ON NEARLY QUASI-EINSTEIN WARPED PRODUCTS

Buddhadev Pal\textsuperscript{1} and Arindam Bhattacharyya\textsuperscript{2}

Abstract. We study nearly quasi-Einstein warped product manifolds for arbitrary dimension $n \geq 3$. In the last section we also give an example of warped product on nearly quasi-Einstein manifold.

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1. Introduction

A Riemannian manifold $(M^n, g)$, $(n > 2)$ is Einstein if its Ricci tensor $S$ of type $(0,2)$ is of the form $S = \alpha g$, where $\alpha$ is smooth function, which turns into $S = \frac{r}{n} g$, $r$ being the scalar curvature of the manifold. Let $(M^n, g)$, $(n > 2)$ be a Riemannian manifold and $U_S = \{ x \in M : S \neq \frac{r}{n} g \text{ at } x \}$, then the manifold $(M^n, g)$ is said to be quasi-Einstein manifold \cite{1,2} if on $U_S \subset M$, we have

\begin{equation}
S - \alpha g = \beta A \otimes A,
\end{equation}

where $A$ is a 1-form on $U_S$ and $\alpha$ and $\beta$ some functions on $U_S$. It is clear that the 1-form $A$ as well as the function $\beta$ are nonzero at every point on $U_S$. From the above definition, it follows that every Einstein manifold is quasi-Einstein. In particular, every Ricci-flat manifold (e.g., Schwarzschild spacetime) is quasi-Einstein. The scalars $\alpha$, $\beta$ are known as the associated scalars of the manifold. Also, the 1-form $A$ is called the associated 1-form of the manifold defined by $g(X, \rho) = A(X)$ for any vector field $X$, $\rho$ being a unit vector field, called the generator of the manifold. Such an $n$-dimensional quasi-Einstein manifold is denoted by $(QE)_n$.

In \cite{3}, De and Gazi introduced nearly quasi-Einstein manifold, denoted by $N(QE)_n$ and gave an example of a 4-dimensional Riemannian nearly quasi Einstein manifold, where the Ricci tensor $S$ of type $(0,2)$ which is not identically zero satisfies the condition

\begin{equation}
S(X,Y) = l g(X,Y) + m D(X,Y),
\end{equation}

where $l$ and $m$ are non-zero scalars and $D$ is a non-zero symmetric tensor of type $(0,2)$.

\textsuperscript{1}Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi - 221 005, India. e-mail: pal.buddha@gmail.com

\textsuperscript{2}Department of Mathematics, Jadavpur University, Kolkata 700032, e-mail: bhattachar1968@yahoo.co.in
Also in [2], De and Gazi introduced the notion of a Riemannian manifold \((M, g)\) of a nearly quasi-constant sectional curvature as a Riemannian manifold with the curvature tensor satisfies the condition

\[
R(X, Y, Z, W) = a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + b[g(Y, Z)D(X, W)
- g(X, Z)D(Y, W) + g(X, W)D(Y, Z) - g(Y, W)D(X, Z)],
\]

where \(a, b\) are scalar functions with \(b \neq 0\) and \(D\) is nonzero symmetric \((0,2)\) tensor.

Let \(M\) be an \(m\)-dimensional, \(m \geq 3\), Riemannian manifold and \(p \in M\). Denote by \(K(\pi)\) or \(K(u \wedge v)\) the sectional curvature of \(M\) associated with a plane section \(\pi \subset T_p M\), where \(\{u, v\}\) is an orthonormal basis of \(\pi\). For any \(n\)-dimensional subspace \(L \subseteq T_p M\), \(2 \leq n \leq m\), its scalar curvature \(\tau(L)\) is denoted in [4] by \(\tau(L) = 2\Sigma_{1 \leq i < j \leq n} K(e_i \wedge e_j)\), where \(\{e_1, e_2, ..., e_n\}\) is any orthonormal basis of \(L\) [2]. When \(L = T_p M\), the scalar curvature \(\tau(L)\) is just the scalar curvature \(\tau(p)\) of \(M\) at \(p\).

\section{Warped product manifolds}

The notion of warped product generalizes that of a surface of revolution. It was introduced in [3] for studying manifolds of negative curvature. Let \((B, g_B)\) and \((F, g_F)\) be two Riemannian manifolds and let \(f\) be a positive differentiable function on \(B\). Consider the product manifold \(B \times F\) with its projections \(\pi : B \times F \to B\) and \(\sigma : B \times F \to F\). The warped product \(B \times_f F\) is the manifold \(B \times F\) with the Riemannian structure such that \(||X||^2 = ||\pi^*(X)||^2 + f^2(\pi(p))||\sigma^*(X)||^2\), for any vector field \(X\) on \(M\). Thus we have

\[
g = g_B + f^2 g_F
\]

holds on \(M\). The function \(f\) is called the warping function of the warped product [5].

