ALMOST AUTOMORPHIC 
GENERALIZED FUNCTIONS

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Abstract. The paper deals with a new algebra of generalized functions. This algebra contains Bochner almost automorphic functions and almost automorphic distributions. Properties of this algebra are studied.

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1. Introduction

The concept of almost automorphy is a generalization of Bohr almost periodicity, it has been introduced by S. Bochner, see [1] and [2]. For a general study of almost automorphic functions see Veech’s paper [7]. There is a considerable amount of papers and books on almost periodic functions and also almost automorphic functions.

L. Schwartz introduced and studied in [6] almost periodic distributions. The study of almost automorphic Schwartz distributions is done in the work [4].

An algebra of generalized functions containing Bohr almost periodic functions as well as Schwartz almost periodic distributions has been introduced and studied in [3]. In this work, we introduce and study a new algebra of generalized functions containing not only Bochner almost automorphic functions and almost automorphic distributions, but also the algebra of almost periodic generalized functions of [3]. So, naturally this paper can be seen as a continuation of our works on almost periodic generalized functions and almost automorphic distributions.

2. Regular almost automorphic functions

We consider functions and distributions defined on the whole space $\mathbb{R}$. Denote by $C_b$ the space of bounded and continuous complex valued functions on $\mathbb{R}$ endowed with the norm $\|\cdot\|_{L^\infty}$ of uniform convergence on $\mathbb{R}$, the space $(C_b, \|\cdot\|_{L^\infty})$ is a Banach algebra.

For the definition and properties of almost automorphic functions see [1], [2] and [7].

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Definition 1. A complex-valued function $f$, defined and continuous on $\mathbb{R}$, is called almost automorphic if for any sequence of real numbers $(s_n)_{n \in \mathbb{N}}$ one can extract a subsequence $(s_{n_k})_k$ such that

$$g(x) := \lim_{k \to +\infty} f(x + s_{n_k})$$

exists for every $x \in \mathbb{R}$ and

$$\lim_{k \to +\infty} g(x - s_{n_k}) = f(x)$$

for every $x \in \mathbb{R}$.

Denote by $C_{aa}$ the space of almost automorphic functions on $\mathbb{R}$.

Remark 1. The space $C_{aa}$ is a Banach subalgebra of $C_b$.

Remark 2. The space of Bohr almost periodic functions is denoted by $C_{ap}$. Every Bohr almost periodic function is an almost automorphic function, and we have $C_{ap} \subset C_{aa} \subset C_b$.

Let $p \in [1, +\infty]$ and recall the Fréchet space

$$\mathcal{D}_{L^p} = \left\{ \varphi \in C^\infty : \forall j \in \mathbb{Z}_+, \, \varphi^{(j)} \in L^p \right\}$$

endowed with the countable family of norms

$$|\varphi|_{k,p} = \sum_{j \leq k} \left\| \varphi^{(j)} \right\|_{L^p}, \, k \in \mathbb{Z}_+.$$

Definition 2. The space of almost automorphic infinitely differentiable functions on $\mathbb{R}$, denoted by $B_{aa}$, is

$$B_{aa} := \left\{ \varphi \in C^\infty : \forall j \in \mathbb{Z}_+, \, \varphi^{(j)} \in C_{aa} \right\}.$$

Example 1. The space $B_{ap}$ of regular almost periodic functions, see [3], is a Frechet subalgebra of $B_{aa}$.

Some properties of $B_{aa}$ are summarized in the following proposition.

Proposition 1. 1. $B_{aa}$ is a subalgebra of $C_{aa}$.

2. $B_{aa}$ is a Frechet subalgebra of $\mathcal{D}_{L^\infty}$.

3. $B_{aa} = C_{aa} \cap \mathcal{D}_{L^\infty}$.

4. $B_{aa} \ast L^1 \subset B_{aa}$.

Proof. See [4].

A consequence of Proposition 1 is the following result.

Corollary 1. Let $u \in \mathcal{D}_{L^\infty}$, then the following statements are equivalent:

(i) $u \in B_{aa}$.

