+ \{\eta(X)\Omega(Y, W) - \eta(Y)\Omega(X, W)\}\eta(Z)
+ \Omega(X, W)\Omega(Y, Z) - \Omega(Y, W)\Omega(X, Z)\xi
+ \eta(Z)R(X, Y)\phi W + \eta(Y)R(X, \phi W)Z - \eta(X)R(Y, \phi W)Z
- \frac{1}{2}\{\eta(Z)\Omega(Y, W)X - \Omega(X, W)Y\}
+ \eta(Y)\Omega(W, Z)X - \eta(X)\Omega(W, Z)Y
+ \eta(Z)\{\alpha(Y, \phi W)X - \alpha(X, \phi W)Y\}
- \eta(Y)\alpha(Z, \phi W)X + \eta(X)\alpha(Z, \phi W)Y
+ \eta(X)\alpha(Y, Z)W + \frac{1}{2}\{\eta(X)g(Y, Z)W - \eta(Y)g(X, Z)W\}.

Thus in a locally $\phi$-symmetric LP-Sasakian manifold with respect to a semi-symmetric metric connection, the relation (3.22) holds for any $X, Y, Z, W \in \chi(M)$.

Next, if the relation (3.22) holds in an LP-Sasakian manifold with respect to semi-symmetric metric connection then for any horizontal vector fields $X, Y, Z, W$, we obtain the relation (3.23) and hence the manifold is locally $\phi$-symmetric with respect to semi-symmetric metric connection. Thus we can state the following:

**Theorem 3.4.** An LP-Sasakian manifold $(M^n, g)$ is locally $\phi$-symmetric with respect to a semi-symmetric metric connection if and only if the relation (3.22) holds for any vector fields $X, Y, Z, W \in \chi(M)$.

In view of (3.21), (3.22) yields

\begin{equation}
(\nabla_\nu R)(X, Y)Z
= \left[\Omega(W, Y) - g(W, Y)\right]\eta(Z)X + \left[\Omega(W, Z) - g(W, Z)\right]\eta(Y)X
+ 2\eta(Z)\eta(W)\left[\eta(X)Y - \eta(Y)X\right] + [\alpha(Y, Z)\eta(X) - \alpha(X, Z)\eta(Y)]W
+ \left[g(W, X) - \Omega(W, X)\right]\eta(Z)Y + \left[g(W, Z) - \Omega(W, Z)\right]\eta(X)Y
+ g(Y, Z)\eta(X)\left[\frac{1}{2}W - \phi W\right] - g(X, Z)\eta(Y)\left[\frac{1}{2}W - \phi W\right]
+ \eta(Z)R(X, Y)\phi W + \eta(Y)R(X, \phi W)Z - \eta(X)R(Y, \phi W)Z
+ \frac{1}{2}\{\eta(X)\Omega(W, Z)Y - \eta(Y)\Omega(W, Z)X
- \eta(Z)\{\Omega(Y, W)X - \Omega(X, W)Y\}\}
+ \eta(Z)\{\alpha(Y, \phi W)X - \alpha(X, \phi W)Y\}
- \eta(Y)\alpha(Z, \phi W)X + \eta(X)\alpha(Z, \phi W)Y
+ \eta(X)\alpha(Y, Z)W + \frac{1}{2}\{\eta(X)g(Y, Z)W - \eta(Y)g(X, Z)W\}
+ \left[2\{\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\}\eta(Y) - g(R(X, Y)\phi W, Z)\right].
\end{equation}
to a semi-symmetric metric connection, the relation 

\[ \phi \]

satisfies

\[ \Omega(X, W) \alpha(X, Z) - \alpha(X, \phi W) g(Y, Z) - \alpha(X, \phi W) g(Y, Z) \]

\[ + \Omega(Y, W) \alpha(X, Z) - \Omega(X, W) \alpha(Y, Z) + \frac{3}{2} \{ \eta(X) \Omega(Y, W) \}

\[ - \eta(Y) \Omega(X, W) \} \eta(Z) + g(Y, W) \Omega(X, Z) - g(X, W) \Omega(Y, Z) \]

\[ + \{ \eta(Y) \Omega(X, Z) - \eta(X) \Omega(Y, Z) \} \eta(W) + 2 \{ \Omega(Y, W) g(X, Z) \]

\[ - \Omega(X, W) g(Y, Z) \} + \Omega(X, W) \Omega(Y, Z) - \Omega(Y, W) \Omega(X, Z) \]

This leads to the following:

**Theorem 3.5.** In a locally \( \phi \)-symmetric LP-Sasakian manifold with respect to a semi-symmetric metric connection, the relation (3.23) holds for any vector fields \( X, Y, Z, W \in \chi(M) \).

