SCORE LISTS OF ORIENTED TRIPARTITE GRAPHS

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Abstract

We characterize the scores of oriented tripartite graphs and give a constructive and existence criterion for lists of non-negative integers to be score lists of some oriented tripartite graph.

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1. Introduction

An oriented tripartite graph is the result of assigning a direction to each edge of a simple tripartite graph. Thus it has no loops or parallel arcs. Suppose $X = \{x_1, x_2, \ldots, x_r\}$, $Y = \{y_1, y_2, \ldots, y_s\}$ and $Z = \{z_1, z_2, \ldots, z_t\}$ be the parts of an oriented tripartite graph $D$, and let $odx(ody$ and $odz)$ and $idx(idy$ and $idx)$ be the outdegree and indegree respectively of vertex $x$ in $X$ ($y$ in $Y$ and $z$ in $Z$). Define $a_x = q + r + odx - idx, b_y = p + r + ody - idy$ and $c_z = p + q + odz - idx$ as the scores of $x, y$ and $z$ respectively. Clearly, $0 \leq a_x \leq 2(q + r), 0 \leq b_y \leq 2(p + r)$ and $0 \leq c_z \leq 2(p + q)$. So, the lists $A = \{a_1, a_2, \ldots, a_r\}, B = \{b_1, b_2, \ldots, b_s\}$ and $C = \{c_1, c_2, \ldots, c_t\}$ in non-decreasing order are the score lists of $D$. We can interpret an oriented tripartite graph
as a result of a competition between three teams in which each player of one team competes against everyone on the other two teams with ties (draws) being allowed. A player receives two points for each win and one point for each tie with this scoring system, player \( z(x) \) and \( z(y) \) receives a total of \( a + b + c \) points. The presence of an arc from \( z \) to \( y \) is denoted by \( z \rightarrow y \), and means \( z \) defeats \( y \). Also \( z \rightarrow y \) or \( y \rightarrow z \) means neither \( z \rightarrow y \) nor \( y \rightarrow z \). The scores of oriented graphs have been characterized by Avery [1] and those of oriented bipartite graphs by Pirzada and Merajuddin [3].

2. Criteria for realizability

The lists \( A, B \) and \( C \) of non-negative integers are said to be realizable if there exists an oriented tripartite graph with score lists \( A, B \) and \( C \).

A triple in an oriented tripartite graph is an induced subgraph with one vertex from each part.

Now we have the following result.

**Theorem 1.** Let \( D \) and \( D' \) be two oriented tripartite graphs with the same score lists. Then \( D \) can be transformed to \( D' \) by successively transforming appropriate triples in one of the following ways:

- either (a) by changing a cyclic triple \( z \rightarrow y \rightarrow z \rightarrow x \) to a transitive triple \( z \rightarrow y \rightarrow z \rightarrow x \) which has the same score lists, or vice versa;
- or (b) by changing an intransitive triple \( z \rightarrow y \rightarrow z \rightarrow x \) to a transitive triple \( z \rightarrow y \rightarrow z \rightarrow x \) which has the same score lists, or vice versa.

**Proof.** Suppose \( A, B \) and \( C \) are the score lists (in non-decreasing order) of a \( p \times q \times r \) oriented tripartite graph \( D \) whose parts are \( X, Y \) and \( Z \). Let \( D' \) be the oriented tripartite graph having parts \( X', Y' \) and \( Z' \). To prove the result, it is sufficient to show that \( D' \) can be obtained from \( D \) by transforming triples in any one of the ways given in (a) or (b).

Let the order of the second and third part be fixed as \( q \) and \( r \) respectively. We use induction on the order of the first part. The result is obvious if there is just one vertex in the first part. Suppose the result holds when there are fewer than \( p \) vertices in the first part. Assume that \( j \) and \( k \) are such that for \( t > j \) and \( m > k \), \( 1 \leq j < t \leq q \) and \( 1 \leq k < m \leq r \) the corresponding arcs
have the like orientations in $D$ and $D'$. For $j$ and $k$ we have the following cases to consider (i) $x_p \rightarrow y_j \rightarrow z_k$ and $x_j^* N y_j^* N z_k^*$, (ii) $x_p N y_j \rightarrow z_k$ and $x_j^* \rightarrow y_j^* N z_k^*$, and (iii) $x_p \rightarrow y_j N z_k$ and $x_j^* N y_j^* \rightarrow z_k^*$. 