(ii) $u \ast \varphi \in C_{aa}, \forall \varphi \in \mathcal{D}$. 
3. Almost automorphic distributions

The spaces of $L^p$-distributions, introduced in [6] and denoted by $\mathcal{D}'_{L^p}$, are the topological dual spaces of $\mathcal{D}_{L^q}$, with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq q < +\infty$. In particular, $\mathcal{D}'_{L^1}$ is the topological dual of the space $\hat{\mathcal{B}}$ defined as the closure in $\mathcal{D}'_{L^\infty}$ of the space of smooth functions with compact support. A distribution in $\mathcal{D}'_{L^1}$ is called an integrable distribution and a distribution in $\mathcal{D}'_{L^\infty}$ is called a bounded distribution. L. Schwartz provided the following characterization of $L^p$-distributions.

**Proposition 2.** Let $p \in [1, +\infty]$. A tempered distribution $T$ belongs to $\mathcal{D}'_{L^p}$ if and only if there exists $(f_j)_{j \leq k} \subset L^p$ such that

$$T = \sum_{j=0}^{k} f_j^{(j)}.$$  

A study of almost automorphic Schwartz distributions is done in the work [4]. The following result gives characterizations of almost automorphic distributions.

**Theorem 1.** Let $T \in \mathcal{D}'_{L^\infty}, T$ is said to be an almost automorphic distribution if it satisfies one of the following equivalent statements:

1. $T * \varphi \in \mathcal{C}_{aa}, \forall \varphi \in \mathcal{D}$.
2. $\exists (f_j)_{j \leq k} \subset \mathcal{C}_{aa}, T = \sum_{j \leq k} f_j^{(j)}$.

**Definition 3.** Denote by $\mathcal{B}'_{aa}$ the space of almost automorphic distributions.

**Example 2.** The space $\mathcal{B}'_{ap}$ of almost periodic distributions of Schwartz is a proper subspace of $\mathcal{B}'_{aa}$.

Some properties of $\mathcal{B}'_{aa}$ are summarized in the following proposition.

**Proposition 3.**

1. If $T \in \mathcal{B}'_{aa}$, then $\forall i \in \mathbb{Z}_+, T^{(i)} \in \mathcal{B}'_{aa}$.
2. $\mathcal{B}_{aa} \times \mathcal{B}'_{aa} \subset \mathcal{B}'_{aa}$.
3. $\mathcal{B}'_{aa} \ast \mathcal{D}'_{L^1} \subset \mathcal{B}'_{aa}$.

**Proof.** See [4].

4. Almost automorphic generalized functions

Let $I = [0, 1]$, and recall the algebra of bounded generalized functions, denoted by $\mathcal{G}_{L^\infty}$,

$$\mathcal{G}_{L^\infty} := \frac{\mathcal{M}_{L^\infty}}{\mathcal{N}_{L^\infty}},$$

where
\[ M_{L_\infty} := \left\{ (u_\epsilon) \in (D_{L_\infty})^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\epsilon|_{k,\infty} = O(\epsilon^{-m}), \epsilon \to 0 \right\} \]

and
\[ N_{L_\infty} := \left\{ (u_\epsilon) \in (D_{L_\infty})^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, |u_\epsilon|_{k,\infty} = O(\epsilon^m), \epsilon \to 0 \right\} \]

**Remark 3.** See [2] for the references on the introduction and the study of the algebras \( G_{L_p} \) constructed on the Banach spaces \( L^p \).

**Definition 4.** The space of almost automorphic moderate elements is defined as
\[
M_{aa} := \left\{ (u_\epsilon) \in (B_{aa})^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\epsilon|_{k,\infty} = O(\epsilon^{-m}), \epsilon \to 0 \right\}
\]
and the space of almost automorphic negligible elements by
\[
N_{aa} := \left\{ (u_\epsilon) \in (B_{aa})^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, |u_\epsilon|_{k,\infty} = O(\epsilon^m), \epsilon \to 0 \right\}.
\]

The main properties of \( M_{aa} \) and \( N_{aa} \) are given in the following proposition.