Since \(B \times_f F\) is a warped product, then we have \(\nabla_X Z = \nabla_Z X = (Xlnf)Z\) for unit vector fields \(X, Z\) on \(B\) and \(F\), respectively. Hence, we find \(K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = (1/f)\{\nabla_X X_f - X^2 f\}\). If we chose a local orthonormal frame \(e_1, ..., e_n\) such that \(e_1, ..., e_{n_1}\) are tangent to \(B\) and \(e_{n_1+1}, ..., e_n\) are tangent to \(F\), then we have

\[
\frac{\Delta f}{f} = \sum_{i=1}^{n} K(e_i \wedge e_j),
\]

for each \(s = n_1 + 1, ..., n\) [3]. We need the following two lemmas from [4], for later use:

\textbf{Lemma 2.1.} Let \(M = B \times_f F\) be a warped product, with Riemannian curvature tensor \(R_M\). Given field \(X, Y, Z\) on \(B\) and \(U, V, W\) on \(F\), then:

1. \(R_M(X, Y)Z = R_B(X, Y)Z\),
2. \(R_M(V, X)Y = -(H^f(X, Y)/f)V\), where \(H^f\) is the Hessian of \(f\),
constant scalar curvature 

\[ X \]

Einstein manifold. Also in the last section we give an example of warped product on nearly quasi-Einstein warped products to generalize Theorems 2.2.

Theorem 2.2.

Let \( M = B \times_f F \) be a warped product, with Ricci tensor \( S_M \).

Given fields \( X, Y \) on \( B \) and \( V, W \) on \( F \), then:

1. \( S_M(X, Y) = S_B(X, Y) - \frac{d}{f^2} H^f(X, Y) \), where \( d = \dim F \)

2. \( S_M(X, V) = 0 \)

3. \( S_M(V, W) = S_F(V, W) - g(V, W) f^# \), \( f^# = \frac{\Delta f}{f} + \frac{d-1}{f^2} \| \grad f \|^2 \), where \( \Delta f \) is the Laplacian of \( f \) on \( B \).

Moreover, the scalar curvature \( \tau_M \) of the manifold \( M \) satisfies the condition

\[ \tau_M = \tau_B + \frac{\tau_F}{f^2} - 2d \frac{\Delta f}{f} - d(d-1) \frac{\| \grad f \|^2}{f^2}, \]

where \( \tau_B \) and \( \tau_F \) are the scalar curvatures of \( B \) and \( F \), respectively.

In [8], Gebarowski studied Einstein warped product manifolds and proved the following three theorems.

Theorem 2.1.

Let \( (M, g) \) be a warped product \( I \times_f F \), \( \dim I = 1 \), \( \dim F = n-1 \) \((n \geq 3)\). Then \( (M, g) \) is an Einstein manifold if and only if \( F \) is Einstein with constant scalar curvature \( \tau_F \) in the case \( n = 3 \) and \( f \) is given by one of the following formulae, for any real number \( b \),

\[ f^2(t) = \begin{cases} \frac{4}{a} K \sinh^2 \frac{\sqrt{a}(t+b)}{2} , & a > 0 \\ K(t + b)^2 , & a = 0 \\ -\frac{4}{a} K \sin^2 \frac{\sqrt{a}(t+b)}{2} , & a < 0 \end{cases} \]

for \( K > 0 \), \( f^2(t) = b \exp(at) \ (a \neq 0) \), for \( K = 0 \), \( f^2(t) = -\frac{4}{a} K \cosh^2 \frac{\sqrt{a}(t+b)}{2} \), \((a > 0)\), for \( K < 0 \), where \( a \) is the constant appearing after first integration of the equation \( q^{''} e^q + 2K = 0 \) and \( K = \frac{\tau_F}{(n-1)(n-2)} \).

Theorem 2.2.

Let \( (M, g) \) be a warped product \( B \times_f F \) of a complete connected \( r \)-dimensional \((1 < r < n)\) Riemannian manifold \( B \) and an \((n-r)\)-dimensional Riemannian manifold \( F \). If \( (M, g) \) is a space of constant sectional curvature \( K > 0 \), then \( B \) is a sphere of radius \( \frac{1}{\sqrt{K}} \).

Theorem 2.3.

