Abstract. The object of the present paper is to study a transformation called D-homothetic deformation of \(LP\)-Sasakian manifolds. Among others it is shown that in an \(LP\)-Sasakian manifold, the Ricci operator \(Q\) commutes with the structure tensor \(\phi\). We also discuss about the invariance of \(\eta\)-Einstein manifolds, \(\phi\)-sectional curvature, the locally \(\phi\)-Ricci symmetry and \(\eta\)-parallelity of the Ricci tensor under the D-homothetic deformation. Finally, we give an example of such a manifold.

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1. Introduction

The notion of Lorentzian para-Sasakian manifold was introduced by Matsumoto in 1989. Then Mihai and Rosca defined the same notion independently and they obtained several results on this manifold. \(LP\)-Sasakian manifolds have also been studied by Matsumoto and Mihai, De and Shaikh, Ozgur and others.

An \(LP\)-Sasakian manifold is said to be \(\eta\)-Einstein if its Ricci tensor \(S\) is of the form

\[
(1.1) \quad S = \lambda g + \mu \eta \otimes \eta
\]

where \(\lambda\) and \(\mu\) are smooth functions on the manifold.

The notion of local \(\phi\)-symmetry was first introduced by Takahashi on a Sasakian manifold. Again in a recent paper De and Sarkar introduced the notion of locally \(\phi\)-Ricci symmetric Sasakian manifolds. Also \(\phi\)-Ricci symmetric Kenmotsu manifolds have been studied by Shukla and Shukla.

An \(LP\)-Sasakian manifold is said to be locally \(\phi\)-Ricci symmetric if

\[
(1.2) \quad \phi^2(\nabla_X Q)(Y) = 0,
\]

where \(Q\) is the Ricci operator defined by \(g(QX, Y) = S(X, Y)\) and \(X, Y\) are orthogonal to \(\xi\).

The Ricci tensor \(S\) of an \(LP\)-Sasakian manifold is said to be \(\eta\)-parallel if it satisfies

\[
(1.3) \quad (\nabla_X S)(\phi Y, \phi Z) = 0,
\]

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for all vector fields $X, Y$ and $Z$. The notion of $\eta$-parallelity in a Sasakian manifold was introduced by Kon [1].

Let $M(\phi, \xi, \eta, g)$ be an almost contact metric manifold with dim $M = m = 2n + 1$. The equation $\eta = 0$ defines an $(m - 1)$-dimensional distribution $D$ on $M$ [11]. By an $(m - 1)$-homothetic deformation or $D$-homothetic deformation [12] we mean a change of structure tensors of the form

$$
\tilde{\eta} = a\eta, \quad \tilde{\xi} = \frac{1}{a}\xi, \quad \tilde{\phi} = \phi, \quad \tilde{g} = ag + a(a - 1)\eta \otimes \eta,
$$

where $a$ is a positive constant. If $M(\phi, \xi, \eta, g)$ is an almost contact metric structure with contact form $\eta$, then $M(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is also an almost contact metric structure [12]. Denoting by $W^i_{jk}$ the difference $\Gamma^i_{jk} - \Gamma^i_{jk}$ of Christoffel symbols we have in an almost contact metric manifold [12]

$$
W(X, Y) = (1 - a)[\eta(Y)\phi X + \eta(X)\phi Y] + \frac{1}{2}(1 - \frac{1}{a})[(\nabla_X \eta)(Y) + (\nabla_Y \eta)(X)]\xi
$$

(1.4)

for all $X, Y \in \chi(M)$. If $R$ and $\tilde{R}$ denote respectively the curvature tensor of the manifold $M(\phi, \xi, \eta, g)$ and $M(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, then we have [12]

$$
$$

(1.5)

for all $X, Y, Z \in \chi(M)$.

A plane section in the tangent space $T_p(M)$ is called a $\phi$-section if there exists a unit vector $X$ in $T_p(M)$ orthogonal to $\xi$ such that $\{X, \phi X\}$ is an orthonormal basis of the plane section. Then the sectional curvature

$$
K(X, \phi X) = g(R(X, \phi X)X, \phi X)
$$

is called a $\phi$-sectional curvature. A para contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be of constant $\phi$-sectional curvature if at any point $p \in M$, the sectional curvature $K(X, \phi X)$ is independent of the choice of non-zero $X \in D_p$, where $D$ denotes the contact distribution of the para contact metric manifold defined by $\eta = 0$.

