ON APPROXIMATION TO FUNCTIONS IN THE $W(L^p, \xi(t))$ CLASS BY A NEW MATRIX MEAN

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Abstract. In this paper we shall present results related to trigonometric approximation of functions belonging to the weighted generalized Lipschitz class by the $(C^1 \cdot T)$ matrix means of their Fourier series. The results of Lal in [8] will be extended to a more general summability method. Moreover, we present the results on degree of approximation to conjugates of functions belonging to a weighted generalized Lipschitz class by the $(C^1 \cdot T)$ matrix means of their conjugate Fourier series.

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1. Introduction and Notations

Summability methods have been used in various fields of mathematics. For example, summability methods are applied in function theory in connection with the analytic continuation of holomorphic functions and the boundary behaviour of a power series, in applied analysis for generation of iteration methods for the solution of a system of linear equations, and for acceleration of convergence in approximation theory, in the theory of Fourier series both for creation and acceleration of convergence of a Fourier series, and in other fields of mathematics like probability theory (Markov chains) and number theory (prime number theorem) [11]. In this work we are interested in a summability method in the theory of Fourier series. For this aim, we shall give the following notations to be used in this paper.

Let $L := L(0,2\pi)$ denote the space of functions that are $2\pi$-periodic and Lebesque integrable on $[0,2\pi]$ and let

$$S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(f; x)$$

be the Fourier series of a function $f \in L$; i.e., for any $k = 0, 1, 2, \cdots$

$$a_k = a_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos ktdt, \quad b_k = b_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin ktdt.$$
Let
\[ s_n(f; x) = \frac{1}{2} a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \equiv \sum_{k=1}^{n} A_k(f; x) \]
denote the partial sum of the first \((n + 1)\) terms of the Fourier series of \(f \in L\) at a point \(x\). The conjugate series of (1.1) is given by
\[ \tilde{S}[f] = \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx) \equiv \sum_{k=1}^{\infty} \tilde{A}_k(f; x). \]
Note that there is no free term in \(\tilde{S}[f]\). Therefore, the series conjugate to the series \(\tilde{S}[f]\) is the series \(S[f]\) without free term.

The function \(\tilde{f} \in L\) for which \(S[\tilde{f}] = \tilde{S}[f]\) is called trigonometrically conjugate, or simply conjugate, to \(f(\cdot)\). It can be shown that the functions \(f(\cdot)\) and \(\tilde{f}(\cdot)\) are connected by the equality
\[ \tilde{f}(x) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + t) \cot \frac{t}{2} dt \]
\[ = -\frac{1}{2\pi} \int_{0}^{\pi} \eta(t) \cot \frac{t}{2} dt = -\frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\pi} \eta(t) \cot \frac{t}{2} dt \]
where \(\eta(t) := \eta(x, t) = f(x + t) - f(x - t)\). If \(f \in L\), then equality (1.2) exists for almost all \(x\) [25].

Let
\[ \tau_n(f; x) = \tau_n(f, T; x) := \sum_{k=0}^{n} a_{n,k} s_k(f; x), \quad \forall n \geq 0 \]
where \(T \equiv (a_{n,k})\) is a lower triangular infinite matrix satisfying the Silverman-Toeplitz [24] condition of regularity such that:
\[ a_{n,k} = \begin{cases} 
\geq 0, & k \leq n; \\
0, & k > n 
\end{cases} \quad (k, n = 0, 1, 2, \ldots) \]
and
\[ \sum_{k=0}^{n} a_{n,k} = 1, \quad (n = 0, 1, 2, \ldots). \]

The Fourier series of a function \(f\) is said to be \(T\)-summable to \(s\), if \(\tau_n(f; x) \to s(x)\) as \(n \to \infty\). The Fourier series of \(f\) is called Cesàro-\(T\) \((C^1 \cdot T)\) summable to \(s(x)\) if
\[ t_n^{CT} := \frac{1}{n+1} \sum_{m=0}^{n} \tau_m(f; x) = \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=0}^{m} a_{m,k} s_k(f; x) \to s(x), \]
as \(n \to \infty\).

