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SCORE LISTS OF ORIENTED TRIPARTITE GRAPHS

Pirzada Shariefuddin, Pirzada Merajuddin
Department of Applied Mathematics
Faculty of Eng. and Tech.
A.M.U., Aligarh-202002, INDIA

Abstract

We characterize the scores of oriented tripartite graphs and give a constructive and existence criterion for lists of non-negative integers to be score lists of some oriented tripartite graph.

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1. Introduction

An oriented tripartite graph is the result of assigning a direction to each edge of a simple tripartite graph. Thus it has no loops or parallel arcs. Suppose $X = \{x_1, x_2, ..., x_p\}$, $Y = \{y_1, y_2, ..., y_q\}$ and $Z = \{z_1, z_2, ..., z_r\}$ be the parts of an oriented tripartite graph D, and let $odx(ody$ and $odz)$ and $idx(idy$ and $idz)$ be the outdegree and indegree respectively of vertex x in X (y in Y and z in Z). Define $a_x = q + r + odx - idx, b_y = p + r + ody - idy$ and $c_z = p + q + odz - idz$ as the scores of x, y and z respectively. Clearly, $0 \leq a_x \leq 2(q + r), 0 \leq b_y \leq 2(p + r)$ and $0 \leq c_z \leq 2(p + q)$. So, the lists $A = [a_1, a_2, ..., a_p], B = [b_1, b_2, ..., b_q]$ and $C = [c_1, c_2, ..., c_r]$ in non-decreasing order are the score lists of D. We can interpret an oriented tripartite graph as a result of a competition between three teams in which each player of one team competes against everyone on the other two teams with ties (draws) being allowed. A player receives two points for each win and one point for each tie and with this scoring system, player $x(y$ and $z)$ receives a total of a_x (b_y and c_z) points. The presence of an arc from x to y is denoted by $x \rightarrow y$ and means x defeats y. Also xNy or yNx means neither $x \rightarrow y$ nor $y \rightarrow x$. The scores of oriented graphs have been characterized by Avery [1] and those of oriented bipartite graphs by Pirzada and Merajuddin [3].

2. Criteria for realizability

The lists A, B and C of non-negative integers are said to be realizable if there exists an oriented tripartite graph with score lists A, B and C.

A triple in an oriented tripartite graph is an induced subdigraph with one vertex from each part.

Now we have the following result.

Theorem 1. *Let D and D^* be two oriented tripartite graphs with the same score lists. Then D can be transformed to D^* by successively transforming appropriate triples in one of the following ways:*

either (a) by changing a cyclic triple $x \rightarrow y \rightarrow z \rightarrow x$ to a transitive triple $xNyNzNx$ which has the some score lists, or vice versa;

or (b) by changing an intransitive triple $x \rightarrow y \rightarrow zNx$ to a transitive triple $xNyNz \leftarrow x$ which has the same score lists, or vice versa.

Proof. Suppose A, B and C are the score lists (in non-decreasing order) of a $p \times q \times r$ oriented tripartite graph D whose parts are X, Y and Z. Let D^* be the oriented tripartite graph having parts X^*, Y^* and Z^*. To prove the result, it is sufficient to show that D^* can be obtained from D by transforming triples in any one of the ways given in (a) or (b).

Let the order of the second and third part be fixed as q and r respectively. We use induction on the order of the first part. The result is obvious if there is just one vertex in the first part. Suppose the result holds when there are fewer than p vertices in the first part. Assume that j and k are such that for $\ell > j$ and $m > k, 1 \leq j < \ell \leq q$ and $1 \leq k < m \leq r$ the corresponding arcs have the like orientations in D and D^*. For j and k we have the following cases to consider $(i) x_p \to y_j \to z_k$ and $x_p^* N y_j^* N z_k^*$, $(ii) x_p N y_j \leftarrow z_k$ and $x_p^* \to y_j^* N z_k^*$, and $(iii) x_p \to y_j N z_k$ and $x_p^* N y_j^* \leftarrow z_k^*$.

