UNIFICATION OF $\lambda$-CLOSED SETS VIA GENERALIZED TOPOLOGIES

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Abstract. In this paper we introduce and study a new type of sets called $(\wedge, \mu\nu)$-closed sets by using the concept of generalized topology introduced by A. Császár.

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1. Introduction

For the last couple of years, different forms of open sets are being studied. Recently, a significant contribution to the theory of generalized open sets has been presented by A. Császár $^{10,11,12}$. Especially, the author defined some basic operators on generalized topological spaces. It is observed that a large number of papers are devoted to the study of generalized open sets like open sets of a topological space, containing the class of open sets and possessing properties more or less similar to those of open sets.

We recall some notions defined in $^{11}$. Let $X$ be a non-empty set and let $expX$ denote the power set of $X$. We call a class $\mu \subseteq expX$ a generalized topology $^{10}$, (briefly, GT) if $\emptyset \in \mu$ and unions of elements of $\mu$ belong to $\mu$. A set $X$ with a GT $\mu$ on it is called a generalized topological space (briefly, GTS) and is denoted by $(X, \mu)$. The $\theta$-closure, $cl_\theta(A)$ $^{23}$ (resp. $\delta$-closure, $cl_\delta(A)$ $^{23}$) of a subset $A$ of a topological space $(X, \tau)$ is defined by $\{x \in X : clU \cap A \neq \emptyset \text{ for all } U \in \tau \text{ with } x \in U\}$ (resp. $\{x \in X : A \cap U \neq \emptyset \text{ for all regular open sets } U \text{ containing } x\}$, where a subset $A$ is said to be regular open if $A = int(cl(A))$.

A is said to be $\delta$-closed $^{23}$ (resp. $\theta$-closed $^{23}$) if $A = cl_\delta A$ (resp. $A = cl_\theta A$) and the complement of a $\delta$-closed set (resp. $\theta$-closed) set is known as a $\delta$-open (resp. $\theta$-open) set. A subset $A$ of a topological space $(X, \tau)$ is said to be preopen $^{21}$ (resp. semi-open $^{17}$, $\alpha$-open $^{21}$, $b$-open $^{11}$) if $A \subseteq int(cl(A))$ (resp. $A \subseteq cl(int(A))$, $A \subseteq int(cl(int(A)))$, $A \subseteq cl(int(A)) \cup int(cl(A)))$. The complement of a semi-open set is called a semi-closed set. The semi-closure $^{18}$ of $A$, denoted by $scl(A)$, is the intersection of all semi-closed sets containing $A$. A point $x \in X$ is called a semi-$\theta$-cluster point $^{18}$ of a set $A$ if $sclU \cap A \neq \emptyset$ for each semi-open set $U$ containing $x$. The set of all semi-$\theta$-cluster points of $A$ is denoted by $scl_{\theta}A$. If $A = scl_{\theta}A$, then $A$ is known as semi-$\theta$-closed and
the complement of a semi-$\theta$-closed set is called a semi-$\theta$-open set [IL]. We note that for any topological space $(X, \tau)$, the collection of all open (resp. preopen, semi-open, $\delta$-open, $\alpha$-open, $b$-open, $\theta$-open, semi-$\theta$-open) sets is denoted by $\tau$ (resp. $PO(X)$, $SO(X)$, $\delta O(X)$, $\alpha O(X)$, $BO(X)$ or $\gamma O(X)$, $\theta O(X)$, $S\theta O(X)$). Each of these collections is a generalized topology on $X$.

For a GTS $(X, \mu)$, the elements of $\mu$ are called $\mu$-open sets and the complements of $\mu$-open sets are called $\mu$-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all $\mu$-closed sets containing $A$, i.e., the smallest $\mu$-closed set containing $A$; and by $i_\mu(A)$ the union of all $\mu$-open sets contained in $A$, i.e., the largest $\mu$-open set contained in $A$ (see [II, III]).