The Cesàro-\(T\) \((C^1 \cdot T)\) means give us the following means for some important cases:
On approximation to functions in the $W(L^p, \xi(t))$ class

- Cesàro–Nörlund ($C^1 \cdot N_p$) means, with
  $a_{m,k} = \begin{cases} \frac{p_{m-k}}{p_m}, & k \leq m; \\ 0, & k > m \end{cases}$ for $(k, m = 0, 1, 2, \ldots), P_m = \sum_{k=0}^{m} p_k \neq 0$;

- $(C, 1)(E, 1)$ Product means, with $a_{m,k} = \frac{1}{2m} \binom{m}{k}$;

- $(C, 1)(E, q)$ Product means, with $a_{m,k} = \frac{1}{(1+q)^m} \binom{m}{k} q^{m-k}$;

- Generalized Nörlund means, with $a_{m,k} = \frac{p_{m-k} q_k}{r_m}$ where $r_m = \sum_{k=0}^{m} p_{m-k} q_k$.

The degree of approximation of a function $f : \mathbb{R} \to \mathbb{R}$ by a trigonometric polynomial $T_n$ of degree $n$ is defined by

$$\|T_n - f\|_\infty = \sup \{|T_n(x) - f(x)|, x \in \mathbb{R}\}$$

with respect to the supremum norm $\| \cdot \|_\infty$. The degree of approximation of a function $f \in L_p$ ($p \geq 1$) is given by

$$E_n(f) = \min_n \|T_n - f\|_p$$

where $\| \cdot \|_p$ denotes the $L_p$-norm with respect to $x$ and will be defined by

$$\|f\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p dx \right\}^{\frac{1}{p}}.$$

This method of approximation is called the trigonometric Fourier approximation.

We recall the following definitions:

1. A function $f$ is said to belong to the $Lip\alpha$ class if $|f(x + t) - f(x)| = O(|t^\alpha|), \ 0 < \alpha \leq 1$;

2. A function $f$ is said to belong to the $Lip(\alpha, p)$ class if $\omega_p(\delta, f) = O(\delta^\alpha)$, where
   $$\omega_p(\delta, f) = \sup \left\{ \frac{1}{2\pi} \int_{|t| \leq \delta} |f(x + t) - f(x)|^p dx \right\}^{\frac{1}{p}}, \ 0 < \alpha \leq 1; \ p \geq 1;$$

3. A function $f$ is said to belong to the $Lip(\xi(t), p)$ class if $\omega_p(\delta, f) = O(\xi(t))$ where $\xi(t)$ is a positive increasing function and $p \geq 1$;

4. We write $f \in W(L^p, \xi(t))$ if $\|(f(x + t) - f(x)) \sin \beta (x/2)\|_p = O(\xi(t))$, $\beta \geq 0$ and $p \geq 1$. 


If $\beta = 0$, then $W(L^p, \xi(t))$ reduces to $\text{Lip}(\xi(t), p)$; and, if $\xi(t) = t^\alpha$, the $\text{Lip}(\xi(t), p)$ class reduces to the $\text{Lip}(\alpha, p)$ class. If $p \to \infty$ then the $\text{Lip}(\alpha, p)$ class coincides with the $\text{Lip}_\alpha$ class. Accordingly, we have the following inclusions:

$$\text{Lip}_\alpha \subset \text{Lip}(\alpha, p) \subset \text{Lip}(\xi(t), p) \subset W(L^p, \xi(t))$$

for all $0 < \alpha \leq 1$ and $p \geq 1$.

2. Approximation by matrix means of a Fourier series

The degree of approximation, using the various summability methods in the $\text{Lip}_\alpha$ class, has been determined by many mathematicians such as Bernstein [25], de la Valle-Poussin [25], Jackson [25], Mcfadden [1]. Similar problems for the $\text{Lip}(\alpha, p)$ class have been studied by researchers like Quade [18], Khan [6], Qureshi [21], Chandra [2], Leindler [11]. Other research related to the $\text{Lip}(\alpha, p)$ class can also be found in [3], [12], [13], [14] and [15].