For case (i), since x_p and x_p^* have equal scores, therefore $x_p \leftarrow z_k$ and $x_p^* N z_k^*$, or $x_p N z_k$ and $x_p^* \to z_k^*$. So, there is a triple $x_p \to y_j \to z_k \to x_p$, or $x_p \to y_j \to z_k N x_p$ in D and corresponding to these $x_p^* N y_j^* N z_k^* N x_p^*$, or $x_p^* N y_j^* N z_k^* \leftarrow x_p^*$ respectively is a triple in D^*. For (ii) we have $x_p \to z_k$ and $x_p^* N z_k^*$ as x_p and x_p^* are of equal scores. Thus, there is a triple $x_p N y_j \leftarrow z_k \leftarrow x_p$ in D and corresponding to this there is a triple $x_p^* \to y_j^* N z_k^* N x_p^*$ in D^*. Finally, for (iii), since x_p and x_p^* have equal scores, so $x_p \leftarrow z_k$ and $x_p^* N z_k^*$. Thus $x_p \to y_j N z_k \to x_p$ is a triple in D and corresponding to this $x_p^* N y_j^* \leftarrow z_k^* N x_p^*$ is a triple in D^*.

It follows from (i), (ii) and (iii) that there is an oriented tripartite graph that can be obtained from D by any one of the transformations (a), or (b) with the scores remaining unchanged. Hence the result follows by induction hypothesis. □

The following observation immediately follows from Theorem 1.

Corollary 1. *Among all oriented tripartite graphs with given score lists, those with the fewest arcs are transitive.*

A transmitter is a vertex with indegree zero. In a transitive oriented tripartite graph with score lists $A = [a_1, a_2, ..., a_p], B = [b_1, b_2, ..., b_q]$ and $C = [c_1, c_2, ..., c_r]$, any of the vertices with scores a_p, or b_q, or c_r may act as a transmitter.

The next result provides a useful recursive test whether the lists of non-negative integers form the score lists of some oriented tripartite graph.

Theorem 2. *Let $A = [a_1, a_2, ..., a_p], B = [b_1, b_2, ..., b_q]$ and $C = [c_1, c_2, ..., c_r]$ be three lists of non-negative integers in non-decreasing order. Let A' be obtained from A by deleting one entry a_p and let B' and C' be obtained by reducing $2(q + r) - a_p$ largest entries of B and C by 1 each, provided $a_p \geq q + r, b_q \leq 2(p + r) - 1$ and $c_r \leq 2(p + q) - 1$. Then A, B and C are score lists of some oriented tripartite graph if and only if A', B' and C' are.*

Proof. Let A', B' and C' be the score lists of some oriented tripartite graph D' with parts X', Y' and Z'. Then an oriented tripartite graph D with score lists A, B and C can be obtained by adding a transmitter to X' that is adjacent to just those vertices of Y' and Z' whose scores are not reduced in going from A, B and C to A', B' and C'.

Conversely, suppose A, B and C are the score lists of an oriented tripartite graph D with parts X, Y and Z. By Corollary 1 any of the vertices x, or y, or z with scores a_p, or b_q, or c_r respectively can be a transmitter. Let the vertex x with score a_p be a transmitter. Assume $a_p \geq q+r, b_q \leq 2(p+r)-1$ and $c_r \leq 2(p+q)-1$ because (i) if $a_p < q+r$, then by deleting a_p we have to reduce more than $q+r$ entries from B and C, which is absurd (ii) if $b_q > 2(p+r)-1$ and $c_r > 2(p+q)-1$, then on reduction $b'_q = b_q - 1 > 2(p+r)-1-1 = 2(p-1+r)$ and $c'_r = c_r - 1 > 2(p+q)-1-1 = 2(p-1+q)$ respectively, which in both cases is impossible. Let V be the set of $2(q+r)-a_p$ vertices of greatest scores in Y and Z and let $W = (Y \cup Z) - V$. Suppose x is adjacent to $v_1, v_2, ..., v_j$ vertices of V in D. Then there exist exactly j vertices $w_1, w_2, ..., w_j$ of W not adjacent to x. Since D is transitive v_i cannot be adjacent to w_i. Also, w_i cannot be adjacent to v_i because taken together with the transitivity of D this would imply that the score of w_i is greater than the score of v_i, which is contrary to our assumption. So $w_i N v_i$ for all i, with v_i and w_i not belonging to the same part Y or Z. Thus we have $x \rightarrow v_i N w_i N x$ and if these are changed to $x N v_i \leftarrow w_i \leftarrow x$, the result is an oriented tripartite graph with the same score lists, but in which the transmitter x is adjacent to all vertices of W and to none of V, i.e. the transmitter x with score a_p is adjacent to $q+r-(2(q+r)-a_p) = a_p-(q+r)$ vertices of Y and Z with least scores, completing the proof. □