Let \( (M, g) \) be a warped product \( B \times_f F \) of a complete connected \( n-1 \)-dimensional Riemannian manifold \( B \) and an one-dimensional Riemannian manifold \( F \). If \( (M, g) \) is an Einstein manifold with scalar curvature \( \tau_M > 0 \) and the Hessian of \( f \) is proportional to the metric tensor \( g_B \), then

1. \((B, g_B)\) is an \((n-1)\)-dimensional sphere of radius \( \rho = \left(\frac{\tau_B}{(n-1)(n-2)}\right)^{-\frac{1}{2}} \).

2. \((M, g)\) is a space of constant sectional curvature \( K = \frac{\tau_M}{n(n-1)} \).

Motivated by the above study by Gebarowski, in the present paper our aim is to generalize Theorems 2.1, 2.2 and 2.3 for nearly quasi-Einstein manifolds. Also in the last section we give an example of warped product on nearly quasi-Einstein manifold.
3. Nearly quasi-Einstein warped products

In this section, we consider nearly quasi-Einstein warped product manifolds and prove some results concerning these type manifolds.

**Theorem 3.1.** Let \((M, g)\) be a warped product \(I \times_f F\), \(\dim I = 1\), \(\dim F = n - 1\) \((n \geq 3)\). If \((M, g)\) is nearly quasi-Einstein manifold with associated scalars \(l, m\), then \(F\) is a nearly quasi-Einstein manifold.

**Proof.** Let us consider \((dt)^2\) to be the metric on \(I\). Taking \(f = \exp\{\frac{q}{2}\}\) and making use of Lemma 2.2, we can write

\[
S_M\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -\frac{n-1}{4}[2q'' + (q')^2]
\]

and

\[
S_M(V, W) = S_F(V, W) - \frac{1}{4}e^q[2q'' + (n-1)(q')^2]g_F(V, W),
\]

for all vector fields \(V, W\) on \(F\).

Since \(M\) is nearly quasi-Einstein, from (2) we have

\[
S_M\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = lg\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) + mD\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right),
\]

and

\[
S_M(V, W) = lg(V, W) + mD(V, W).
\]

On the other hand, using (3), the equations (7) and (11) reduce to

\[
S_M\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = l + mD\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)
\]

and

\[
S_M(V, W) = le^q g_F(V, W) + mD_F(V, W).
\]

Comparing the right hand side of the equations (11) and (14) we get

\[
l + mD\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -\frac{n-1}{4}[2q'' + (q')^2].
\]

Similarly, comparing the right hand sides of (8) and (12) we obtain

\[
S_F(V, W) = \frac{1}{4}e^q[2q'' + (n-1)(q')^2 + 4l]g_F(V, W) + mD_F(V, W).
\]

which implies that \(F\) is a nearly quasi-Einstein manifold. This completes the proof of the theorem. \(\square\)
Theorem 3.2. Let \((M, g)\) be a warped product \(B \times_f F\) of a complete connected \(r\)-dimensional \((1 < r < n)\) Riemannian manifold \(B\) and an \((n-r)\)-dimensional Riemannian manifold \(F\).

If \((M, g)\) is a space of nearly quasi-constant sectional curvature, the Hessian of \(f\) is proportional to the metric tensor \(g_B\), then \(B\) is a nearly quasi-Einstein manifold.

Proof. Assume that \(M\) is a space of nearly quasi-constant sectional curvature. Then from equation (3), we can write

\[
R(X, Y, Z, W) = a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + b[g(Y, Z)D(X, W) - g(X, Z)D(Y, W) + g(X, W)D(Y, Z) - g(Y, W)D(X, Z)],
\]

for all vector fields \(X, Y, Z, W\) on \(B\).

In view of Lemma 2.1 and by using (4) in equation (15) and then after a contraction over \(X\) and \(W\) (we put \(X = W = e_i\)), we get

\[
S_B(Y, Z) = [a(r - 1) + bD_B(e_i, e_i)]g_B(Y, Z) + brD_B(Y, Z),
\]

which shows us \(B\) is a nearly quasi-Einstein manifold. Contracting from (16) over \(Y\) and \(Z\), we can write

\[
\tau_B = ar(r - 1) + 2rbD_B(e_i, e_i).
\]

Since \(M\) is a space of nearly quasi-constant sectional curvature, in view of (3) and (15) we get

\[
\Delta f = \frac{ar + brD_B(e_i, e_i)}{2}.
\]

On the other hand, since the Hessian of \(f\) is proportional to the metric tensor \(g_B\), it can be written as follows

\[
H^f(X, Y) = \frac{\Delta f}{r}g_B(X, Y).
\]

Then by use of (17) and (18) in (19) we obtain \(H^f + Kf g_B(X, Y) = 0\), where \(K = \frac{r(3-r)bD_B(e_i, e_i) - \tau_B}{2r(r-1)}\) holds on \(B\). So by Obata’s theorem [14], \(B\) is isometric to the sphere of radius \(\frac{1}{\sqrt{K}}\) in the \((r+1)\)-dimensional Euclidean space. This gives us that \(B\) is a nearly quasi-Einstein manifold. Since \(b \neq 0\) and also \(r \neq 0\), therefore \(B\) is a nearly quasi-Einstein manifold of dimension \(n \geq 2\).