For case (i), since $x_j$ and $x_j^*$ have equal scores, therefore $x_p \sim z_k$ and $x_j^* N z_k^*$, or $x_p N z_k$ and $x_j^* \sim z_k^*$. So, there is a triple $x_p \rightarrow y_j \rightarrow z_k \rightarrow x_p$, or $x_p \rightarrow y_j \rightarrow z_k \rightarrow x_p$ in $D$ and corresponding to this $x_j^* N y_j^* N z_k^* N x_j^*$, or $x_j^* N y_j^* N z_k^* \sim x_j^*$ respectively is a triple in $D'$. For (ii) we have $x_p \rightarrow z_k$ and $x_j^* N z_k^*$ as $x_p$ and $x_j^*$ are of equal scores. Thus, there is a triple $x_p N y_j \rightarrow z_k \rightarrow x_p$ in $D$ and corresponding to this there is a triple $x_j^* \rightarrow y_j^* N z_k^* N x_j^*$ in $D'$. Finally, for (iii), since $x_p$ and $x_j^*$ have equal scores, so $x_p \sim z_k$ and $x_j^* N z_k^*$. Thus $x_p \rightarrow y_j N z_k \rightarrow x_p$ is a triple in $D$ and corresponding to this $x_j^* N y_j^* \rightarrow z_k^* N x_j^*$ is a triple in $D'$.

It follows from (i), (ii) and (iii) that there is an oriented tripartite graph that can be obtained from $D$ by any one of the transformations (a), or (b) with the scores remaining unchanged. Hence the result follows by induction hypothesis. 

The following observation immediately follows from Theorem 1.

**Corollary 1.** Among all oriented tripartite graphs with given score lists, those with the fewest arcs are transitive.

A transmitter is a vertex with indegree zero. In a transitive oriented tripartite graph with score lists $A = [a_1, a_2, ..., a_n], B = [b_1, b_2, ..., b_n]$ and $C = [c_1, c_2, ..., c_n]$ of any of the vertices with scores $a_p$, or $b_q$, or $c_r$ may act as a transmitter.

The next result provides a useful recursive test whether the lists of non-negative integers form the score lists of some oriented tripartite graph.

**Theorem 2.** Let $A = [a_1, a_2, ..., a_n], B = [b_1, b_2, ..., b_n]$ and $C = [c_1, c_2, ..., c_n]$ be three lists of non-negative integers in non-decreasing order. Let $A'$ be obtained from $A$ by deleting one entry $a_p$, and let $B'$ and $C'$ be obtained by reducing $2(q + r) - a_p$ largest entries of $B$ and $C$ by 1 each, provided $a_p \geq q + r, b_i \leq 2(p + r) - 1$ and $c_r \leq 2(p + q) - 1$. Then $A, B$ and $C$ are score lists of some oriented tripartite graph if and only if $A', B'$ and $C'$ are.

**Proof.** Let $A', B'$ and $C'$ be the score lists of some oriented tripartite graph $D'$ with parts $X', Y'$ and $Z'$. Then an oriented tripartite graph $D$ with score
lists \( A, B \) and \( C \) can be obtained by adding a transmitter to \( X' \) that is adjacent to just those vertices of \( Y' \) and \( Z' \) whose scores are not reduced in going from \( A, B \) and \( C \) to \( A', B' \) and \( C' \).