**Proposition 4.**
1. The space \( M_{aa} \) is a subalgebra of \( (B_{aa})^I \).
2. The space \( N_{aa} \) is an ideal in \( M_{aa} \).

**Proof.**
1. Easy by the results on the algebra \( B_{aa} \), see Proposition 1.
2. Let \((w_\epsilon)\in M_{aa}\), i.e.
\[
\forall k \in \mathbb{Z}_+, \exists m_0 \in \mathbb{Z}_+, \exists c_0 > 0, \exists \epsilon_0 \in I, \forall \epsilon < \epsilon_0, |w_\epsilon|_{k,\infty} < c_0 \epsilon^{-m_0},
\]
and \((v_\epsilon)\in N_{aa}\), i.e.
\[
\forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, \exists c_1 > 0, \exists \epsilon_1 \in I, \forall \epsilon < \epsilon_1, |v_\epsilon|_{k,\infty} < c_1 \epsilon^m.
\]
By using the Leibniz formula, we find \( c_k > 0 \) such that
\[
|w_\epsilon v_\epsilon|_{k,\infty} \leq c_k |w_\epsilon|_{k,\infty} |v_\epsilon|_{k,\infty} \leq c_k c_0 c_1 \epsilon^{-m_0 + m}.
\]
Take \( m \in \mathbb{Z}_+ \) such that \(-m_0 + m = m_1 \in \mathbb{Z}_+\), so we obtain \( \forall k \in \mathbb{Z}_+, \forall m_1 \in \mathbb{Z}_+, \exists C = c_0 c_1 c_k > 0, \exists \epsilon_2 = \inf (\epsilon_0, \epsilon_1) \in I, \forall \epsilon < \epsilon_2, \)
\[
|w_\epsilon v_\epsilon|_{k,\infty} < C \epsilon^{m_1},
\]
which gives \((w_\epsilon v_\epsilon)_\epsilon \in N_{aa}\). \(\square\)

Following the well-known classical construction of algebras of generalized functions of Colombeau type, see [3], we introduce the algebra of almost automorphic generalized functions.
Definition 5. The algebra of almost automorphic generalized functions is defined as the quotient
\[ G_{aa} := \frac{\mathcal{M}_{aa}}{\mathcal{N}_{aa}}. \]

Notation 1. If \( u \in G_{aa} \), then \( u = [(u_\epsilon)_\epsilon] = (u_\epsilon)_\epsilon + \mathcal{N}_{aa} \), where \( (u_\epsilon)_\epsilon \) is a representative of \( u \).

Remark 4. The algebra of almost automorphic generalized functions \( G_{aa} \) is embedded into \( G_{L^\infty} \) canonically.

The following characterization of elements of \( G_{aa} \) is similar to the result of Theorem 1-(1).

Proposition 5. Let \( u = [(u_\epsilon)_\epsilon] \in G_{L^\infty} \). Then the following statements are equivalent

1. \( u \in G_{aa} \).
2. \( u_\epsilon * \varphi \in B_{aa}, \forall \epsilon \in I, \forall \varphi \in \mathcal{D} \).

Proof. If \( u = [(u_\epsilon)_\epsilon] \in G_{aa} \), then \( u_\epsilon \in B_{aa}, \forall \epsilon \in I \), and due to (4) of Proposition 1, \( u_\epsilon * \varphi \in B_{aa}, \forall \epsilon \in I, \forall \varphi \in \mathcal{D} \). Conversely, let \( u = [(u_\epsilon)_\epsilon] \in G_{L^\infty} \) and \( u_\epsilon * \varphi \in B_{aa}, \forall \epsilon \in I, \forall \varphi \in \mathcal{D} \), so \( u_\epsilon \in D_{L^\infty}, \forall \epsilon \in I \), and \( u_\epsilon * \varphi \in B_{aa}, \forall \epsilon \in I, \forall \varphi \in \mathcal{D} \). Corollary 1 gives that \( u_\epsilon \in B_{aa}, \forall \epsilon \in I \). Since \( u \in G_{L^\infty} \), we have

\[ \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\epsilon|_{k,\infty} = O(\epsilon^{-m}), \epsilon \to 0, \]

consequently \( (u_\epsilon)_\epsilon \in \mathcal{M}_{aa} \) and thus \( u \in G_{aa} \).