It is easy to observe that $i_\mu$ and $c_\mu$ are idempotent and monotonic, where the operator $\gamma : \exp X \to \exp X$ is said to be idempotent if $A \subseteq X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic if $A \subseteq B \subseteq X$ implies $\gamma(A) \subseteq \gamma(B)$. It is also well known from [II, I2] that if $\mu$ is a GT on $X$, $x \in X$ and $A \subseteq X$, then $x \in c_\mu(A)$ iff $x \in M \in \mu \Rightarrow M \cap A \neq \emptyset$ and $c_\mu(X \setminus A) = X \setminus i_\mu(A)$.

As the final prerequisites, we wish to recall a few definitions and results from [I3].

**Definition 1.1.** [I3] Let $(X, \mu)$ be a GTS and $A \subseteq X$. Then, the subset $\wedge_\mu(A)$ is defined as follows:

$$\wedge_\mu(A) = \left\{ \bigcap X \{G : A \subseteq G, G \in \mu\} \right\}, \quad \text{if there exists } G \in \mu \text{ such that } A \subseteq G;$$

otherwise.

**Proposition 1.2.** [I3] Let $A, B$ and $\{B_\alpha : \alpha \in \Omega\}$ be subsets of a GTS $(X, \mu)$. Then the following properties hold:

(a) $B \subseteq \wedge_\mu(B)$;
(b) If $A \subseteq B$, then $\wedge_\mu(A) \subseteq \wedge_\mu(B)$;
(c) $\wedge_\mu(\lambda_\alpha(B)) = \wedge_\mu(B)$;
(d) $\wedge_\mu[\bigcup_{\alpha \in \Omega} B_\alpha] = \bigcup_{\alpha \in \Omega} [\wedge_\mu(B_\alpha)]$;
(e) If $A \in \mu$, then $A = \wedge_\mu(A)$;
(f) $\wedge_\mu[\bigcap_{\alpha \in \Omega} B_\alpha] \subseteq \bigcap_{\alpha \in \Omega} [\wedge_\mu(B_\alpha)]$;

**Definition 1.3.** [I4] In a GTS $(X, \mu)$, a subset $B$ is called a $\wedge_\mu$-set if $B = \wedge_\mu(B)$.

**Theorem 1.4.** [I4] If $(X, \mu)$ is a GTS, then the intersection of $\wedge_\mu$-sets is a $\wedge_\mu$-set.

2. $(\wedge, \mu\nu)$-closed sets and associated separation axioms

**Definition 2.1.** Let $\mu$ and $\nu$ be two GT’s on $X$. A subset $A$ of $X$ is said to be $(\wedge, \mu\nu)$-closed if $A = U \cap F$, where $U$ is a $\wedge_\mu$-set and $F$ is a $\nu$-closed set.

The family of all $(\wedge, \mu\nu)$-closed sets of $(X, \mu, \nu)$ is denoted by $\wedge_{\mu\nu}\mu$.

**Remark 2.2.** In a topological space $(X, \tau)$, if $\mu = \nu = \tau$ (resp. $SO(X)$, $\alpha O(X)$, $\theta O(X)$, $\delta O(X)$, $S\theta O(X)$), then a $(\wedge, \mu\nu)$-closed set reduces to a $\lambda$-closed set [2].
Lemma 2.3. Let $\mu$ and $\nu$ be two GT’s on $X$, then the following properties are equivalent:

(a) $A$ is $(\wedge, \mu\nu)$-closed;
(b) $A = U \cap c_\nu(A)$, where $U$ is a $\bigwedge_\mu$-set;
(c) $A = \bigwedge_\mu(A) \cap c_\nu(A)$.

Proof. (a) $\Rightarrow$ (b): Let $A = U \cap F$, where $U$ is a $\bigwedge_\mu$-set and $F$ is a $\nu$-closed set of $X$. Since $A \subseteq F$, we have $c_\nu(A) \subseteq F$. Thus $A \subseteq U \cap c_\nu(A) \subseteq U \cap F = A$.

(b) $\Rightarrow$ (c): Let $A = U \cap c_\nu(A)$, where $U$ is a $\bigwedge_\mu$-set. Since $A \subseteq U$, we have by Proposition 2.6, $\bigwedge_\mu(A) \subseteq \bigwedge_\mu(U) = U$ and hence, $A \subseteq \bigwedge_\mu(A) \cap c_\nu(A) \subseteq U \cap c_\nu(A) = A$. Thus, we obtain $A = \bigwedge_\mu(A) \cap c_\nu(A)$.