From (3.23) and (3.24), we can state the following:

**Theorem 3.6.** A locally \( \phi \)-symmetric LP-Sasakian manifold is invariant under a semi-symmetric metric connection if and only if the relation

\[ [\Omega(W, Y) - g(W, Y)] \eta(Z) X + [\Omega(W, Z) - g(W, Z)] \eta(Y) X \]

\[ + 2 \eta(Z) \eta(W) [\eta(X) Y - \eta(Y) X] + [\alpha(Y, Z) \eta(X) - \alpha(X, Z) \eta(Y)] W \]

\[ + [g(W, X) - \Omega(W, X)] \eta(Z) Y + [g(W, Z) - \Omega(W, Z)] \eta(X) Y \]

\[ + \frac{1}{2} [g(Y, Z) \eta(X) - g(X, Z) \eta(Y)] W + \eta(Z) [g(Y, W) \phi X - g(X, W) \phi Y] \]

\[ - \frac{1}{2} [\eta(X) \Omega(W, Z) Y - \eta(Y) \Omega(W, Z) X - \eta(Z) \{ \Omega(Y, W) X - \Omega(X, W) Y \}] \]

\[ + \eta(Z) \{ \alpha(Y, \phi W) X - \alpha(X, \phi W) Y \} - \eta(Y) \alpha(Z, \phi W) X + \eta(X) \alpha(Z, \phi W) Y \]

\[ + \eta(X) \alpha(Y, Z) W + \frac{1}{2} \{ \eta(X) g(Y, Z) W - \eta(Y) g(X, Z) W \} \]

\[ + \left[ 2 \{ \eta(X) g(Y, Z) - \eta(Y) g(X, Z) \} \eta(Y) + \frac{1}{2} \{ \eta(Y) g(X, W) - \eta(X) g(Y, W) \} \eta(Z) \right. \]

\[ - \eta(Y) \eta(W) \alpha(X, Z) + \alpha(Y, \phi W) g(X, Z) - \alpha(X, \phi W) g(Y, Z) + \Omega(Y, W) \alpha(X, Z) \]

\[ - \Omega(X, W) \alpha(Y, Z) + \frac{3}{2} \{ \eta(X) \Omega(Y, W) - \eta(Y) \Omega(X, W) \} \eta(Z) + g(Y, W) \Omega(X, Z) \]

\[ - g(X, W) \Omega(Y, Z) + \{ \eta(Y) \Omega(X, Z) - \eta(X) \Omega(Y, Z) \} \eta(W) + \Omega(Y, W) g(X, Z) \]

\[ - \Omega(X, W) g(Y, Z) + \Omega(X, W) \Omega(Y, Z) - \Omega(Y, W) \Omega(X, Z) \]

\[ \xi = 0 \]

holds for arbitrary vector fields \( X, Y, Z, W \in \chi(M) \).

4. **Locally \( \phi \)-Ricci symmetric LP-Sasakian manifolds admitting semi-symmetric connection**

**Definition 4.1.** An LP-Sasakian manifold \( M \) is said to be locally \( \phi \)-Ricci symmetric with respect to the semi-symmetric metric connection if its satisfies
the condition
\begin{equation}
\phi^2((\tilde{\nabla}_X \tilde{Q})(Y)) = 0
\end{equation}
for horizontal vector fields \( X \) and \( Y \), where \( \tilde{Q} \) is the Ricci-operator with respect to the semi-symmetric metric connection \( \tilde{\nabla} \), i.e. \( g(\tilde{Q}X, Y) = \tilde{S}(X, Y) \) for all vector fields \( X, Y \).

Let us take an LP-Sasakian manifold, which is \( \phi \)-Ricci symmetric with respect to semi-symmetric metric connection \( \tilde{\nabla} \). Then by the virtue of (2.1) it follows from (4.1) that
\begin{equation}
(\tilde{\nabla}_X \tilde{Q})(Y) + \eta((\tilde{\nabla}_X \tilde{Q})(Y))\xi = 0
\end{equation}
from which it follows that
\begin{equation}
(\tilde{\nabla}_X \tilde{S})(Y, Z) = 0
\end{equation}
for all horizontal vector fields \( X \) and \( Y \) and \( Z \).