Conversely, suppose \( A, B \) and \( C \) are the score lists of an oriented tripartite graph \( D \) with parts \( X, Y \) and \( Z \). By Corollary 1 any of the vertices \( x, y, \) or \( z \) with scores \( a_x, b_y, \) or \( c_z \), respectively can be a transmitter. Let the vertex \( x \) with score \( a_x \) be a transmitter. Assume \( a_x > q + r, b_y > 2(p + r) - 1 \) and \( c_z > 2(\frac{\alpha}{p + q}) - 1 \) because (i) \( a_x < q + r \), then by deleting \( a_x \) we have to reduce more than \( q + r \) entries from \( B \) and \( C \), which is absurd (ii) if \( b_y > 2(p + r) - 1 \) and \( c_z > 2(p + q) - 1 \), then on reduction \( b' = b_y - 1 > 2(p + r) - 1 \) and \( c' = c_z - 1 > 2(p + q) - 1 \), respectively, which in both cases is impossible. Let \( V \) be the set of \( 2(q + r) - a_x \) vertices of greatest scores in \( Y \) and \( Z \) and let \( W = (Y \cup Z) - V \). Suppose \( x \) is adjacent to \( v_1, v_2, \ldots, v_j \) vertices of \( V \) in \( D \). Then there exist exactly \( j \) vertices \( w_1, w_2, \ldots, w_j \) of \( W \) not adjacent to \( x \). Since \( D \) is transitive \( w_i \) cannot be adjacent to \( w_z \). Also, \( w_i \) cannot be adjacent to \( v_j \) because taken together with the transitivity of \( D \) this would imply that the score of \( w_i \) is greater than the score of \( v_j \), which is contrary to our assumption. So \( w_i w_z \) for all \( i \), with \( v_i \) and \( w_i \) not belonging to the same part \( Y \) or \( Z \). Thus we have \( x \rightarrow v_i w_i w_z \) and if these are changed to \( x \rightarrow z \), \( z \rightarrow z \), the result is an oriented tripartite graph with the same score lists, but in which the transmitter \( z \) is adjacent to all vertices of \( W \) and to none of \( Y \), i.e. the transmitter \( x \) with score \( a_x \) is adjacent to \( q + r - (2(q + r) - a_x) = a_x - (q + r) \) vertices of \( Y \) and \( Z \) with least scores, completing the proof. \( \square \)

Theorem 2 gives an algorithm for determining whether the lists \( A, B \) and \( C \) of non-negative integers in non-decreasing order are the score lists, and for constructing a corresponding oriented tripartite graph. Let \( A = (a_1, a_2, \ldots, a_n), B = (b_1, b_2, \ldots, b_m) \) and \( C = (c_1, c_2, \ldots, c_l) \) with \( a_x > q + r, b_y \leq 2(p + r) - 1 \) and \( c_z \leq 2(p + q) - 1 \), be the score lists of an oriented tripartite graph with parts \( X = \{x_1, x_2, \ldots, x_n\}, Y = \{y_1, y_2, \ldots, y_m\}, \) and \( Z = \{z_1, z_2, \ldots, z_l\} \) respectively. Deleting \( a_x \) and reducing \( 2(q + r) - a_x \) greatest entries of \( B \) and \( C \) by \( 1 \) each to form \( B' = (b'_1, b'_2, \ldots, b'_m) \) and \( C' = (c'_1, c'_2, \ldots, c'_l) \). Then arcs are defined by \( x_p \rightarrow y_q \) and \( x_p \rightarrow z_q \) for which \( b'_p > b'_q \) and \( c'_q \) respectively. Now, if at least one of the conditions \( a_x > q + r, \) or \( b'_p \leq 2(p + r) - 1, \) or \( c'_q \leq (p + q) - 1 \) does not hold, then we delete \( b'_p \) or \( c'_q \) for which the conditions get satisfied and the same argument is used for defining arcs. If this method is applied successively, then (a) it tests whether \( A, B \) and \( C \) are score lists, and if \( A, B \) and \( C \) are score lists
(b) an oriented tripartite graph $\Delta(A, B, C)$ with score lists $A$, $B$ and $C$ is constructed.

We illustrate this reduction and the resulting construction with the following example, beginning with lists $A_1$, $B_1$ and $C_1$:

$A_1 = [7, 8, 9]$  $B_1 = [8, 8, 8, 9]$  $C_1 = [6, 7, 7, 8, 9]$  $y_4 \rightarrow z_1$

$A_2 = [7, 8]$  $B_2 = [7, 7, 7, 8]$  $C_2 = [5, 6, 6, 7, 8]$  $y_4 \rightarrow z_1$

$A_3 = [6, 7]$  $B_3 = [7, 7, 7]$  $C_3 = [5, 5, 5, 6, 7]$  $y_4 \rightarrow z_1$