Theorem 2 gives an algorithm for determining whether the lists A, B and C of non-negative integers in non-decreasing order are the score lists, and for constructing a corresponding oriented tripartite graph. Let $A = [a_1, a_2, ..., a_p], B = [b_1, b_2, ..., b_q]$ and $C = [c_1, c_2, ..., c_r]$ with $a_p \geq q+r$, $b_q \leq 2(p+r)-1$ and $c_r \leq 2(p+q)-1$, be the score lists of an oriented tripartite graph with parts $X = \{x_1, x_2, ..., x_p\}$, $Y = \{y_1, y_2, ..., y_q\}$ and $Z = \{z_1, z_2, ..., z_r\}$ respectively. Deleting a_p and reducing $2(q+r)-a_p$ greatest entries of B and C by 1 each to form $B' = [b'_1, b'_2, ..., b'_q]$ and $C' = [c'_1, c'_2, ..., c'_r]$. Then arcs are defined by $x_p \rightarrow y_j$ and $x_p \rightarrow x_k$ for which $b_j = b'_j$ and $c_k = c'_k$ respectively. Now, if at least one of the conditions $a_p \geq q+r$, or $b_q \leq 2(p+r)-1$, or $c_r \leq (p+q)-1$ does not hold, then we delete b_q, or c_r for which the conditions get satisfied and the same argument is used for defining arcs. If this method is applied successively, then (a) it tests whether A, B and C are score lists, and if A, B and C are score lists

(b) an oriented tripartite graph $\Delta(A, B, C)$ with score lists A, B and C is constructed.

We illustrate this reduction and the resulting construction with the following example, beginning with lists A_1, B_1 and C_1,

$A_1 = [7, 8, 9]$	$B_1 = [8, 8, 8, 9]$	$C_1 = [6, 7, 7, 8, 9]$	
$A_2 = [7, 8]$	$B_2 = [7, 7, 7, 8]$	$C_2 = [5, 6, 6, 7, 8]$	$y_4 \rightarrow z_1$
$A_3 = [6, 7]$	$B_3 = [7, 7, 7]$	$C_3 = [5, 5, 5, 6, 7]$	
$A_4 = [5, 6]$	$B_4 = [7, 7]$	$C_4 = [4, 4, 4, 5, 6]$	
$A_5 = [4, 5]$	$B_5 = [7]$	$C_5 = [3, 3, 3, 4, 5]$	
$A_6 = [3, 4]$	$B_6 = \emptyset$	$C_6 = [2, 2, 2, 3, 4]$	$z_5 \rightarrow x_1, x_2$
$A_7 = [3, 4]$	$B_7 = \emptyset$	$C_7 = [2, 2, 2, 3]$	
$A_B = [3]$	$B_8 = \emptyset$	$C_8 = [1, 1, 1, 2]$	$z_4 \rightarrow x_1$
$A_9 = [3]$	$C_9 = \emptyset$	$C_9 = [1, 1, 1]$	
$A_{10} = \emptyset$	$B_{10} = \emptyset$	$C_{10} = [0, 0, 0]$	

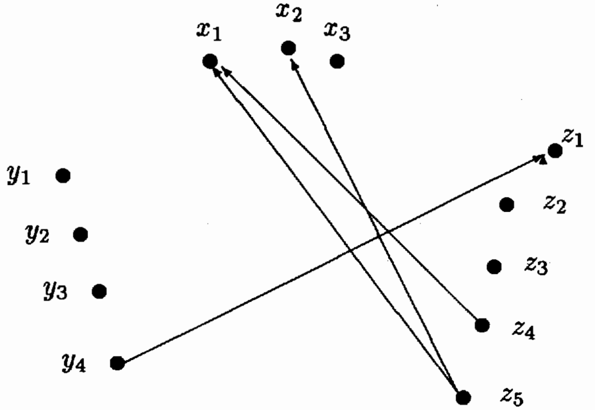

Figure 1

The oriented tripartite graph $\Delta(A, B, C)$ has the following property.