(c) $\Rightarrow$ (a): We know that $c_\nu(A)$ is a $\nu$-closed set and by Proposition 2.6(c), we have $\bigwedge_\mu(A)$ is a $\bigwedge_\mu$-set. Thus by (c), we have $A = \bigwedge_\mu(A) \cap c_\nu(A)$ and hence $A$ is a $(\wedge, \mu\nu)$-closed set. \qed

Remark 2.4. Every $\bigwedge_\mu$-set is $(\wedge, \mu\nu)$-closed and every $\nu$-closed set is $(\wedge, \mu\nu)$-closed.

Example 2.5. Let $X = \{a, b, c\}$, $\mu = \emptyset, \{a\}, \{a, b\}$ and $\nu = \emptyset, \{b\}, \{a, b\}$. Then, $\mu$ and $\nu$ are two GT’s on $X$. It is easy to see that $\{a, c\}$ is a $(\wedge, \mu\nu)$-closed set but it is not a $\bigwedge_\mu$-set and $\{a, b\}$ is a $(\wedge, \mu\nu)$-closed set but it is not a $\nu$-closed set.

Proposition 2.6. Let $\mu$ and $\nu$ be two GT’s on a set $X$. Then $\bigwedge_{\mu\nu}$ is closed under arbitrary intersections.

Proof. Suppose that $\{A_\alpha : \alpha \in I\}$ is a family of $(\wedge, \mu\nu)$-closed subsets of $X$. Then, for each $\alpha \in I$ there exist a $\bigwedge_\mu$-set $U_\alpha$ and a $\nu$-closed $F_\alpha$ such that $A_\alpha = U_\alpha \cap F_\alpha$. Hence we have $\bigcap_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (U_\alpha \cap F_\alpha) = \bigcap_{\alpha \in I} U_\alpha \cap \bigcap_{\alpha \in I} F_\alpha$.

We note that $\bigcap_{\alpha \in I} U_\alpha$ is a $\bigwedge_\mu$-set (by Theorem 2.3) and $\bigcap_{\alpha \in I} F_\alpha$ is $\nu$-closed. Thus by Definition 2.4, it follows that $\bigcap_{\alpha \in I} A_\alpha$ is a $(\wedge, \mu\nu)$-closed set. \qed

Example 2.7. Let $X = \{a, b, c\}$. Consider two GT’s on $X$ as $\mu = \emptyset, \{a\}, \{a, b\}$ and $\nu = \emptyset, \{a, b\}$. It is easy to see that $\{a\}$ and $\{c\}$ are two $(\wedge, \mu\nu)$-closed subsets of $X$ but their union $\{a, c\}$ is not a $(\wedge, \mu\nu)$-closed set.

Definition 2.8. Let $\mu$ and $\nu$ be two GT’s on $X$. Then a subset $A$ of $X$ is said to be generalized $\mu\nu$-closed (briefly, $\mu\nu g$-closed) if $c_\nu(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \mu$. 

Unification of $\lambda$-closed sets via generalized topologies

53
Observation 2.9. Let $\mu$ and $\nu$ be two GT’s on $X$ and $A$, $B$ be two subsets of $X$.

(i) If $A$ is $\nu$-closed, then $A$ is $\mu \nu g$-closed.
(ii) If $A$ is $\mu \nu g$-closed and $\mu$-open, then $A$ is $\nu$-closed.
(iii) If $A$ is $\mu \nu g$-closed and $A \subseteq B \subseteq c_\nu(A)$, then $B$ is $\mu \nu g$-closed.
(iv) $A$ is $\mu \nu g$-closed if and only if $c_\nu(A) \subseteq \bigwedge_\mu(A)$.