The present paper is organized as follows:

After preliminaries in section 2, we prove some important lemmas. Section 4 deals with the study of $(2n + 1)$-dimensional $\eta$–Einstein LP-Sasakian manifolds and prove that these manifolds are invariant under a $D$-homothetic deformation. Also we study $\phi$-sectional curvature, locally $\phi$-Ricci symmetry and $\eta$-parallelity of the Ricci tensor in a $(2n + 1)$-dimensional LP-Sasakian manifold under a $D$-homothetic deformation. Finally in section 5, we cited an example of LP-Sasakian manifold which validates a theorem of section 4.
2. Preliminaries

Let $M^{2n+1}$ be an $2n + 1$-dimensional differentiable manifold endowed with a $(1,1)$ tensor field $\phi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and a Lorentzian metric $g$ of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p: T_pM \times T_pM \rightarrow \mathbb{R}$ is a non-degenerate inner product of signature $(-,+,+,...,+)$, where $T_pM$ denotes the tangent space of $M$ at $p$ and $\mathbb{R}$ is the real number space which satisfies

$$\phi^2(X) = X + \eta(X)\xi, \eta(\xi) = -1,$$

$$g(X, \xi) = \eta(X), g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for all vector fields $X, Y$. Then such a structure $(\phi, \xi, \eta, g)$ is termed as Lorentzian almost paracontact structure and the manifold $M^{2n+1}$ with the structure $(\phi, \xi, \eta, g)$ is called Lorentzian almost paracontact manifold [5]. In the Lorentzian almost paracontact manifold $M^{2n+1}$, the following relations hold [5] :

$$\phi \xi = 0, \eta(\phi X) = 0,$$

$$\Omega(X, Y) = \Omega(Y, X),$$

where $\Omega(X, Y) = g(X, \phi Y)$.

Let $\{e_i\}$ be an orthonormal basis such that $e_1 = \xi$. Then the Ricci tensor $S$ and the scalar curvature $r$ are defined by

$$S(X, Y) = \sum_{i=1}^{n} \epsilon_i g(R(e_i, X)Y, e_i)$$

and

$$r = \sum_{i=1}^{n} \epsilon_i S(e_i, e_i),$$

where we put $\epsilon_i = g(e_i, e_i)$, that is, $\epsilon_1 = -1, \epsilon_2 = \cdots = \epsilon_n = 1$.

A Lorentzian almost paracontact manifold $M^n$ equipped with the structure $(\phi, \xi, \eta, g)$ is called Lorentzian paracontact manifold if

$$\Omega(X, Y) = \frac{1}{2} \{(\nabla_X \eta)Y + (\nabla_Y \eta)X\}.$$

A Lorentzian almost paracontact manifold $M^n$ equipped with the structure $(\phi, \xi, \eta, g)$ is called an LP-Sasakian manifold [5] if

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X.$$
In an $LP$-Sasakian manifold the 1-form $\eta$ is closed. Also in [5], it is proved that if an $n$-dimensional Lorentzian manifold $(M^n, g)$ admits a timelike unit vector field $\xi$ such that the 1-form $\eta$ associated to $\xi$ is closed and satisfies

$$(\nabla_X \nabla_Y \eta)Z = g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z),$$

then $M^n$ admits an $LP$-Sasakian structure.

Further, on such an $LP$-Sasakian manifold $M^n (\phi, \xi, \eta, g)$, the following relations hold [5]:

\begin{equation}
(2.5) \quad \eta(R(X,Y)Z) = [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)],
\end{equation}

\begin{equation}
(2.6) \quad S(X,\xi) = 2n\eta(X),
\end{equation}

\begin{equation}
(2.7) \quad R(X,Y)\xi = [\eta(Y)X - \eta(X)Y],
\end{equation}

\begin{equation}
(2.8) \quad R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X,
\end{equation}

\begin{equation}
(2.9) \quad (\nabla_X \phi)(Y) = [g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X],
\end{equation}

for all vector fields $X, Y, Z$, where $R, S$ denote respectively the curvature tensor and the Ricci tensor of the manifold. Also since the vector field $\eta$ is closed in an $LP$-Sasakian manifold, we have ([4], [3])

\begin{equation}
(2.10) \quad (\nabla_X \eta)Y = \Omega(X,Y),
\end{equation}

\begin{equation}
(2.11) \quad \Omega(X,\xi) = 0,
\end{equation}

\begin{equation}
(2.12) \quad \nabla_X \xi = \phi X,
\end{equation}

for any vector field $X$ and $Y$.