Theorem 3. *The oriented tripartite graph $\Delta(A, B, C)$ is transitive for any score lists A, B and C.*

Proof. Let $\Delta(A, B, C)$ be an oriented tripartite graph of the order $p \times q \times r$ with parts $X = \{x_1, x_2, ..., x_p\}, Y = \{y_1, y_2, ..., y_q\}$ and $Z = \{z_1, z_2, ..., z_r\}$. To prove this result, we fix the order of second and third part and use induction on the order of the first part, that is on p. Let $\Delta(A, B, C)$ be transitive with fewer than p entries, which is clearly true for $p = 1$. If $\Delta(A, B, C)$ is intransitive for some score lists A, B, and C with p entries in the first part, then by induction hypothesis the induced oriented tripartite graph $\Delta(A', B', C') = \Delta(A, B, C) - x_p$ is transitive. So, any intransitive triple in $\Delta(A, B, C)$ includes the vertex x_p. Let y_j, z_k and $x_p (j \leq q$ and $k \leq r)$ form such a triple. Thus, it must be $x_p \to z_k \to y_j N x_p$ in $\Delta(A, B, C)$, since other cases are not possible. From the definition of $\Delta(A, B, C)$, $x_p \to z_k$ and $x_p N y_j$ implies that $s_{z_k} \leq s_{y_j}$. Also $s'_{z_k} = s_{z_k}$ and $s'_{y_j} = s_{y_j} - 1$. So $s'_{z_k} \leq s'_{y_j} + 1$. Since $\Delta(A, B, C) - x_p$ is transitive, therefore $z_k \to y_j$ gives $odz_k \geq ody_j + 1$ and $idz_k \leq idy_j - 1$ in $\Delta(A, B, C) - x_p$. Thus $s'_{z_k} \geq s'_{y_j} + 2$, which is contrary to the earlier conclusion that $s'_{z_k} \leq s'_{y_j} + 1$. Thus $\Delta(A, B, C)$ cannot have an intransitive triple and the result follows by induction. □

Let D_i be the oriented tripartite graphs with disjoint parts X_i, Y_i and Z_i for $1 \leq i \leq t$ and let $X = \cup_{i=1}^{t} X_i, Y = \cup_{i=1}^{t} Y_i$ and $Z = \cup_{i=1}^{t} Z_i$. Define the oriented tripartite graph $D = [D_1, D_2, ..., D_t]$ with parts X, Y and Z obtained from D_i for $1 \leq i \leq t$ such that the arcs of D are the arcs of D_i and each vertex of X_i is adjacent to every vertex of Y_j and Z_k for $i > j$ and $i > k$, each vertex of Y_j is adjacent to every vertex of X_i and Z_k for $j > i$ and $j > k$, and each vertex of Z_k is adjacent to every vertex of X_i and Y_j for $k > i$ and $k > j$.

The next result gives a simple criterion for determining whether three lists of non-negative integers are realizable as scores.

Theorem 4. *Let $A = [a_1, a_2, ..., a_p], B = [b_1, b_2, ..., b_q]$ and $C = [c_1, c_2, ..., c_r]$ be the lists of non-negative integers in a non-decreasing order. Then A, B and C are score lists of some oriented tripartite graph if and only if*

$$(1) \qquad \sum_{i=1}^{\ell} a_i + \sum_{j=1}^{m} b_j + \sum_{k=1}^{n} c_k \geq 2(\ell m + mn + n\ell)$$

for $1 \leq \ell \leq p, 1 \leq m \leq q$ and $1 \leq n \leq r$, with the equality when $\ell = p$, $m = q$ and $n = r$.

Proof. The necessity of the condition follows from the fact that the oriented subtripartite graph induced by ℓ vertices from the first part, m vertices from the second part, and n vertices from the third part has a sum of scores $2(\ell m + mn + n\ell)$.

For sufficiency, let A, B and C be the lists of non-negative integers in non-decreasing order satisfying the conditions (1) but A, B and C do not form the score lists of any oriented tripartite graph. Let these lists be chosen such that p, q and r are the smallest possible and a_ℓ is the least for the choice of p, q, and r. We have two cases to consider.

Case (i) Suppose the equality in (1) holds for same $\ell < p$, $m \leq q$ and $n \leq r$. That is

$$\sum_{i=1}^{\ell} a_i + \sum_{j=1}^{m} b_j + \sum_{k=1}^{n} c_k = 2(\ell m + mn + n\ell) \tag{2}$$