Proof. The proofs of (i), (ii) and (iii) are straightforward, and we shall only prove (iv). Let $A$ be a $\mu \nu g$-closed set and $U$ be any $\mu$-open set such that $A \subseteq U$. Then $c_\nu(A) \subseteq U$ and hence we obtain $c_\nu(A) \subseteq \bigwedge_\mu(A)$.

Conversely, suppose that $c_\nu(A) \subseteq \bigwedge_\mu(A)$ and $A \subseteq U \in \mu$. Then $c_\nu(A) \subseteq \bigwedge_\mu(A) \subseteq U$. This shows that $A$ is $\mu \nu g$-closed.

Example 2.10. Let $\mu = \{\varnothing, \{a\}, \{a, b\}, \{b, c\}, X\}$ and $\nu = \{\varnothing, \{a\}, \{a, c\}\}$ be two GT’s on a set $X = \{a, b, c\}$. Then it is easy to see that $\{c\}$ is a $\mu \nu g$-closed set which is not a $\nu$-closed set. Also, $\{b\}$ is a $\nu$-closed set which is not a $\mu$-open set.

Proposition 2.11. Let $\mu$ and $\nu$ be two GT’s on a set $X$. Then a subset $A$ of $X$ is $\nu$-closed if and only if $A$ is $\mu \nu g$-closed and $(\bigwedge, \mu \nu)$-closed.

Proof. One part follows from Observation 2.9(i) and Remark 2.9. Conversely, let $A$ be a $\mu \nu g$-closed as well as a $(\bigwedge, \mu \nu)$-closed set. Then by Observation 2.9(iv), $c_\nu(A) \subseteq \bigwedge_\mu(A)$. Thus by hypothesis and Lemma 2.9, $A = \bigwedge_\mu(A) \cap c_\nu(A) = c_\nu(A)$. So $A$ is a $\nu$-closed set.

Definition 2.12. Let $\mu$ and $\nu$ be two GT’s on a set $X$. Then $(X, \mu, \nu)$ is said to be

(i) $\mu \nu T_0$ if for any two distinct points $x, y \in X$, there exists a $\mu$-open set $U$ of $X$ containing $x$ but not $y$ or a $\nu$-open set $V$ of $X$ containing $y$ but not $x$.
(ii) $\mu \nu T_{1/2}$ if every singleton $\{x\}$ is either $\nu$-open or $\mu$-closed.

Theorem 2.13. Let $\mu$ and $\nu$ be two GT’s on a set $X$. Then $(X, \mu, \nu)$ is $\mu \nu T_0$ if and only if for each $x \in X$, the singleton $\{x\}$ is $(\bigwedge, \mu \nu)$-closed.

Proof. Suppose that $(X, \mu, \nu)$ be $\mu \nu T_0$. For each $x \in X$, we have $\{x\} \subseteq \bigwedge_\mu(\{x\}) \cap c_\nu(\{x\})$. Let $y \neq x$. Then there exists a $\mu$-open set $U$ of $X$ containing $x$ but not $y$ or a $\nu$-open set $V$ of $X$ containing $y$ but not $x$. In the first case, $y \notin \bigwedge_\mu(\{x\})$ and we have $y \notin \bigwedge_\mu(\{x\}) \cap c_\nu(\{x\})$. In the second case, $y \notin c_\nu(\{x\})$ and we have $y \notin \bigwedge_\mu(\{x\}) \cap c_\nu(\{x\})$. Thus $\bigwedge_\mu(\{x\}) \cap c_\nu(\{x\}) \subseteq \{x\}$. Hence we have $\bigwedge_\mu(\{x\}) \cap c_\nu(\{x\}) = \{x\}$. Hence by Lemma 2.9, $\{x\}$ is a $(\bigwedge, \mu \nu)$-closed set.

Conversely, suppose that $(X, \mu, \nu)$ is not $\mu \nu T_0$. Thus there exist distinct points $x, y \in X$ such that (i) $y \in U$ for every $\mu$-open set $U$ containing $x$ and (ii) $x \in V$ for every $\nu$-open set $V$ containing $y$. Thus by (i) and (ii), $y \in \bigwedge_\mu(\{x\})$ and $y \in c_\nu(\{x\})$, respectively. Then by Lemma 2.9, $y \in \bigwedge_\mu(\{x\}) \cap c_\nu(\{x\}) = \{x\}$. This contradicts the fact that $x \neq y$.