The weighted $W(L^p, \xi(t))$ class is a generalization of the classes $\text{Lip}_\alpha$, $\text{Lip}(\alpha, p)$ and $\text{Lip}(\xi(t), p)$. The degree of approximation of a function belonging to the weighted $W(L^p, \xi(t))$ class has been studied by Qureshi in [21]. In [6] and [8] Lal has considered the degree of approximation of functions belonging to the weighted $W(L^p, \xi(t))$ class by the $(C, 1)(E, 1)$ means and $(C^1 \cdot N_p)$ means, respectively. Nigam has studied the same problem for the $(C, 1)(E, q)$ means, which are much more general than the $(C, 1)(E, 1)$ means in [16]. Singh, Mittal and Sonker have generalized the results of Lal [8] in [23]. Therefore, taking into account this generalization of the function classes, we shall give two theorems on degree of approximation to functions belonging to the classes $W(L^p, \xi(t))$ and $\text{Lip}_\alpha$ by the $(C^1 \cdot T)$ matrix means, being more general than $(C, 1)(E, 1)$, $(C^1 \cdot N_p)$ and $(C, 1)(E, q)$ means given in [3], [8, 22] and [16], respectively.

Also, throughout this section, we shall use the following notations:

$$\Psi(x, t) := \Psi(t) = f(x + t) + f(x - t) - 2f(x)$$

and

$$K_T(n, t) := \frac{1}{2\pi(n + 1)} \sum_{m=0}^{n} \sum_{k=0}^{m} a_{m,k} \frac{\sin(k + \frac{1}{2})t}{\sin(\frac{t}{2})}.$$ 

Before stating the theorems, we develop the following auxiliary results needed in the proofs of both of them.

**Lemma 2.1.** For $0 < t \leq \pi/n$, we have $K_T(n, t) = O(n)$.

**Proof.** For $0 < t \leq \pi/n$, from $(\sin(t/2))^{-1} \leq \pi/t$ and $\sin(n + 1)t \leq (n + 1)t$, we have

$$|K_T(n, t)| \leq \frac{1}{2\pi(n + 1)} \sum_{m=0}^{n} \sum_{k=0}^{m} a_{m,k} \left| \frac{\sin(k + \frac{1}{2})t}{\sin(\frac{t}{2})} \right|$$

$$\leq \frac{2n + 1}{2\pi(n + 1)} \sum_{m=0}^{n} \sum_{k=0}^{m} a_{m,k} = O(n)$$

by considering [16]. \qed
Lemma 2.2. For $\pi/n < t \leq \pi$ and any $n$, we have

$$K_T(n, t) = O\left(\frac{t^{-2}}{n + 1}\right) + O(t^{-1}).$$

Proof.

$$K_T(n, t) = \frac{1}{2\pi(n + 1)\sin(\frac{t}{2})} \sum_{m=0}^{n} \sum_{k=0}^{m} a_{m,k} \sin(k + \frac{1}{2})t$$

$$= \frac{1}{2\pi(n + 1)\sin(\frac{t}{2})} \left\{ \sum_{m=0}^{\tau} + \sum_{m=\tau+1}^{n} \right\} \sum_{k=0}^{m} a_{m,k} \sin(k + \frac{1}{2})t$$

$$= : I_1 + I_2,$$

where $\tau$ denotes the integer part of $1/t$. Owing to (1.3) and Jordan’s inequality, $(\sin(t/2))^{-1} \leq \pi/t$, for $0 < t \leq \pi$, we obtain

(2.1) $|I_1| = O\left(\frac{1}{(n + 1)t}\right) \sum_{m=0}^{\tau} \sum_{k=0}^{m} a_{m,k} = O\left(\frac{\tau t^{-1}}{n + 1}\right) = O\left(\frac{t^{-2}}{n + 1}\right).$

We now estimate $I_2$. By using (1.3) again and the Jordan inequality $(\sin(t/2))^{-1} \leq \pi/t$, for $0 < t \leq \pi$, we get