Let $A_1 = [a_1, a_2, ..., a_p], B_1 = [b_1,\ b_2, ..., b_q]$ and $C_1 = [c_1, c_2, ..., c_r]$. Clearly, A_1, B_1 and C_1 satisfy the conditions (1) and by the minimallity of p, q and r the lists A_1, B_1 and C_1 are the score lists of some oriented tripartite graph D_1. Define

$$A_2 = [a_{\ell+1} - 2(m+n), a_{\ell+2} - 2(m+n), ..., a_p - 2(m+n)].$$

$$B_2 = [b_{m+1} - 2(n+\ell), b_{m+2} - 2(n+\ell), ..., b_q - 2(n+\ell)],$$

and

$$C_2 = [c_{n+1} - 2(\ell+m), c_{n+2} - 2(\ell+m), ..., c_r - 2(\ell+m)].$$

Consider the sum

$$\sum_{i=1}^{L}(a_{\ell+i} - 2(m+n) + \sum_{j=1}^{M}(b_{m+j} - 2(n+\ell) + \sum_{k=1}^{N}(c_{n+k} - 2(\ell+m))$$

$$= \sum_{i=1}^{\ell+L} a_i + \sum_{j=1}^{m+M} b_j + \sum_{k=1}^{n+N} c_k - (\sum_{i=1}^{\ell} a_i + \sum_{j=1}^{m} b_j + \sum_{k=1}^{n} c_k)$$

$$-2L(m+n) - 2M(n+\ell) = 2N(\ell+m)$$

$$\geq 2((\ell+L)(m+M) + (m+M)(n+N) + (\ell+L)(n+N)) - 2(\ell m + mn + n\ell)$$

$$-2L(m+n) - 2M(n+\ell) - 2N(\ell+m)$$

$$= 2(LM + MN + NL)$$

for $1 \leq L \leq p - \ell, 1 \leq M \leq q - m$ and $1 \leq N \leq r - n$, with equality for $L = p - \ell$, $M = q - m$ and $N = r - n$. Thus the lists A_2, B_2 and C_2 satisfy conditions (1) and by the minimallity of p,q and r, the lists A_2, B_2 and C_2 form the score lists of some oriented tripartite graph D_2.

Let X_i, Y_i and Z_i be the parts of the oriented tripartite graph D_i with the score lists A_i, B_i and C_i for $i = 1, 2$. Suppose $X = X_1 \cup X_2, Y = Y_1 \cup Y_2$ and $Z = Z_1 \cup Z_2$. Define an oriented tripartite graph D with parts X, Y and Z and such that arcs of D are the arcs of D_i and each vertex of X_2 is adjacent to every vertex of Y_1 and Z_1, each vertex of Y_2 is adjacent to every vertex of X_1 and Z_1 and each vertex of Z_2 is adjacent to every vertex of X_1 and Y_1. Thus we get an oriented tripartite graph D with the score lists A, B and C, which is a contradiction.

Case (ii) Suppose the strict inequality holds in (1) for $n \neq p, m \neq q$ and $n \neq r$. Also assume that $a_1 \neq 0$. Let $A_1 = [a_1 - 1, a_2, ..., a_p + 1].B_1 = [b_1, b_2, ..., b_q]$ and $C_1 = [c_1, c_2, ..., c_r]$. Obviously, A_1, B_1 and C_1 satisfy (1) and by the minimallity of a_1, the lists $A_1 B_1$ and C_1 are the score lists of some oriented tripartite graph D_1 with parts X_1, Y_1 and Z_1 (say). Let $s_{x_1} = a_1 - 1$ and $s_{x_p} = a_p + 1$. Since $s_{x_p} > s_{x_1}$, there exists a vertex u either in Y_1, or in Z_1 such that either $x_p \to u \to x_1$, or $x_p N u \to x_1$, or $x_p \to u N x_1$, or $x_p N u N x_1$ in D_1 and if these are changed to $x_p N u N x_1$, or $x_p \leftarrow u N x_1$, or $x_p N u \leftarrow x_1$, or $x_p \leftarrow N u \leftarrow x_1$ respectively, the result is an oriented tripartite graph D with the score lists A, B and C. This is again a contradiction, which completes the proof. □

In case of tripartite tournaments [2], the lists $A = [a_1, a_2, ..., a_p]$, $B = [b_1, b_2, ..., b_q]$ and $C = [c_1 c_2, ..., c_r]$ of non-negative integers in non-decreasing order are realizable as scores if and only if

$$\sum_{i=1}^{\ell} a_i + \sum_{j=1}^{m} b_j + \sum_{k=1}^{n} c_k \geq \ell m + mn + n\ell$$

for $1 \leq \ell \leq p,\ 1 \leq m \leq q$ and $1 \leq n \leq r$, with equality when $\ell = p,\ m = q$, and $n = r$.

This result now follows from Theorem 4.

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