Theorem 2.14. Let $\mu$ and $\nu$ be two GT’s on a set $X$. Then the following statements are equivalent:
Proof. (a) ⇒ (b): Let \((X, \mu, \nu)\) be \(\mu\nu\)-closed. Suppose that there exists a \(\mu\nu\)-closed set \(A\) of \(X\) which is not \(\nu\)-closed. So, there exists \(x \in c_\nu(A) \setminus A\). If \(\{x\}\) is \(\nu\)-open, then \(x \in A\), which is a contradiction. In the case \(\{x\}\) is \(\mu\)-closed, we have \(x \in X \setminus \{x\}\) and so \(A \subseteq X \setminus \{x\} \in \mu\). So, by \(\mu\nu\)-closedness of \(A\), \(c_\nu(A) \subseteq X \setminus \{x\}\), which is a contradiction.

(b) ⇒ (a): Suppose that \(\{x\}\) is not \(\mu\)-closed. If \(X\) is not \(\mu\)-open, then we have nothing to show. If \(X \in \mu\), then the only \(\mu\)-open set containing \(X \setminus \{x\}\) is \(X\). Thus \(c_\nu(X \setminus \{x\}) \subseteq X\) and hence \(X \setminus \{x\}\) is \(\mu\nu\)-closed. Thus, by (b), \(X \setminus \{x\}\) is \(\nu\)-closed. So \(\{x\}\) is \(\nu\)-open. Therefore, \((X, \mu, \nu)\) is \(\mu\nu\)-closed.

(a) ⇒ (c): Suppose that \((X, \mu, \nu)\) is \(\mu\nu\)-closed and \(A \subseteq X\). Then, for each \(x \in X\), \(\{x\}\) is \(\nu\)-open or \(\mu\)-closed. Let \(B_\nu = \cap\{X \setminus \{x\} : x \in X \setminus A, \{x\}\) is \(\nu\)-open\} and \(C_\mu = \cap\{X \setminus \{x\} : x \in X \setminus A, \{x\}\) is \(\mu\)-closed\}. Then, \(B_\nu\) is \(\nu\)-closed, \(C_\mu\) is a \(\Lambda_\mu\)-set and \(A = B_\nu \cap C_\mu\). Therefore, \(A\) is \((\Lambda, \mu\nu)\)-closed.

(c) ⇒ (a): Suppose that \(A\) is a \(\mu\nu\)-closed subset of \(X\). Then, by the hypothesis, \(A\) is \((\Lambda, \mu\nu)\)-closed. Thus, by Proposition \(\mathbf{[3]}\), \(A\) is \(\nu\)-closed. Therefore, \((X, \mu, \nu)\) is \(\mu\nu\)-closed \((\text{by } (a) \iff (b))\).

3. \(g \Lambda_{\mu\nu}\) - sets

**Definition 3.1.** Let \(\mu\) and \(\nu\) be two GT’s on a set \(X\). Then a subset \(A\) of \(X\) is called a \(g \Lambda_{\mu\nu}\)-set if \(\Lambda_\mu(A) \subseteq F\) whenever \(A \subseteq F\) and \(F\) is a \(\nu\)-closed set.

The family of all \(g \Lambda_{\mu\nu}\)-sets is denoted by \(g \Lambda_{\mu\nu}\). The complement of a \(g \Lambda_{\mu\nu}\)-set is called \(g \Lambda_{\mu\nu}^c\)-set.

**Remark 3.2.** Let \((X, \tau)\) be a topological space. If \(\mu = \nu = \tau\) (resp. \(\text{SO}(X), \text{PO}(X), \text{BO}(X), \delta\text{O}(X)\)) then a \(g \Lambda_{\mu\nu}\)-set is a generalized \(\Lambda\)-set [19] (resp. generalized \(\Lambda\)-set [3], generalized pre-\(\Lambda\)-set [15], \(g \Lambda_{\delta}\)-set [15], \(g \Lambda_{\theta}\)-set [1]).