(2.2) $|I_2| = O\left(\frac{1}{(n + 1)t}\right) \sum_{m=\tau+1}^{n} \sum_{k=0}^{m} a_{m,k} = O\left(\frac{(n - \tau) t^{-1}}{n + 1}\right) = O(1/t).$

Combining (2.1) and (2.2), we have

$$K_T(n, t) = O\left(\frac{t^{-2}}{n + 1}\right) + O(t^{-1}).$$

Theorem 2.3. Let $f \in L$ and let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with nonnegative entries and row sums 1. If $f \in \text{Lip}_{\alpha} (0 < \alpha \leq 1)$, then the degree of approximation by the $(C^1 \cdot T)$ means of its Fourier series is given by

$$\|t^C_n f - f(x)\|_{\infty} = \left\{ \begin{array}{ll}
O(n^{-\alpha}), & 0 < \alpha < 1; \\
O(\frac{\log n}{n}), & \alpha = 1.
\end{array} \right.$$  

Proof. We know that

(2.3) $s_n(f, x) - f(x) = \frac{1}{2\pi} \int_{0}^{\pi} \Psi(t) \left( \frac{\sin(n + \frac{1}{2})t}{\sin(t/2)} \right) dt.$
Taking into account (2.3) and the definitions of $t_n^{CT}(f)$ and the $(C^1 \cdot T)$ means of $s_n(f)$, we write

$$|t_n^{CT}(f) - f(x)| = \frac{1}{n+1} \left| \sum_{m=0}^{n} \sum_{k=0}^{m} a_{m,k} (s_k(f;x) - f(x)) \right|$$

$$= \frac{1}{2\pi(n+1)} \left| \int_{0}^{\pi} \Psi(t) \sum_{m=0}^{n} \sum_{k=0}^{m} a_{m,k} \left( \frac{\sin(k + \frac{1}{2})t}{\sin(\frac{t}{2})} \right) dt \right|$$

$$\leq \int_{0}^{\pi} |\Psi(t) K_T(n,t)| dt = \left[ \int_{0}^{\pi/n} + \int_{\pi/n}^{\pi} \right] |\Psi(t) K_T(n,t)| dt$$

$$= : J_1 + J_2.$$

Since $f \in Lip_\alpha$, $\Psi(t)$ belongs to the $Lip_\alpha$ class. Therefore, from Lemma 2.1, we obtain

(2.4) \quad J_1 = \int_{0}^{\pi/n} |\Psi(t) K_T(n,t)| dt = O(n) \int_{0}^{\pi/n} t^\alpha dt = O(n^{-\alpha})$

for $0 < \alpha \leq 1$.

By using Lemma 2.2, then we have

$$J_2 = \int_{\pi/n}^{\pi} |\Psi(t) K_T(n,t)| dt = O \left\{ \int_{\pi/n}^{\pi} t^\alpha \left( \frac{t^{2}}{n+1} + t^{-1} \right) dt \right\}$$

$$= O \left\{ \int_{\pi/n}^{\pi} \frac{t^{\alpha-2}}{n+1} dt \right\} + O \left\{ \int_{\pi/n}^{\pi} t^{\alpha-1} dt \right\} =: J_2^1 + J_2^2.$$

Accordingly,

(2.5) \quad J_2^1 = O \left\{ \int_{\pi/n}^{\pi} \frac{t^{\alpha-2}}{n+1} dt \right\} = \left\{ \begin{array}{ll} O(n^{-\alpha}), & 0 < \alpha < 1; \\
O\left(\frac{\log n}{n}\right), & \alpha = 1. \end{array} \right.$$

and

(2.6) \quad J_2^2 = O \left\{ \int_{\pi/n}^{\pi} t^{\alpha-1} dt \right\} = O \left\{ n^{-\alpha} \right\}.$$

Taking into account (2.3), (2.5) and (2.6), we obtain

$$\|t_n^{CT}(f) - f(x)\|_\infty = \sup_{x \in [0,2\pi]} |t_n^{CT}(f) - f(x)| = \left\{ \begin{array}{ll} O(n^{-\alpha}), & 0 < \alpha < 1; \\
O\left(\frac{\log n}{n}\right), & \alpha = 1. \end{array} \right.$$
by using $1/n \leq \log n/n$, for large values of $n$. Therefore the proof of Theorem 2.3 is completed.