3. Some Lemmas

In this section we shall state and prove some Lemmas which will be needed to prove the main results.

**Lemma 3.1.** [1] In an $LP$-Sasakian manifold, the following relation holds

\begin{equation}
g(R(\phi X, \phi Y)\phi Z, \phi W) = g(R(X,Y)Z,W) + g(X,W)\eta(Y)\eta(Z) - g(X,Z)\eta(W)\eta(Y) + g(Y,Z)\eta(X)\eta(W) - g(Y,W)\eta(X)\eta(Z).
\end{equation}
Lemma 3.2. Let \((M^{2n+1}, g)\) be an \(LP\)-Sasakian manifold. Then the Ricci operator \(Q\) commutes with \(\phi\).

Proof. From (3.1), it follows that

\[\phi R(\phi X, \phi Y)Z = R(X, Y)Z - [\eta(Z)Y - g(Y, Z)\xi]\eta(X) + [X\eta(Z) - g(X, Z)\xi]\eta(Y).\] (3.2)

Let \(\{e_i, \phi e_i, \xi\}, i = 1, 2, \ldots, n\) be an orthonormal frame at any point of the manifold. Then putting \(Y = Z = e_i\) in (3.2) and taking summation over \(i\) and using \(\eta(e_i) = 0\), we get

\[\sum_{i=1}^{n} \epsilon_i \phi R(\phi X, \phi e_i)\phi e_i = \sum_{i=1}^{n} \epsilon_i R(X, e_i) e_i - n\eta(X)\xi,\] (3.3)

where \(\epsilon_i = g(e_i, e_i)\).

Again setting \(Y = Z = \phi e_i\) in (3.2), taking summation over \(i\) and using \(\eta.\phi = 0\), we get

\[\sum_{i=1}^{n} \epsilon_i \phi R(\phi X, \phi e_i)e_i = \sum_{i=1}^{n} \epsilon_i R(X, \phi e_i)\phi e_i - n\eta(X)\xi.\] (3.4)

Adding (3.3) and (3.4) and using the definition of the Ricci tensor, we obtain

\[\phi(Q\phi X - R(\phi X, \xi)\xi) = QX - R(X, \xi)\xi - 2n\eta(X)\xi.\]

Using (2.1) and \(\phi\xi = 0\) in the above relation, we have

\[\phi(Q\phi X) = QX - 2n\eta(X)\xi.\]

Operating both sides by \(\phi\) and using (2.1), symmetry of \(Q\) and \(\phi\xi = 0\), we get \(\phi Q = Q\phi\). This proves the lemma.

\[\square\]

Proposition 3.1. In an \(2n+1\)-dimensional \(\eta\)-Einstein \(LP\)-Sasakian manifold, the Ricci tensor \(S\) is expressed as

\[S(X, Y) = [\frac{r}{2n} - 1]g(X, Y) - [\frac{r}{2n} - 2n - 1]\eta(X)\eta(Y).\] (3.5)
4. Main results

In this section we study \( \eta \)-Einstein \( LP \)-Sasakian manifolds, \( \phi \)-sectional curvature, locally \( \phi \)-Ricci symmetry and \( \eta \)-parallelity of the Ricci tensor of an odd dimensional \( LP \)-Sasakian manifold under a D-homothetic deformation.