**Proposition 3.3.** Let \(\mu\) and \(\nu\) be two GT’s on a set \(X\) and \(A\) and \(B\) be two subsets of \(X\), then the following properties hold:

(a) If \(A\) is a \(\Lambda_\mu\)-set, then \(A\) is a \(g \Lambda_{\mu\nu}\)-set.

(b) If \(A\) is a \(g \Lambda_{\mu\nu}\)-set and \(\nu\)-closed, then \(A\) is a \(\Lambda_\mu\)-set.

(c) If \(A\) is a \(g \Lambda_{\mu\nu}\)-set and \(A \subseteq B \subseteq \Lambda_\mu(A)\), then \(B\) is a \(g \Lambda_{\mu\nu}\)-set.

Proof. (a) Suppose that \(A\) is a \(\Lambda_\mu\)-set and \(A \subseteq F\), where \(F\) is a \(\nu\)-closed set. Then \(\Lambda_\mu(A) = A \subseteq F\). Thus \(A\) is a \(g \Lambda_{\mu\nu}\)-set.

(b) Let \(A\) be a \(g \Lambda_{\mu\nu}\)-set and \(\nu\)-closed. Then \(\Lambda_\mu(A) \subseteq A\). Thus, by Proposition \(\mathbf{[3]}(a)\), \(\Lambda_\mu(A) = A\) i.e., \(A\) is a \(\Lambda_\mu\)-set.

(c) Let \(B \subseteq F\), where \(F\) is a \(\nu\)-closed set. Then, \(A \subseteq F\) and \(A\) is a \(g \Lambda_{\mu\nu}\)-set. Therefore, \(\Lambda_\mu(A) \subseteq F\). Now, by Proposition \(\mathbf{[3]}\) we have, \(\Lambda_\mu(A) \subseteq \)
\[ \bigwedge_\mu(B) \subseteq \bigwedge_\mu(\bigwedge_\mu(A)) = \bigwedge_\mu(A). \] Thus \( \bigwedge_\mu(A) = \bigwedge_\mu(B) \) and hence \( \bigwedge_\mu(B) \subseteq F \).

Therefore, \( B \) is a \( g_\bigwedge_\mu \)-set.

**Example 3.4.** Let \( X = \{a, b, c\} \), \( \mu = \{\emptyset, \{a, b\}\} \) and \( \nu = \{\emptyset, \{c\}, \{a, c\}\} \). Then \( \mu \) and \( \nu \) are two GT’s on \( X \). It is easy to check that \( \{a\} \) is a \( g_\bigwedge_\mu \)-set which is not a \( \bigwedge_\mu \)-set. We also note that \( \{a, b\} \) and \( \{b, c\} \) are two \( g_\bigwedge_\mu \)-sets but their intersection \( \{b\} \) is not a \( g_\bigwedge_\mu \)-set.

**Proposition 3.5.** Let \( \mu \) and \( \nu \) be two GT’s on a set \( X \). Then a subset \( A \) is a \( g_\bigwedge_\mu \)-set if and only if \( \bigwedge_\mu(A) \cap U = \emptyset \) whenever \( A \cap U = \emptyset \) and \( U \in \nu \).

**Proof.** Suppose that \( A \) is a \( g_\bigwedge_\mu \)-set. Let \( A \cap U = \emptyset \) and \( U \in \nu \). Then \( A \subseteq X \setminus U \) and \( X \setminus U \) is \( \nu \)-closed. Therefore, \( \bigwedge_\mu(A) \subseteq X \setminus U \) and hence \( \bigwedge_\mu(A) \cap U = \emptyset \).

Conversely, let \( A \subseteq F \) and \( F \) be \( \nu \)-closed. Then \( A \cap (X \setminus F) = \emptyset \) and \( X \setminus F \in \nu \). So, by the hypothesis we have \( \bigwedge_\mu(A) \cap (X \setminus F) = \emptyset \) and hence \( \bigwedge_\mu(A) \subseteq F \). This shows that \( A \) is a \( g_\bigwedge_\mu \)-set. \( \square \)

**Proposition 3.6.** Let \( \mu \) and \( \nu \) be two GT’s on a set \( X \). Then a subset \( A \) of \( X \) is a \( g_\bigwedge_\mu \)-set if and only if \( \bigwedge_\mu(A) \subseteq c_\nu(A) \).