**Theorem 2.4.** Let $f \in L$ and $\xi(t)$ be a positive increasing function. If $f \in W(L^p, \xi(t))$ with $0 \leq \beta \leq 1 - 1/p$, the degree of approximation by $(C^1 \cdot T)$ means of its Fourier series is given by

$$
\| t_n^{CT}(f) - f(x) \|_p = O(n^{\beta+1/p} \xi(\frac{1}{n})),
$$

provided that the function $\xi(t)$ satisfies the following conditions:

$$
\left\{ \frac{\xi(t)}{t} \right\}
$$

is a decreasing function and

$$
\left\{ \int_0^{\pi/n} \left( \frac{\left| \Psi(t) \right| \sin^\beta(t/2)}{\xi(t)} \right)^p dt \right\}^{1/p} = O(1)
$$

(2.7)

$$
\left\{ \int_{\pi/n}^{\pi} \left( \frac{\left| \Psi(t) \right| t^{-\delta}}{\xi(t)} \right)^p dt \right\}^{1/p} = O(n^\delta),
$$

(2.8)

where $\delta$ is an arbitrary number such that $q(\beta - \delta) - 1 > 0$, $p^{-1} + q^{-1} = 1$, $p \geq 1$, and (2.7) and (2.8) hold uniformly in $x$.

**Proof.** Proceeding as above, we have

$$
| t_n^{CT}(f) - f(x) | = \frac{1}{n+1} \left| \sum_{m=0}^{n} \sum_{k=0}^{m} a_{m,k} \left( s_k(f; x) - f(x) \right) \right|
$$

$$
\leq \left| \int_0^{\pi/n} \Psi(t) K_T(n, t) dt \right| + \left| \int_{\pi/n}^{\pi} \Psi(t) K_T(n, t) dt \right|
$$

(2.9)

$$
= : J_3 + J_4.
$$

By considering Hölder’s inequality, condition (2.8), Lemma 2.1, Jordan’s in-
equality, and \( \Psi(t) \in W(L^p, \xi(t)) \), we get

\[
J_3 = \left. \left| \int_0^{\pi/n} \frac{\Psi(t) \sin^\beta(t/2) \xi(t) K_T(n, t)}{\sin^\beta(t/2)} \, dt \right| \right|^{1/p} \left( \lim_{\varepsilon \to 0} \int_\varepsilon^{\pi/n} \frac{\xi(t) K_T(n, t)}{\sin^\beta(t/2)} \, dt \right)^{1/q}
\]

\[
= O(1) \left( \lim_{\varepsilon \to 0} \int_\varepsilon^{\pi/n} \left( \frac{\xi(t) \sin^\beta(t/2)}{\sin^\beta(t/2)} \right)^q \, dt \right)^{1/q} = O(n \xi(\pi/n)) \left( \lim_{\varepsilon \to 0} \int_\varepsilon^{\pi/n} \frac{t^{-\beta q}}{dt} \right)^{1/q}
\]

in view of \( p^{-1} + q^{-1} = 1 \) and \( \xi(\pi/n)/(\pi/n) \leq \xi(1/n)/(1/n) \).