Let \( X, Y, Z \) be arbitrary vector fields of \( \chi(M) \). We now compute
\begin{equation}
(\tilde{\nabla}_{\phi^2 X} \tilde{S})(\phi^2 Y, \phi^2 Z)
\end{equation}
in two different ways. Since \( \phi^2 X, \phi^2 Y, \phi^2 Z \) are horizontal vector fields for all \( X, Y, Z \in \chi(M) \), from (E.4) we have
\begin{equation}
(\tilde{\nabla}_{\phi^2 X} \tilde{S})(\phi^2 Y, \phi^2 Z) = 0
\end{equation}
for all \( X, Y, Z \in \chi(M) \). By the virtue of (4.4) we get
\begin{equation}
(\tilde{\nabla}_{\phi^2 X} \tilde{S})(\phi^2 Y, \phi^2 Z) = (\tilde{\nabla}_X \tilde{S})(\phi^2 Y, \phi^2 Z) + \eta(X)(\tilde{\nabla}_\xi \tilde{S})(\phi^2 Y, \phi^2 Z).
\end{equation}
Now for any horizontal vector fields \( Y \) and \( Z \) we have from (E.4) that
\begin{equation}
(\tilde{\nabla}_\xi \tilde{S})(Y, Z) = 0,
\end{equation}
which implies that
\begin{equation}
(\tilde{\nabla}_\xi \tilde{S})(\phi^2 Y, \phi^2 Z) = 0
\end{equation}
for arbitrary vector fields \( Y, Z \in \chi(M) \).

Using (E.2) in (E.3) we get
\begin{equation}
(\tilde{\nabla}_{\phi^2 X} \tilde{S})(\phi^2 Y, \phi^2 Z) = (\tilde{\nabla}_X \tilde{S})(\phi^2 Y, \phi^2 Z).
\end{equation}
In view of (E.1), we get
\begin{equation}
(\tilde{\nabla}_X \tilde{S})(\phi^2 Y, \phi^2 Z) = (\tilde{\nabla}_X \tilde{S})(Y, Z) + \eta(Y)(\tilde{\nabla}_X \tilde{S})(Z, \xi) + \eta(Z)(\tilde{\nabla}_X \tilde{S})(Z, \xi) + \eta(Y)\eta(Z)(\tilde{\nabla}_X \tilde{S})(\xi, \xi).
\end{equation}
Using (2.37) in (4.7) we get

\[(\nabla_X \tilde{S})(\phi^2 Y, \phi^2 Z) = (\nabla_X S)(Y, Z) - \eta(Z)S(Y, \phi X)
+ \eta(Y)[S(X, Z) - S(Z, \phi X)] + \eta(Y)\alpha(X, Z)
+ [(2n - 1)\eta(X) - da(X)]\eta(Y)\eta(Z)
+ (n - 1)\eta(Z)\Omega(X, Y) - (n - 3)\eta(Y)\Omega(X, Z)
+ (n - \frac{1}{2})\eta(Y)g(X, Z) - da(X)g(Y, Z).\]

By the virtue of (4.3) and (4.8) we obtain from (4.7) that

\[(\nabla_X S)(Y, Z) = \eta(Z)S(Y, \phi X) - \eta(Y)[S(X, Z) - S(Z, \phi X)]
- \eta(Y)\alpha(X, Z) - [(2n - 1)\eta(X) - da(X)]\eta(Y)\eta(Z)
- (n - 1)\eta(Z)\Omega(X, Y) + (n - 3)\eta(Y)\Omega(X, Z)
- (n - \frac{1}{2})\eta(Y)g(X, Z) + da(X)g(Y, Z).\]

Thus in a locally \(\phi\)-Ricci symmetric LP-Sasakian manifold with respect to a semi-symmetric metric connection, the relation (4.9) holds for any \(X, Y, Z \in \chi(M)\).

Next if the relation (4.9) holds in an LP-Sasakian manifold with respect to a semi-symmetric metric connection then for any horizontal vector fields \(X, Y, Z\) with \(\text{tr.} \alpha = \text{constant}\), we obtain \((\nabla_X S)(Y, Z) = 0\) and hence the manifold is locally \(\phi\)-Ricci symmetric with respect to a semi-symmetric metric connection. Thus we can state the following:

**Theorem 4.1.** An LP-Sasakian manifold \((M^n, g)\) is locally \(\phi\)-Ricci symmetric with respect to a semi-symmetric metric connection with \(\text{tr.} \alpha = \text{constant}\) if and only if the relation (4.9) holds for any vector fields \(X, Y, Z \in \chi(M)\).

Putting \(Y = \xi\) in (4.4) and using (2.38), we get

\[S(X, Z) = 2(n - 2)\Omega(X, Z) - \alpha(X, Z)
- (n - \frac{1}{2})g(X, Z) + (2n - 1)\eta(X)\eta(Z)\]

for any vector fields \(X, Z \in \chi(M)\).

This leads to the following:

**Theorem 4.2.** In a locally \(\phi\)-Ricci symmetric LP-Sasakian manifold with respect to a semi-symmetric metric connection, the Ricci tensor is of the form (4.10).

**References**

On $\phi$-symmetric LP-Sasakian manifolds


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