**Proof.** Suppose that \( A \) is a \( g_\bigwedge_\mu \)-set and \( x \not\in c_\nu(A) \). Then there exists a \( \nu \)-open set \( U \) containing \( x \) such that \( A \cap U = \emptyset \). Thus by Proposition 3.4, \( \bigwedge_\mu(A) \cap U = \emptyset \) (as \( A \) is a \( g_\bigwedge_\mu \)-set). Hence \( x \not\in \bigwedge_\mu(A) \) and so we obtain \( \bigwedge_\mu(A) \subseteq c_\nu(A) \).

Conversely, suppose that \( \bigwedge_\mu(A) \subseteq c_\nu(A) \) and \( A \subseteq F \), where \( F \) is \( \nu \)-closed. Then \( \bigwedge_\mu(A) \subseteq c_\nu(A) \subseteq F \) and thus \( A \) is a \( g_\bigwedge_\mu \)-set. \( \square \)

**Proposition 3.7.** Let \( \mu \) and \( \nu \) be two GT’s on a set \( X \). If \( A_\alpha \in g_\bigwedge_\mu \) for each \( \alpha \in I \), then \( \bigcup_{\alpha \in I} A_\alpha \in g_\bigwedge_\mu \).

**Proof.** Let \( \bigcup_{\alpha \in I} A_\alpha \subseteq F \) and \( F \) be \( \nu \)-closed. Then \( A_\alpha \subseteq F \) and hence \( \bigwedge_\mu(A_\alpha) \subseteq F \) for each \( \alpha \in I \), since \( A_\alpha \) is a \( g_\bigwedge_\mu \)-set. Thus by Proposition 3.4, we have \( \bigwedge_\mu(\bigcup_{\alpha \in I} A_\alpha) = \bigcup_{\alpha \in I} \bigwedge_\mu(A_\alpha) \subseteq F \). This shows that \( \bigcup_{\alpha \in I} A_\alpha \in g_\bigwedge_\mu \). \( \square \)

**Proposition 3.8.** Let \( \mu \) and \( \nu \) be two GT’s on a set \( X \) and \( A \) be a \( g_\bigwedge_\mu \)-set of \( X \). Then, for every \( \nu \)-closed set \( F \) such that \( (X \setminus \bigwedge_\mu(A)) \cup A \subseteq F \), \( F = X \) holds.

**Proof.** Let \( A \) be a \( g_\bigwedge_\mu \)-set and \( F \) a \( \nu \)-closed set such that \( (X \setminus \bigwedge_\mu(A)) \cup A \subseteq F \). Since \( A \subseteq F \), \( \bigwedge_\mu(A) \subseteq F \) and \( X = (X \setminus \bigwedge_\mu(A)) \cup \bigwedge_\mu(A) \subseteq F \). Therefore, we have \( X = F \). \( \square \)

**Proposition 3.9.** Let \( \mu \) and \( \nu \) be two GT’s on a set \( X \) and \( A \) a \( g_\bigwedge_\mu \)-set of \( X \). Then, \( (X \setminus \bigwedge_\mu(A)) \cup A \) is \( \nu \)-closed if and only if \( A \) is a \( \bigwedge_\mu \)-set.

**Proof.** By Proposition 3.8, \( (X \setminus \bigwedge_\mu(A)) \cup A = X \). Thus, \( \bigwedge_\mu(A) \cap (X \setminus A) = \emptyset \) i.e., \( \bigwedge_\mu(A) \subseteq A \). Thus by Proposition 3.4(a), \( \bigwedge_\mu(A) = A \) i.e., \( A \) is a \( \bigwedge_\mu \)-set.