In virtue of (2.11), the relation (1.4) reduces to

\[
W(X, Y) = (1 - a)[\eta(Y)\phi X + \eta(X)\phi Y] + (1 - \frac{1}{a})g(\phi X, Y)\xi.
\]

In view of (2.9), (2.10) and (2.12), the relation (4.1) yields

\[
(\nabla_Z W)(X, Y) = (1 - a)[\{g(\phi Z, Y)\phi X + g(X, Z)\eta(Y)\xi + 2\eta(X)\eta(Y)Z + 4\eta(X)\eta(Y)\eta(Z)\xi
\]
\[
+ g(\phi Z, Y)\phi Y + \eta(X)g(Y, Z)\xi]
\]
\[
+ \frac{a - 1}{a}g(\phi X, Y)\phi Z.
\]

(4.2)

Using (4.1) and (4.2) into (1.5), we obtain by virtue of (2.7) and (2.10) that

\[
R(X, Y)Z = R(X, Y)Z + (1 - a)[g(X, Z)\eta(Y)\xi
\]
\[
- g(Y, Z)\eta(X)\xi + 2\eta(Y)\eta(Z)X
\]
\[
- 2\eta(X)\eta(Z)Y
\]
\[
+ g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X
\]
\[
+ \frac{a - 1}{a}[g(\phi Z, Y)\phi X - g(\phi Z, X)\phi Y]
\]
\[
+ (1 - a)^2[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X]
\]
\[
- \frac{(1 - a)^2}{a}[g(\phi Z, X)\phi Y - g(\phi Z, Y)\phi X].
\]

(4.3)

Putting \( Y = Z = \xi \) in (1.3) and using (2.11) we obtain

\[
\tilde{R}(X, Y)\xi = R(X, Y)\xi + 2(1 - a)[-X + \eta(X)\xi] - (1 - a)^2\phi^2 X.
\]

(4.4)

Let \( \{e_i, \phi e_i, \xi\} \), \( i = 1, 2, \ldots, n \) be an orthonormal frame at any point of the manifold. Then putting \( Y = Z = e_i \) in (1.3) and taking summation over \( i \) and using \( \eta(e_i) = 0 \), we get

\[
\sum_{i=1}^{n} \epsilon_i \tilde{R}(X, e_i)e_i = \sum_{i=1}^{n} \epsilon_i R(X, e_i)e_i - (1 - a)n\eta(X)\xi,
\]

where \( \epsilon_i = g(e_i, e_i) \).

Again setting \( Y = Z = \phi e_i \) in (1.3) and taking summation over \( i \) and using \( \eta.\phi = 0 \), we get

\[
\sum_{i=1}^{n} \epsilon_i \tilde{R}(X, \phi e_i)\phi e_i = \sum_{i=1}^{n} \epsilon_i R(X, \phi e_i)\phi e_i - (1 - a)n\eta(X)\xi.
\]

(4.5)
Adding (4.5) and (4.6) and using the definition of Ricci operator we have

\[(4.7) \quad \bar{Q}X - \bar{R}(X, \xi)\xi = QX - R(X, \xi)\xi - 2(1 - a)n\eta(X)\xi.\]

In view of (4.4) we get from (4.7)

\[(4.8) \quad S(X, Y) = S(X, Y) - [2(1 - a) + (1 - a)^2]g(X, Y) - [2(1 - a)(n - 1) + (1 - a)^2]\eta(X)\eta(Y),\]

which implies that

\[(4.9) \quad \bar{Q}X = QX - [2(1 - a) + (1 - a)^2]X - [2(1 - a)(n - 1) + (1 - a)^2]\eta(X)\xi.\]

Operating \(\bar{\phi} = \phi\) on both sides of (4.8) from the left we have

\[(4.10) \quad \bar{\phi}QX = \phi QX - [2(1 - a) + (1 - a)^2]\phi X.\]

Again, putting \(\bar{\phi}X = \phi X\) in (4.8) from the right we have

\[(4.11) \quad \bar{Q}\phi X = Q\phi X - [2(1 - a) + (1 - a)^2]\phi X.\]

Subtracting (4.10) and (4.11) we get

\[(4.12) \quad (\bar{\phi}\bar{Q} - \bar{Q}\bar{\phi})X = (\phi Q - Q\phi)X.\]

Therefore using Lemma 3.2 we can state the following:

**Theorem 4.1.** Under a D-homothetic deformation, the expression \(\bar{Q}\bar{\phi} = \bar{\phi}\bar{Q}\) holds in an \((2n + 1)\)-dimensional LP-Sasakian manifold.