Now let us estimate \( J_4 \). By using Lemma 2.2, we write

\[
J_4 = O \left( \int_\pi^{\pi/n} |\Psi(t)| \left( \frac{t^{-2}}{n+1} \right) \, dt \right) + O \left( \int_\pi^{\pi/n} |\Psi(t)| (t^{-1}) \, dt \right) =: J_4^1 + J_4^2.
\]

We shall evaluate \( J_4^1 \) and \( J_4^2 \) in a manner similar to the evaluation of \( J_3 \), respectively. Using Hölder’s inequality, the (2.8) and Jordan’s inequality, we have

\[
J_4^1 = O(n^{-1}) \left( \int_\pi^{\pi/n} \left( \frac{\xi(t) t^{-\delta} \sin^\beta(t/2)}{\xi(t)} \right)^p \, dt \right)^{1/p} \left( \int_\pi^{\pi/n} \left( \frac{\xi(t) t^{-\delta-2}}{\sin^\beta(t/2)} \right)^q \, dt \right)^{1/q}
\]

\[
= O(n^{\delta-1}) \left( \int_\pi^{\pi/n} \left( \frac{\xi(t) t^{-\delta-2}}{\sin^\beta(t/2)} \right)^q \, dt \right)^{1/q} = O(n^{\delta-1}) \left( \int_\pi^{\pi/n} \left( \xi(t) t^{-\beta-2} \right)^q \, dt \right)^{1/q}
\]

\[
= O \left( \int_{1/\pi}^{n/\pi} \left( \xi(1/x) \frac{x^{\beta-\delta+2}}{x^{\beta-\delta-2}} \right)^q \, dx \right)^{1/q} \quad ; \ x \xi(1/x) < \frac{n}{\pi} \xi(\pi/n)
\]

\[
= O \left( \xi(\frac{\pi}{n}) n^{\delta} \int_{1/\pi}^{n/\pi} x^{\beta q-\delta q+q-2} \, dx \right)^{1/q} = O \left( \xi(\frac{\pi}{n}) n^{\delta} n^{1-\delta+\beta-(1/q)} \right)
\]

\[
J_4^2 = O \left( \frac{1}{n} \right),
\]

(2.10)
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since $p^{-1} + q^{-1} = 1$ and $\xi(\pi/n)/(\pi/n) \leq \xi(1/n)/(1/n)$.

\[
J_4^2 = O(1) \left( \int_{\pi/n}^{\pi} \left( \frac{|\Psi(t)|t^{-\delta} \sin^\beta(t/2)}{\xi(t)} \right)^p dt \right)^{1/p} \left( \int_{\pi/n}^{\pi} \left( \frac{\xi(t)t^{-\delta - 1 - \beta}}{\sin^\beta(t/2)} \right)^q dt \right)^{1/q}
\]

\[
= O(n^\delta) \left( \int_{\pi/n}^{\pi} (\xi(t)t^{-\delta - 1 - \beta})^q dt \right)^{1/q}
\]

\[
= O(n^\delta) \left( \frac{n^\beta}{\pi} \int_{1/\pi}^{n/\pi} (\xi(1/x)x^{\beta - \delta + 1})^q x^{-2} dx \right)^{1/q} ; x\xi(1/x) < \frac{n}{\pi} \xi(\pi/n)
\]

\[
= O \left( \xi(\pi/n)^{n^\delta + 1} \left( \int_{1/\pi}^{n/\pi} x^{\beta q - \delta q - 2} dx \right) \right)^{1/q} = O \left( \xi(\pi/n)^{n^\delta + 1} n^{\beta - \delta - (1/q)} \right)
\]

(2.13) \quad = O \left( n^{\beta + 1/p}\xi(\frac{1}{n}) \right),

since $p^{-1} + q^{-1} = 1$ and $\xi(\pi/n)/(\pi/n) \leq \xi(1/n)/(1/n)$. Combining (2.13)-(2.15), we get

\[
\|t_n^{CT}(f) - f(x)\|_p = O(n^{\beta + 1/p}\xi(\frac{1}{n})�).
\]

3. In case of conjugate Fourier series

As mentioned above, the problems on determining the degree of approximation by summability methods have been studied by many mathematicians. Qureshi has determined the degree of approximation to functions which belong to the classes $Lip_\alpha$ and $Lip(\alpha, p)$ by means of conjugate series in [13] and [22], respectively. In subsequent years, similar investigations have been made in researches such as in [7], [9], [16] and [17].