Conversely, if \( A \) is a \( \bigwedge_\mu \)-set, then \( A = \bigwedge_\mu(A) \). So \( (X \setminus \bigwedge_\mu(A)) \cup A = (X \setminus A) \cup A = X \) which is \( \nu \)-closed. \( \square \)
Proposition 3.10. Let $\mu$ and $\nu$ be two GT's on a set $X$. Then, for each $x \in X$,
(a) $\{x\}$ is either $\nu$-open or $X \setminus \{x\}$ is a $g \bigwedge_{\mu \nu}$-set in $X$;
(b) $\{x\}$ is either a $\nu$-open set or a $g \bigwedge^*_{\mu \nu}$-set in $X$.

Proof. (a) Suppose that $\{x\}$ is not $\nu$-open. Then, the only $\nu$-closed set $F$ containing $X \setminus \{x\}$ is $X$. Thus, $\bigwedge_\mu(X \setminus \{x\}) \subseteq F = X$ and hence $X \setminus \{x\}$ is a $g \bigwedge_{\mu \nu}$-set.

(b) Follows from (a) and Definition 3.11.

Theorem 3.11. Let $\mu$ and $\nu$ be two GT's on a set $X$. Then $(X, \mu, \nu)$ is $\mu \nu$-$T_{1/2}$ if and only if every $g \bigwedge_{\mu \nu}$-set is a $\bigwedge_\mu$-set.

Proof. Let $(X, \mu, \nu)$ be $\mu \nu$-$T_{1/2}$. Suppose that there exists a $g \bigwedge_{\mu \nu}$-set $A$ in $X$ which is not a $\bigwedge_\mu$-set. Then, there exists $x \in \bigwedge_\mu(A)$ such that $x \notin A$. Now since $(X, \mu, \nu)$ is $\mu \nu$-$T_{1/2}$, $\{x\}$ is either $\nu$-open or $\mu$-closed. If $\{x\}$ is $\nu$-open, then $A \subseteq X \setminus \{x\}$, where $X \setminus \{x\}$ is $\nu$-closed. Since $A$ is a $g \bigwedge_{\mu \nu}$-set, $\bigwedge_\mu(A) \subseteq X \setminus \{x\}$, and this is a contradiction. On the other hand, if $\{x\}$ is $\mu$-closed then $A \subseteq X \setminus \{x\}$, where $X \setminus \{x\}$ is $\mu$-open. Thus by Proposition 3.11, $\bigwedge_\mu(A) \subseteq \bigwedge_\mu(X \setminus \{x\}) = X \setminus \{x\}$. This is again a contradiction. Thus, every $g \bigwedge_{\mu \nu}$-set is a $\bigwedge_\mu$-set.

Conversely, assume that every $g \bigwedge_{\mu \nu}$-set is a $\bigwedge_\mu$-set. Suppose that $(X, \mu, \nu)$ is not $\mu \nu$-$T_{1/2}$. Then by Theorem 3.11, there exists a $\mu \nu g$-closed set $A$ which is not $\nu$-closed. Since $A$ is not $\nu$-closed, there exists a point $x \in c_\nu(A)$ such that $x \notin A$. Thus, by Proposition 3.10, the singleton $\{x\}$ is either $\nu$-open or $X \setminus \{x\}$ is a $g \bigwedge_{\mu \nu}$-set.

Case - 1: $\{x\}$ is $\nu$-open: Then, since $x \in c_\nu(A)$, $x \in A$. This is a contradiction.

Case - 2: $X \setminus \{x\}$ is a $g \bigwedge_{\mu \nu}$-set: $\{x\}$ is either $\mu$-closed or not $\mu$-closed. If $\{x\}$ is not $\mu$-closed, $X \setminus \{x\}$ is not $\mu$-open and hence $\bigwedge_\mu(X \setminus \{x\}) = X$. Therefore, $X \setminus \{x\}$ is not a $\bigwedge_\mu$-set, which is a contradiction. If $\{x\}$ is $\mu$-closed, then $A \subseteq X \setminus \{x\} \in \mu$ and $A$ is $\mu \nu g$-closed. Hence, $c_\nu(A) \subseteq X \setminus \{x\}$ (by Definition 2.8). Thus, $x \notin c_\nu(A)$, which is a contradiction.

\begin{thebibliography}{9}
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