Theorem 3.3. Let \((M, g)\) be a warped product \(B \times_f F\) of a complete connected \(n - 1\)-dimensional Riemannian manifold \(B\) and one-dimensional Riemannian manifold \(I\). If \((M, g)\) is a nearly quasi-Einstein manifold with constant associated scalars \(l, m\) and the Hessian of \(f\) is proportional to the metric tensor \(g_B\), then \((B, g_B)\) is an \((n - 1)\)-dimensional sphere of radius \(\rho = \frac{n-1}{\sqrt{\tau_B + 1}}\).
Proof. Assume that $M$ is a warped product manifold. Then by use of Lemma 2.2 we can write

\begin{equation}
S_B(X,Y) = S_M(X,Y) + \frac{1}{f}H^f(X,Y)
\end{equation}

for any vector fields $X$, $Y$ on $B$. On the other hand, since $M$ is a nearly quasi-Einstein manifold we have

\begin{equation}
S_M(X,Y) = \ln g(X,Y) + mD(X,Y).
\end{equation}

In view of (21) the equation (20) can be written as

\begin{equation}
S_B(X,Y) = \ln g_B(X,Y) + mD_B(X,Y) + \frac{1}{f}H^f(X,Y).
\end{equation}

By a contraction from the above equation over $X$, $Y$, we find

\begin{equation}
\tau_B = l(n - 1) + mD_B(e_i, e_i) + \frac{\Delta f}{f}.
\end{equation}

On the other hand, we know from the equation (21) that

\begin{equation}
\tau_M = \ln + mD_B(e_i, e_i).
\end{equation}

By use of (24) in (23) we get $\tau_B = \tau_M - l + \frac{\Delta f}{f}$. In view of Lemma 2.2 we also know that

\begin{equation}
-\frac{\tau_M}{n} = \frac{\Delta f}{f}.
\end{equation}

The last two equations give us $\tau_B = \frac{n-1}{n} \tau_M - l$. On the other hand, since the Hessian of $f$ is proportional to the metric tensor $g_B$, it can be written as follows

\begin{equation}
H^f(X,Y) = \frac{n-1}{n} g_B(X,Y).\end{equation}

As the consequence of the equation (25) we have

\begin{equation}
\frac{\Delta f}{n-1} = - \frac{1}{n(n-1)} \tau_M f,\end{equation}

which implies that

\begin{equation}
H^f(X,Y) + \frac{\tau_B + l}{(n-1)^2} g_B(X,Y) = 0.
\end{equation}

So, $B$ is isometric to the $(n - 1)$-dimensional sphere of radius $\frac{n-1}{\sqrt{\tau_B + l}}$. Hence the Theorem is proved. \hfill \Box

4.  Example of warped product on nearly quasi-Einstein manifold

In [3], De and Gazi established the 4-dimensional example of nearly quasi-Einstein manifold. Let $(M_4, g)$ be a Riemannian manifold endowed with the metric given by

\[
\begin{aligned}
ds^2 = g_{ij}dx^i dx^j = (dx^4)^2 + \frac{4}{3}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]
\end{aligned}
\]
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where \( i, j = 1, 2, 3, 4 \) and \( x^1, x^2, x^3, x^4 \) are the standard coordinates of \( M_4 \). Then they have shown that it is nearly quasi-Einstein manifold with nonzero and nonconstant scalar curvature.

To define warped product on \( N(QE)_4 \), we consider the warping function \( f : \mathbb{R} \to (0, \infty) \) by \( f(x^4) = \sqrt{(x^4)^\frac{4}{3}} \), here we observe that \( f = \sqrt{(x^4)^\frac{4}{3}} > 0 \) and is a smooth function. The line element defined on \( \mathbb{R} \times \mathbb{R}^3 \) which is of the form \( I \times_f F \), where \( I = \mathbb{R} \) is the base and \( F = \mathbb{R}^3 \) is the fibre.

Therefore the metric \( ds^2_M = ds^2_B + f^2 ds^2_F \) that is

\[
ds^2 = g_{ij}dx^idx^j = (dx^4)^2 + (x^4)^\frac{4}{3}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2],\]

is the example of Riemannian warped product on \( N(QE)_4 \).

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