Remark 5. The characterization 2. from the previous Proposition does not depend on representatives.

The following result is easy to prove.

Proposition 6. The algebra of almost periodic generalized functions \( G_{ap} \) of \([3]\) is embedded canonically into \( G_{aa} \).

The following result is well-known.

Lemma 1. There exists \( \rho \in \mathcal{S} \) satisfying

\[ (4.1) \quad \int_{\mathbb{R}} \rho(x) \, dx = 1 \] and \( \int_{\mathbb{R}} x^k \rho(x) \, dx = 0, \forall k \geq 1. \]

Denote by \( \Sigma \) the set of of functions \( \rho \in \mathcal{S} \) satisfying (4.1), and define \( \rho_\epsilon(\cdot) := \frac{1}{\epsilon} \rho\left(\frac{\cdot}{\epsilon}\right), \epsilon > 0. \)

By means of convolution with a mollifier from \( \Sigma \), we embed the space of almost automorphic distributions \( \mathcal{B}_{aa}^\prime \) into the algebra \( G_{aa} \).
Proposition 7. Let $\rho \in \Sigma$, the map

$$i_{aa} : \mathcal{B}_{aa} \to \mathcal{G}_{aa}$$

$$u \mapsto (u * \rho_\epsilon)_\epsilon + \mathcal{N}_{aa}$$

is a linear embedding which commutes with derivatives.

Proof. Let $u \in \mathcal{B}_{aa}'$, then $\exists (f_j)_{j \leq p} \subset \mathcal{C}_{aa}$ and $u = \sum_{j \leq p} f_j^{(j)}$. Let us show that $(u * \rho_\epsilon)_\epsilon \in \mathcal{M}_{aa}$. By (4) of Proposition 1, $u * \rho_\epsilon \in \mathcal{B}_{aa}$, $\forall \epsilon \in I$. Moreover, we have

$$\left|\left( u^{(i)} * \rho_\epsilon \right)(x) \right| \leq \sum_{j \leq p} \frac{1}{\epsilon^{i+j}} \int_{\mathbb{R}} \left| f_j (x - \epsilon y) \rho^{(i+j)}(y) \right| dy,$$

then

$$\sup_{x \in \mathbb{R}} \left|\left( u^{(i)} * \rho_\epsilon \right)(x) \right| \leq \sum_{j \leq p} \frac{1}{\epsilon^{i+j}} \|f_j\|_{L_\infty} \int_{\mathbb{R}} \left| \rho^{(i+j)}(y) \right| dy,$$

consequently $\exists C > 0$ such that

$$|u * \rho_\epsilon|_{k, \infty} \leq \frac{C}{\epsilon^{k+p}}.$$

So, $(u * \rho_\epsilon)_\epsilon \in \mathcal{M}_{aa}$. The linearity of $i_{aa}$ follows from the linearity of convolution. If $(u * \rho_\epsilon)_\epsilon \in \mathcal{N}_{aa}$, then $\lim_{\epsilon \to 0} u * \rho_\epsilon = 0$ in $\mathcal{D}_{L_\infty}'$, but, it is easy to see that $\lim_{\epsilon \to 0} u * \rho_\epsilon = u$ in $\mathcal{D}_{L_\infty}'$, so $u = 0$, which means that $i_{aa}$ is injective. Finally, $i_{aa}(u^{(j)}) = [(u^{(j)} * \rho_\epsilon)_\epsilon]_\epsilon = [(u * \rho_\epsilon)_\epsilon]^{(j)} = (i_{aa}(u))^{(j)}$. □

Defining the canonical embedding

$$\sigma_{aa} : \mathcal{B}_{aa} \to \mathcal{G}_{aa}$$

$$f \mapsto (f)_\epsilon + \mathcal{N}_{aa},$$

we have two ways to embed the space $\mathcal{B}_{aa}$ into $\mathcal{G}_{aa}$ by $i_{aa}$ and also by $\sigma_{aa}$. The following result gives that we have the same result.

