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ON A FINITE DIFFERENCE ANALOGUE OF FORTH ORDER FOR BOUNDARY VALUE PROBLEM

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Abstract. We consider a modification of a well-known finite difference analogue for boundary value problem obtained by a five-point difference scheme on a uniform mesh. For the matrix arising from this analogue some of its properties are derived.

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1. Introduction

In this paper we shall concern ourselves with the boundary value problem

$$-u'' + q(x)u(x) = f(x), \quad x \in [0,1], \tag{1}$$
$$u(0) = \alpha, \qquad u(1) = \beta,$$

where $q(x) \geq 0$ and both $f(x)$ and $q(x)$ posses four derivatives. Under these conditions it follows that the unique solution $u(x)$ of (1) is of class $C^{(6)}[0,1]$.

The numerical solution of two-point boundary value problem (1) is most commonly obtained by finite difference methods. We place a uniform mesh of size $h = 1/(n+1)$ on $[0,1]$, and denote the mesh points of the discrete problem by $x_i = ih, \quad i = 0,1,\ldots,n+1$.

Denoting $u(x_i)$ by u_i and $u''(x_i)$ by u_i'' we have for $i = 2,3,\ldots,n-1$, see [1],

$$-u_i'' = \frac{h^{-2}}{12}\left(u_{i-2} - 16u_{i-1} + 30u_i - 16u_{i+1} + u_{i+2}\right) + r_i, \quad |r_i| \leq Mh^4. \tag{2}$$

Here and throughout the paper M denotes any positive constant independent of n. In order to form a discrete analogue for (1) we use (2) at the points $x_i, i = 2,3,\ldots,n-1$,

$$-u_1'' = \frac{h^{-2}}{12}\left(-14u_0 + 29u_1 - 16u_2 + u_3\right) + \frac{u_0''}{12} + r_1, \quad |r_1| \leq Mh^2, \tag{3}$$

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at the point x_1, and analogously at x_n

$$(4) \qquad -u_n'' = \frac{h^{-2}}{12}\left(u_{n-2} - 16u_{n-1} + 29u_n - 14u_{n+1}\right) + \frac{u_{n+1}''}{12} + r_n, \quad |r_n| \le Mh^2.$$

Using (1), (2), (3), (4), and $u_0 = \alpha$, $u_{n+1} = \beta$, we obtain n equations for n unknowns. In matrix notation, we can write this in the form

$$Bu_h = d + r,$$

where B is an $n \times n$ matrix, and u_h, d and r are column vectors, given by

$$B = \frac{h^{-2}}{12}\begin{bmatrix} 29+12h^2q_1 & -16 & 1 & & & & \\ -16 & 30+12h^2q_2 & -16 & 1 & & & \\ 1 & -16 & 30+12h^2q_3 & -16 & 1 & & \\ & \cdots & \cdots & \cdots & \cdots & & \\ & \cdots & \cdots & \cdots & \cdots & & \\ & 1 & -16 & 30+12h^2q_{n-2} & -16 & 1 \\ & & 1 & -16 & 30+12h^2q_{n-1} & -16 \\ & & & 1 & -16 & 29+12h^2q_n \end{bmatrix},$$

(5)

$$u_h = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-2} \\ u_{n-1} \\ u_n \end{bmatrix}, \quad d = \begin{bmatrix} f_1 + \frac{q_0\alpha}{12} - f_0 + \frac{14\alpha}{12h^2} \\ f_2 - \frac{\alpha}{12h^2} \\ f_3 \\ \vdots \\ f_{n-2} \\ f_{n-1} - \frac{\beta}{12h^2} \\ f_n + \frac{q_{n+1}\beta}{12} - f_{n+1} + \frac{14\beta}{12h^2} \end{bmatrix}, \quad r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{n-2} \\ r_{n-1} \\ r_n \end{bmatrix}.$$

Let us define the solution vector z of

$$Bz = d$$

as our discrete approximation of the solution $u(x)$ for problem (1). In [3] it is proved

$$\|u_h - z\|_\infty \le Mh^4$$

in both cases $q(x) = 0$, $x \in [0,1]$ and $\max_{x\in[0,1]} |q(x)|\, h^2 < 8$.

Let B_0 be the matrix B in the case $q(x) = 0, x \in [0,1]$. In [3], the inverse of the matrix B_0 is given explicitly by $B_0^{-1} = [b_{ij}]$, where

$$b_{ij} = b_{ji}, \qquad b_{ij} = h^2\left(\frac{j(n+1-i)}{n+1} - \frac{\delta_{j-1}\delta_{n-i}}{\delta_n}\right), \quad i \ge j,$$

$$\delta_i = \frac{\sinh((i+1)\theta)}{\sinh(\theta)}, \quad \theta = \operatorname{arccosh}(7).$$

2. Some properties of the matrix B

We begin with some properties of the matrix B_0. To explain and describe these properties we shall use the $n \times n$ matrix

$$A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}. \tag{6}$$

The following results concerning the matrix A are well known, see [4].