### 4.1. \(\eta\)-Einstein LP-Sasakian manifolds

Let \(M(\phi, \xi, \eta, g)\) be a \((2n+1)\)-dimensional \(\eta\)-Einstein LP-Sasakian manifold which reduces to \(M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\) under a D-homothetic deformation. Then from (11.8) it follows by virtue of (11.3) that

\[(4.13) \quad \bar{S}(X, Y) = \bar{\lambda}\bar{g}(X, Y) + \bar{\mu}\bar{\eta}(X)\bar{\eta}(Y),\]

where \(\bar{\lambda}, \bar{\mu}\) are smooth functions given by

\[(4.14) \quad \bar{\lambda} = [\frac{r}{2n} - (a - 2)^2]\]

and

\[(4.15) \quad \bar{\mu} = [\frac{r}{2n} - 4n + 2an - a^2].\]

In view of the relation (11.3) we state the following:

**Theorem 4.2.** Under a D-homothetic deformation, a \((2n + 1)\)-dimensional \(\eta\)-Einstein LP-Sasakian manifold is invariant.
4.2. $\phi$-sectional curvature of $LP$-Sasakian manifolds

In this section we consider the $\phi$-sectional curvature on a $(2n + 1)$-dimensional $LP$-Sasakian manifold.

From (4.3) it can be easily seen that

\[ K(X,\phi X) - K(X,\phi X) = -2(a - 1) \]

(4.16)

and hence we state the following theorem.

**Theorem 4.3.** The $\phi$-sectional curvature of $(2n+1)$-dimensional $LP$-Sasakian manifolds is not an invariant property under $D$-homothetic deformations.

If a $(2n + 1)$-dimensional $LP$-Sasakian manifold $M(\phi, \xi, \eta, \tilde{g})$ satisfies $R(X,Y)\xi = 0$ for all $X,Y$, then it can be easily seen that $K(X,\phi X) = 0$ and hence from (4.16) it follows that

\[ K(X,\phi X) = -2(a - 1) \neq 0, \]

where $X$ is a unit vector field orthogonal to $\xi$ and $K(X,\phi X)$ is the $\phi$-sectional curvature. This implies that the $\phi$-sectional curvature $\tilde{K}(X,\phi X)$ is non-vanishing. Therefore we state the following:

**Theorem 4.4.** There exists $(2n + 1)$-dimensional $LP$-Sasakian manifold with non-zero $\phi$-sectional curvature.

4.3. Locally $\phi$-Ricci symmetric $LP$-Sasakian manifolds

In this section we study locally $\phi$-Ricci symmetry on an $LP$-Sasakian manifold.

Differentiating (4.9) covariantly with respect to $W$ we obtain

\[ (\nabla_W Q)(X) = (\nabla_W Q)(X) \\
- [2(1 - a)(n - 1) + (1 - a)^2]\nabla_W \eta(X)\xi \\
- [2(1 - a)(n - 1) + (1 - a)^2]\eta(X)\nabla_W \xi. \]

(4.17)

Operating $\phi^2$ on both sides of (4.17) and taking $X$ as an orthonormal vector to $\xi$ we obtain

\[ \phi^2(\nabla_W Q)(X) = \phi^2(\nabla_W Q)(X). \]

(4.18)

In view of the relation (4.18) we state the following:

**Theorem 4.5.** The local $\phi$-Ricci symmetry on $LP$-Sasakian manifolds is an invariant property under $D$-homothetic deformations.
4.4. $\eta$– parallel Ricci tensor of an $LP$-Sasakian manifolds

Let us consider the $\eta$-parallelity of the Ricci tensor on an $LP$-Sasakian manifold.

Differentiating (4.8) covariantly with respect to $W$ and using (2.10) we obtain

$$
(\nabla_W \tilde{S})(X, Y) = (\nabla_W S)(X, Y)
- [2(1 - a)(n - 1) + (1 - a)^2]
[\eta(\phi W, X)\eta(Y) + \eta(\phi W, Y)\eta(X)].
$$

(4.19)

In (4.19) replacing $X$ by $\phi X$, $Y$ by $\phi Y$ and using (2.3) we get

$$
(\nabla_W \tilde{S})(\phi X, \phi Y) = (\nabla_W S)(\phi X, \phi Y).
$$

(4.20)

Hence we can state the following:

**Theorem 4.6.** The $\eta$-parallelity of the Ricci tensor on $LP$-Sasakian manifolds is an invariant property under D-homothetic deformations.