Proposition 8. The following diagram

$$\begin{array}{ccc}
\mathcal{B}_{aa} & \to & \mathcal{B}_{aa}' \\
\sigma_{aa} \downarrow & & \downarrow i_{aa} \\
\mathcal{G}_{aa} & \to & \mathcal{G}_{aa}
\end{array}$$

commutes.

Proof. It suffices to show that for $f \in \mathcal{B}_{aa}$ we have $(f * \rho_\epsilon - f)_\epsilon \in \mathcal{N}_{aa}$. Applying Taylor’s formula, we obtain, $\forall m \in \mathbb{Z}_+$,

$$\left( f^{(j)} * \rho_\epsilon \right)(x) - f^{(j)}(x) = \int_{\mathbb{R}} \sum_{k=1}^{m} \frac{(-\epsilon y)^k}{k!} f^{(k+j)}(x) \rho(y) dy +$$

$$\int_{\mathbb{R}} \frac{(-\epsilon y)^{m+1}}{(m+1)!} f^{(m+j+1)}(x - \theta(x) \epsilon y) \rho(y) dy.$$
Since \( f \in \mathcal{B}_{aa} \) and \( \rho \in \Sigma \), then

\[
\left\| f^{(j)} * \rho - f^{(j)} \right\|_{L^\infty} \leq \left\| f^{(m+j+1)} \right\|_{L^\infty} \| y^{m+1} \|_{L^1} \frac{\epsilon^{m+1}}{(m+1)!},
\]

consequently \( \forall k \in \mathbb{Z}_+, \forall m' = m + 1, \n |f * \rho - f|_{k,\infty} = O(\epsilon^{m'}), \epsilon \to 0, \)
i.e. \( (f * \rho - f)_{\epsilon} \in \mathcal{N}_{aa}. \)

\[ \square \]

The algebra of tempered generalized functions on \( \mathbb{C} \) is denoted by \( \mathcal{G}_T \), see [5] for the definition and properties of \( \mathcal{G}_T \).

**Proposition 9.** Let \( u = [(u_{\epsilon})_{\epsilon}] \in \mathcal{G}_{aa} \) and \( F = [(f_{\epsilon})_{\epsilon}] \in \mathcal{G}_T \), then

\[
F \circ u := [(f_{\epsilon} \circ u_{\epsilon})_{\epsilon}] + \mathcal{N}_{aa}
\]
is a well-defined element of \( \mathcal{G}_{aa} \).

**Proof.** Since \( (f_{\epsilon})_{\epsilon} \in \mathcal{M}_T \) and \( (u_{\epsilon})_{\epsilon} \in \mathcal{M}_{aa} \), by the classical result of composition of almost automorphic function with continuous function, we have \( f_{\epsilon} \circ u_{\epsilon} \in \mathcal{B}_{aa}, \forall \epsilon \in I. \) The estimates

\[
\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |f_{\epsilon} \circ u_{\epsilon}|_{k,\infty} = O(\epsilon^{-m}), \epsilon \to 0,
\]
are obtained from the fact that \( (u_{\epsilon})_{\epsilon} \in \mathcal{M}_{aa} \) and \( (f_{\epsilon})_{\epsilon} \) is polynomially bounded. It is easy to prove that the composition is independent on representatives. \[ \square \]

The convolution of an almost automorphic distribution with an integrable distribution is an almost automorphic distribution. We extend this result to the case of almost automorphic generalized functions.