Definition 1. *A matrix A is called inverse monotone if A has an inverse $A^{-1} \geq 0$, see [1].*

Theorem 1. *Let the $n \times n$ matrix A be given by (6). Then*
(i) $A^{-1} = [a_{ij}]$ exists and

$$a_{ij} = a_{ji}, \qquad a_{ij} = \frac{i(n+1-j)}{n+1}, \quad 1 \leq i \leq j \leq n.$$

(ii) A has the eigenvalues

$$\lambda_k = 2 - 2\cos(k\pi h), \quad k = 1, 2, \ldots, n,$$

and the corresponding eigenvectors

$$v_k = [\sin(k\pi h), \sin(2k\pi h), \ldots, \sin(nk\pi h)]^\top, \quad k = 1, 2, \ldots, n. \tag{7}$$

(iii) For all $h > 0$

$$\left\|h^2 A^{-1}\right\|_1 = \left\|h^2 A^{-1}\right\|_\infty \leq \frac{1}{8}.$$

(iv) A matrix $A + \sigma E$ is inverse monotone for all real $\sigma \geq 0$, where E is identity matrix.

Using results of Theorem 1 it is easy to show the following

Theorem 2. *Let B_0 be the matrix B given by (5) with $q_i = 0$, $i = 1, 2, \ldots, n$. Then*
(i) The matrix B_0 is an inverse monotone matrix.
(ii) B_0 has eigenvalues

$$\mu_k = \frac{1}{12h^2}\left((8 - 2\cos(k\pi h))^2 - 36\right) > 0, \quad k = 1, 2, \ldots, n,$$

and the corresponding eigenvectors are given by (7).

(iii) *For all* $h > 0$

$$\|B_0^{-1}\|_1 = \|B_0^{-1}\|_\infty \le \frac{1}{8}.$$

(iv) *Let* $\omega \ge 0$. *Then it holds*

$$\left\|(B_0 + \omega E)^{-1}\right\|_2 = \frac{1}{\omega + \mu_1} = \frac{3h^2}{3\omega h^2 + (7 - \cos(\pi h))(1 - \cos(\pi h))}.$$

Proof. It is easy to see that

$$B_0 = \frac{h^{-2}}{12}\left(A^2 + 12A\right).$$

So, we have

$$B_0^{-1} = 12h^2(A + 12E)^{-1}A^{-1}.$$

Since A and $A + 12E$ are inverse monotone matrices, it follows that B_0 is an inverse monotone matrix too.

From

$$B_0 = \frac{h^{-2}}{12}\left(A^2 + 12A\right) = \frac{h^{-2}}{12}\left((A + 6E)^2 - 36E\right)$$

we obtain that the eigenvalues μ_k of the matrix B_0 are for $k = 1, 2, \ldots, n$,

$$\mu_k = \frac{1}{12h^2}\left((\lambda_k + 6)^2 - 36\right) = \frac{1}{12h^2}\left((8 - 2\cos(k\pi h))^2 - 36\right) > 0,$$

where λ_k are the eigenvalues of the matrix A.

From the inequalities, see [6],

$$\|(A + 12E)^{-1}\|_1 = \|(A + 12E)^{-1}\|_\infty \le \frac{1}{12}, \qquad \|h^2 A^{-1}\|_1 = \|h^2 A^{-1}\|_\infty \le \frac{1}{8},$$

we have (iii).

The eigenvalues of the matrix $B_0 + \omega E$ are $\omega + \mu_k$ and all are positive. So, this matrix is regular and the spectral radius of $\left((B_0 + \omega E)^{-1}\right)^2$ is $\left(\frac{1}{\omega + \mu_1}\right)^2$. Now by the definition of the norm $\|\cdot\|_2$ the statement follows directly. □

Theorem 3. *Matrix* B *is inverse monotone if*

$$0 \le q_i \le 3h^{-2}, \quad i = 1, 2, \ldots, n. \tag{8}$$

Proof. In this case we have

$$B = B_0 + Q, \qquad Q = \operatorname{diag}(q_1, q_2, \ldots, q_n).$$

It is convenient to make the following transformations of the matrices B_0 and B:

$$B_0 = \frac{1}{12h^2}\left((A+6E)(A+6E)-36E\right),$$

$$B = B_0 + Q = \frac{1}{12h^2}\left((A+6E)(A+6E)+12h^2Q-36E\right).$$

If we denote by

$$K = (36E - 12h^2Q)(A+6E)^{-1}(A+6E)^{-1},$$

we have

$$B = \frac{1}{12h^2}(E-K)(A+6E)(A+6E),$$

and

$$B^{-1} = 12h^2(A+6E)^{-1}(A+6E)^{-1}(E-K)^{-1}.$$

Since $(A + 6E)^{-1} \geq 0$, Theorem 1, to prove that the matrix B is inverse monotone it is enough to show inverse monotonicity of the matrix $E - K$.