5. Example

We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3\}$, where $(x,y,z)$ are standard coordinates of $\mathbb{R}^3$.

The vector fields

$$
e_1 = e^z \frac{\partial}{\partial y}, \quad e_2 = e^z(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}), \quad e_3 = \frac{\partial}{\partial z}
$$

are linearly independent at each point of $M$.

Let $g$ be the Lorentzian metric defined by

$$
g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,
g(e_1, e_1) = g(e_2, e_2) = 1,
g(e_3, e_3) = -1.
$$

Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$.

Let $\phi$ be the (1,1) tensor field defined by

$$
\phi(e_1) = -e_1, \quad \phi(e_2) = -e_2, \quad \phi(e_3) = 0.
$$

Then using the linearity of $\phi$ and $g$, we have

$$
\eta(e_3) = -1,
\phi^2 Z = Z + \eta(Z)e_3,
g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W),
$$

for any $Z, W \in \chi(M)$. 
Then for \( e_3 = \xi \), the structure \((\phi, \xi, \eta, g)\) defines a Lorentzian paracontact structure on \( M \).

Let \( \nabla \) be the Levi-Civita connection with respect to the Lorentzian metric \( g \) and let \( R \) be the curvature tensor of \( g \). Then we have

\[
[e_1, e_2] = 0, \quad [e_1, e_3] = -e_1 \quad \text{and} \quad [e_2, e_3] = -e_2.
\]

Taking \( e_3 = \xi \) and using Koszul’s formula for the Lorentzian metric \( g \), we can easily calculate

\[
\nabla_{e_1} e_3 = -e_1, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_1} e_1 = -e_3,
\]

\[
\nabla_{e_2} e_3 = -e_2, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_1 = 0,
\]

\( (5.1) \)

\[
\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0.
\]

From the above it can be easily seen that \( M^3(\phi, \xi, \eta, g) \) is an \( LP \)-Sasakian manifold. With the help of the above results it can be easily verified that

\[
R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1,
\]

\[
R(e_1, e_2)e_2 = e_1, \quad R(e_2, e_3)e_2 = -e_3, \quad R(e_1, e_3)e_2 = 0, \\
R(e_1, e_2)e_1 = -e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = -e_3.
\]

From the above expressions of the curvature tensor we obtain

\[
S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) - g(R(e_1, e_3)e_3, e_1) = 2.
\]

Similarly we have

\[
S(e_2, e_2) = 2
\]

and

\[
S(e_3, e_3) = -2.
\]

Therefore,

\[
r = S(e_1, e_1) + S(e_2, e_2) - S(e_3, e_3) = 6.
\]

From [3] we know that in a 3- dimensional \( LP \)-Sasakian manifold

\[
R(X, Y)Z = \left( \frac{r - 4}{2} \right)[g(Y, Z)X - g(X, Z)Y] + \left( \frac{r - 6}{2} \right)[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\eta(Z)Y].
\]

\( (5.2) \)

Now using \( (5.2) \) we get

\[
g(R(X, Y)Z, W) = \left( \frac{r - 4}{2} \right)[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + \left( \frac{r - 6}{2} \right)[g(Y, Z)\eta(X)\eta(W) - g(X, Z)\eta(Y)\eta(W)] + \eta(Y)\eta(Z)g(X, W) - \eta(X)\eta(Z)g(Y, W).
\]

\( (5.3) \)
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From (5.3), it follows that the $\phi$-sectional curvature of the manifold is given by

$$K(X, \phi X) = \frac{r - 4}{2}$$

for any vector field $X$ orthogonal to $\xi$.

In view of the above relation we get

$$K(e_1, \phi e_1) = K(e_2, \phi e_2) = \frac{r - 4}{2}$$

Again it can be easily shown from (4.3) that

$$\bar{K}(e_1, \phi e_1) - K(e_1, \phi e_1) = -2(a - 1)$$

and

$$\bar{K}(e_2, \phi e_2) - K(e_2, \phi e_2) = -2(a - 1)$$

Therefore Theorem 4.3 is verified.

References


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