The following two theorems are related with the degree of approximation to conjugate of functions belonging to the classes $W(L^p, \xi(t))$ and $Lip_\alpha$ by $(C^1,T)$ matrix means of conjugate of their Fourier series and are more general than $(C,1)(E,1)$ and $(C,1)(E,q)$. Not only for $(C,1)(E,1)$ and $(C,1)(E,q)$ means but also different results are obtained for other means.

The following notations will be used throughout this section and auxiliary results:

\[
\rho(x,t) := \rho(t) = f(x + t) + f(x - t)
\]

and

\[
\tilde{K}_T(n,t) := \frac{1}{2\pi(n + 1)} \sum_{m=0}^{n} \sum_{k=0}^{m} a_{m,k} \frac{\cos(k + \frac{1}{2})t}{\sin(\frac{t}{2})}.
\]
Lemma 3.1. For $0 < t \leq \pi/n$, we have $\tilde{K}_T(n, t) = O(1/t)$.

Proof. For $0 < t \leq \pi/n$, by $(\sin(t/2))^{-1} \leq \pi/t$ and $|\cos(2k+1)t| \leq 1$, we have

$$|\tilde{K}_T(n, t)| \leq \frac{1}{2\pi(n+1)} \sum_{m=0}^{n} \sum_{k=0}^{m} a_{m,k} \left| \frac{\cos(k + \frac{1}{2})t}{\sin(\frac{t}{2})} \right|.$$ 

$$\leq \frac{1}{2\pi t(n+1)} \sum_{m=0}^{n} 1 = O(1/t)$$

by (1.3).

Lemma 3.2. For $\pi/n < t \leq \pi$ and any $n$, we have

$$\tilde{K}_T(n, t) = O \left( \frac{t^{-2}}{n+1} \right) + O(t^{-1}).$$

Proof. This lemma can be proved by using an argument similar to that of Lemma 2.2.

Theorem 3.3. Let $f \in L$ and let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with nonnegative entries and row sums 1. If $f \in \text{Lip}_0 (0 < \alpha \leq 1)$, then the degree of approximation of the conjugate function $\tilde{f}$ by the $(C^1 \cdot T)$ means of its conjugate Fourier series is given by

$$\|t_n^{CT}(f) - \tilde{f}(x)\|_\infty = \begin{cases} O(n^{-\alpha}), & 0 < \alpha < 1; \\ O(\log n), & \alpha = 1. \end{cases}$$

Proof. We have, by (3.2),

$$\tilde{s}_n(f, x) - \tilde{f}(x) = \frac{1}{2\pi} \int_0^\pi \rho(t) \left( \frac{\cos(n + \frac{1}{2})t}{\sin(\frac{t}{2})} \right) dt.$$ 

Taking into consideration (3.2) and $t_n^{CT}(f)$ that $(C^1 \cdot T)$ means of $\tilde{s}_n(f)$, we write

$$\|t_n^{CT}(f) - f(x)\| = \frac{1}{2\pi(n+1)} \int_0^\pi \rho(t) \sum_{m=0}^{n} \sum_{k=0}^{m} a_{m,k} \left( \frac{\cos(k + \frac{1}{2})t}{\sin(\frac{t}{2})} \right) dt.$$ 

Using Lemma 3.1 and Lemma 3.2, and proceeding as in the proof of Theorem 2.3 in (3.3), we get (3.1).

Theorem 3.4. Let $f \in L$ and $\xi(t)$ be a positive increasing function. If $f \in W(L^p, \xi(t))$ with $0 \leq \beta \leq 1 - 1/p$, then the degree of approximation of the conjugate function $\tilde{f}$ by the $(C^1 \cdot T)$ means of its conjugate Fourier series is given by
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\begin{equation}
\| \tilde{t_n^{CT}}(f) - \tilde{f}(x) \|_p = O(n^{\beta+1/p} \xi \left( \frac{1}{n} \right)),
\end{equation}

provided that the function $\xi(t)$ satisfies the following conditions:

\begin{align*}
\{ \frac{\xi(t)}{t} \}
\end{align*}

is a decreasing function and

\begin{equation}
\left\{ \frac{\pi/n}{\int_0^\pi \left( \frac{|\rho(t)| \sin^{\beta}(t/2)}{\xi(t)} \right)^p \, dt} \right\}^{1/p} = O(1)
\end{equation}

\begin{equation}
\left\{ \int_{\pi/n}^\pi \left( \frac{|\rho(t)|^{1-\delta}}{\xi(t)} \right)^p \, dt \right\}^{1/p} = O(n^\delta),
\end{equation}

where $\delta$ is an arbitrary number such that $q(\beta - \delta) - 1 > 0$, $p^{-1} + q^{-1} = 1$, $p \geq 1$, and (3.4) and (3.6) hold uniformly in $x$.

Proof. Taking into account Lemma 3.1 and Lemma 3.2, and proceeding as in the proof of Theorem 2.4 in (3.3), we obtain (3.4). \qed

4. Corollaries and remarks

Using results given in Section 2 and Section 3, we observe the following corollaries and remarks.

**Corollary 4.1.** If $\beta = 0$, then the weighted class $W(L^p, \xi(t))$ reduces to the class Lip($\xi(t), p$). Therefore, for $f \in \text{Lip}(\xi(t), p)$, we have

\begin{equation}
\| t_n^{CT}(f) - f(x) \|_p = O(n^{1/p} \xi \left( \frac{1}{n} \right))
\end{equation}

and

\begin{equation}
\| \tilde{t_n^{CT}}(f) - \tilde{f}(x) \|_p = O(n^{1/p} \xi \left( \frac{1}{n} \right))
\end{equation}

with respect to Theorem 2.4 and Theorem 3.4, respectively.

**Corollary 4.2.** If $\beta = 0$ and $\xi(t) = t^\alpha$, $(0 < \alpha \leq 1)$, then the weighted class $W(L^p, \xi(t))$ reduces to the Lip($\alpha, p$) class. Therefore, for $f \in \text{Lip}(\alpha, p)$, $(1/p < \alpha)$, we have

\begin{equation}
\| t_n^{CT}(f) - f(x) \|_p = O(n^{1/p - \alpha})
\end{equation}

and

\begin{equation}
\| \tilde{t_n^{CT}}(f) - \tilde{f}(x) \|_p = O(n^{1/p - \alpha})
\end{equation}

with respect to Theorem 2.4 and Theorem 3.4, respectively.
**Corollary 4.3.** If \( p \to \infty \) in Corollary 4.2, then for \( f \in \text{Lip}_\alpha \), \((0 < \alpha < 1)\) we have
\[
\|t_n^{CT}(f) - f(x)\|_\infty = O(n^{-\alpha})
\]
and
\[
\|\tilde{t}_n^{CT}(f) - \tilde{f}(x)\|_\infty = O(n^{-\alpha})
\]
with respect to Theorem 2.4 and Theorem 3.4, respectively.

**Remark 4.4.** If \( T \equiv (a_{m,k}) \) is a Nörlund matrix, then the \((C^1 \cdot T)\) means give us the Cesàro- Nörlund \((C^1 \cdot N_p)\) means. Accordingly, our main theorems coincide with Theorem 2.3 and Theorem 2.4 in [23]. Moreover, our main results generalize the main results in [8] and [23].

**Remark 4.5.** If \( a_{m,k} = \frac{1}{2^m \binom{m}{k}} \), then the \((C^1 \cdot T)\) means give us the \((C,1)(E,1)\) product means. In this case our main results are reduced to the \((C,1)(E,1)\) product means and the results given in Section 2 and Section 3 coincide with the results in [6] and [10], respectively.

**Remark 4.6.** If \( a_{m,k} = \frac{1}{(1+q)^m \binom{m}{k}} q^{m-k} \), then the \((C^1 \cdot T)\) means give us the \((C,1)(E,q)\) product means. Therefore, the results mentioned in Section 2 and Section 3 are reduced the main results in [11], [15] and [17].

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**References**


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