**Proposition 10.** Let \( u = [(u_{\epsilon})_{\epsilon}] \in \mathcal{G}_{aa} \) and \( v \in \mathcal{D}'_{L^1} \), then the convolution \( u * v \) defined by

\[
uu \star v := [(u_{\epsilon} \star v)_{\epsilon}] \]
is a well-defined element of \( \mathcal{G}_{aa} \).

**Proof.** The characterization \( (11) \) of elements of \( \mathcal{D}'_{L^1} \) gives that there exists \( (f_i)_{j \leq p} \subset L^1 \) such that \( v = \sum_{i \leq p} f_i^{(i)}. \) Let \( (u_{\epsilon})_{\epsilon} \in \mathcal{M}_{aa} \) be a representative of \( u. \) Then \( u_{\epsilon} \in \mathcal{B}_{aa}, \forall \epsilon \in I, \) by Proposition 1, \( u_{\epsilon} \star v = \sum_{i \leq p} u_{\epsilon}^{(i)} \star f_i \in \mathcal{B}_{aa}, \forall \epsilon \in I. \) Moreover, by Young inequality, we have

\[
\left\| (u_{\epsilon} \star v)^{(j)} \right\|_{L^\infty} \leq \sum_{i \leq p} \left\| f_i \right\|_{L^1} \left\| u_{\epsilon}^{(i+j)} \right\|_{L^\infty},
\]
so the fact that \( (u_{\epsilon})_{\epsilon} \in \mathcal{M}_{aa} \) gives that

\[
\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_{\epsilon} \star v|_{k,\infty} = O(\epsilon^{-m}), \epsilon \to 0,
\]
consequently \( (u_{\epsilon} \star v)_{\epsilon} \in \mathcal{M}_{aa}. \) Finally, one shows that the result is independent on representatives by obtaining the same estimates. \[ \square \]
We give an extension of the classical Bohl-Bohr theorem. First, we recall the definition of a primitive of a generalized function.

**Definition 6.** Let \( u = [(u_\epsilon)_\epsilon] \in \mathcal{G}_{aa} \) and \( x_0 \in \mathbb{R} \), a primitive of \( u \) is a generalized function \( U \) defined by

\[
U(x) = \left( \int_{x_0}^{x} u_\epsilon(t) \, dt \right)_\epsilon + \mathcal{N}[\mathbb{C}]
\]

**Proposition 11.** A primitive of an almost automorphic generalized function is almost automorphic if and only if it is a bounded generalized function.

**Proof.** Let \( u = [(u_\epsilon)_\epsilon] \in \mathcal{G}_{aa} \), so \( u_\epsilon \in \mathcal{B}_{aa}, \forall \epsilon \in I \). If \( U \) is a primitive of \( u \) and \( U \in \mathcal{G}_{aa} \), then \( U \in \mathcal{G}_{L^\infty} \) because \( \mathcal{G}_{aa} \subset \mathcal{G}_{L^\infty} \). Conversely, if \( U = [(U_\epsilon)_\epsilon] \in \mathcal{G}_{L^\infty} \), then \( \forall \epsilon \in I, U_\epsilon = \int_{x_0}^{x} u_\epsilon(t) \, dt \in D_{L^\infty}, \) so \( U_\epsilon \) is bounded primitive of \( u_\epsilon \in \mathcal{C}_{aa} \).

By the classical result of Bohl-Bohr we have \( U_\epsilon \in \mathcal{C}_{aa}, \) consequently \( U_\epsilon \in \mathcal{C}_{aa} \cap D_{L^\infty}, \forall \epsilon \in I \). By Proposition 1, \( U_\epsilon \in \mathcal{B}_{aa}, \forall \epsilon \in I \). Moreover \( (U_\epsilon)_\epsilon \in \mathcal{M}_{L^\infty}, \) i.e.

\[
\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |U_\epsilon|_{k,\infty} = O(\epsilon^{-m}), \epsilon \to 0,
\]

so \( (U_\epsilon)_\epsilon \in \mathcal{M}_{aa} \) and \( U \in \mathcal{G}_{aa} \). The result is independent on representatives. \( \square \)

**References**


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