Let

$$C = 36(A+6E)^{-1}(A+6E)^{-1}.$$

Eigenvalues of the matrix $A + 6E$ are $8 - 2\cos(k\pi h)$, $k = 1, 2, \ldots, n$, and it follows that eigenvalues of the matrix C are

$$\gamma_k = \frac{36}{(8-2\cos(k\pi h))^2}, \quad k = 1, 2, \ldots, n.$$

For the spectral radius of the matrix C we obtain

$$\rho(C) = \frac{36}{(8-2\cos(\pi h))^2} < 1.$$

From (8) it follows

$$0 \leq 36E - 12h^2Q \leq 36E,$$

and we have

$$0 \leq K \leq C.$$

Now from well-known theorems, see [4], we conclude

$$\rho(K) \leq \rho(C) < 1$$

and $(E-K)^{-1}$ exists and is nonnegative. So, the matrix B is inverse monotone, as a multiplication of three inverse monotone matrices. □

Corollary 1. *If $Q = \omega E$ with $\omega \in \left[0, 3h^{-2}\right]$, then the matrix $B_0 + Q$ is inverse monotone.*

Theorem 4. *If $Q = \omega E$ and $\omega < 0$, the matrix $B_0 + Q$ is not inverse monotone for all $h > 0$.*

Proof. Let

$$z = [\sin(\pi h), \sin(2\pi h), \dots, \sin(n\pi h)]^\top.$$

Obviously, $z > 0$. The vector z is eigenvector of the matrix B_0 and corresponding eigenvalue is

$$\mu_1 = \frac{1}{12h^2}\left((8 - 2\cos(\pi h))^2 - 36\right) = \frac{1}{3h^2}(7 - \cos(\pi h))(1 - \cos(\pi h)).$$

So, the matrix $B_0 + Q$ has the eigenvalue $\mu_1 + \omega$ and the corresponding eigenvector is z :

$$(B_0 + Q)\, z = (\mu_1 + \omega)\, z.$$

If we suppose the matrix $B_0 + Q$ is inverse monotone, then $(B_0 + Q)^{-1} \geq 0$ and from

$$z = (\mu_1 + \omega)(B_0 + Q)^{-1} z$$

follows that $\mu_1 + \omega$ must be positive. But, $\lim_{h \to 0} \mu_1 = 0$ and for a sufficiently small h we obtain

$$\mu_1 + \omega < 0.$$

This means that $B_0 + Q$ is not an inverse monotone matrix. □

We considered the boundary value problem (1) with the assumption $q(x) \geq 0$. It is known, see [4], that (1) has unique solution if

$$q(x) \geq \eta > -\pi^2, \quad x \in [0,1]. \tag{9}$$

Now we shall prove that the matrix B is regular for a sufficiently small h if (9) is satisfied.

Theorem 5. *The matrix B is regular for sufficiently small h if*

$$-\pi^2 < \eta \leq q_i, \quad i = 1, 2, \dots, n.$$

Proof. The matrices B and Q are both Hermitian. Let β_k and τ_k be eigenvalues of the matrix B and Q respectively and

$$\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n, \qquad \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n.$$

Since $B = B_0 + Q$ we have, see [2],

$$\mu_k + \tau_1 \leq \beta_k \leq \mu_k + t_n, \quad k = 1, 2, \dots, n,$$

where μ_k are the eigenvalues of the matrix B_0. Since

$$\tau_1 = \min_{1 \leq i \leq n} q_i \geq \eta > -\pi^2, \qquad \mu_k \geq \mu_1 = \frac{1}{3h^2}(7 - \cos(\pi h))(1 - \cos(\pi h)),$$

and

$$\lim_{h \to 0} \mu_1 = \pi^2,$$

it follows for sufficiently small h

$$0 < \mu_1 + \eta \le \beta_k, \quad k = 1, 2, \ldots, n.$$

So, all eigenvalues of the matrix B are positive and B is a regular matrix. □

Corollary 2. *The matrix B is regular for all h if $0 \le q_i, \quad i = 1, 2, \ldots, n$.*

Theorem 6. *The matrix B is regular for $n \ge 3$ if*

$$-9.8 \le \eta \le q_i, \quad i = 1, 2, \ldots, n,$$

and it holds that

$$\left\|B^{-1}\right\|_2 \le \frac{1}{\mu_1 + \eta} = \frac{3h^2}{3\eta h^2 + (7 - \cos(\pi h))(1 - \cos(\pi h))}.$$

Proof. For $n \ge 3$ we have $h \le 0.25$. The matrix B is Hermitian and for the smallest eigenvalue β_1 of the matrix B it holds

$$0 < \mu_1 - 9.8 \le \mu_1 + \eta \le \beta_1,$$

since μ_1 is monotone decreasing as function of h and $\mu_1(0.25) \ge 9.83$. So,

$$\left\|B^{-1}\right\|_2 = \frac{1}{\beta_1} \le \frac{1}{\mu_1 + \eta}. \quad \square$$

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