$A_4 = [5, 6]$  $B_4 = [7, 7]$  $C_4 = [4, 4, 4, 5, 6]$  $y_4 \rightarrow z_1$

$A_5 = [4, 5]$  $B_5 = [7]$  $C_5 = [3, 3, 3, 4, 5]$  $y_4 \rightarrow z_1$

$A_6 = [3, 4]$  $B_6 = [6]$  $C_6 = [2, 2, 2, 3]$  $y_4 \rightarrow z_1$

$A_7 = [3, 4]$  $B_7 = [6]$  $C_7 = [2, 2, 2, 3]$  $y_4 \rightarrow z_1$

$A_8 = [3]$  $B_8 = [6]$  $C_8 = [1, 1, 1, 2]$  $y_4 \rightarrow z_1$


$A_{10} = [2]$  $B_{10} = [5]$  $C_{10} = [0, 0, 0]$

The oriented tripartite graph $\Delta(A, B, C)$ has the following property.

**Theorem 3.** The oriented tripartite graph $\Delta(A, B, C)$ is transitive for any score lists $A$, $B$ and $C$.

**Proof.** Let $\Delta(A, B, C)$ be an oriented tripartite graph of the order $p \times q \times r$ with parts $X = \{x_1, x_2, \ldots, x_p\}$, $Y = \{y_1, y_2, \ldots, y_q\}$ and $Z = \{z_1, z_2, \ldots, z_r\}$. To prove this result, we fix the order of second and third part and use induction on the order of the first part, that is on $p$. Let $\Delta(A, B, C)$ be
transitive with fewer than \( p \) entries, which is clearly true for \( p = 1 \). If \( \Delta(A, B, C) \) is intransitive for some score lists \( A, B, \) and \( C \) with \( p \) entries in the first part, then by induction hypothesis the induced oriented tripartite graph \( \Delta(A', B', C') = \Delta(A, B, C) - x_p \) is intransitive. So, any intransitive triple in \( \Delta(A, B, C) \) includes the vertex \( x_p \). Let \( y_1, y_2 \) and \( x_p(j) \leq q \) and \( k \leq r \) form such a triple. Thus, it must be \( x_p \rightarrow z_k \rightarrow y_2, N_{x_p} \Delta(A, B, C) \), since other cases are not possible. From the definition of \( \Delta(A, B, C) \), \( x_p \rightarrow z_k \) and \( z_p \rightarrow y_2 \) implies that \( s_{y_2} = s_{z_k} \). Also \( s_{y_1} = s_{x_p} \) and \( s_{y_2} = s_{x_p} - 1 \). So \( s_{y_1} \leq s_{y_2} + 1 \). Since \( \Delta(A, B, C) - x_p \) is transitive, therefore \( z_k \rightarrow y_2 \) gives \( od_{y_2} \geq od_{y_1} + 1 \) and \( id_{z_k} \leq id_{y_2} - 1 \) in \( \Delta(A, B, C) - x_p \). Thus \( s_{y_1} \geq s_{y_2} + 2 \), which is contrary to the earlier conclusion that \( s_{y_1} \leq s_{y_2} + 1 \). Thus \( \Delta(A, B, C) \) cannot have an intransitive triple and the result follows by induction.

Let \( D_i \) be the oriented tripartite graphs with disjoint parts \( X_i, Y_i \) and \( Z_i \), for \( 1 \leq i \leq t \) and let \( X = \bigcup_{i=1}^{t} X_i, Y = \bigcup_{i=1}^{t} Y_i \) and \( Z = \bigcup_{i=1}^{t} Z_i \). Define the oriented tripartite graph \( D = [D_1, D_2, \ldots, D_t] \) with parts \( X, Y \) and \( Z \) obtained from \( D_i \) for \( 1 \leq i \leq t \) such that the arcs of \( D \) are the arcs of \( D_i \) and each vertex of \( X_i \) is adjacent to every vertex of \( Y_i \) and \( Z_i \) for \( i > j \) and \( i > k \), each vertex of \( Y_j \) is adjacent to every vertex of \( X_i \) and \( Z_k \) for \( j > i \) and \( j > k \), and each vertex of \( Z_k \) is adjacent to every vertex of \( X_i \) and \( Y_j \) for \( k > i \) and \( k > j \).

The next result gives a simple criterion for determining whether three lists of non-negative integers are realizable as scores.

**Theorem 4.** Let \( A = [a_1, a_2, \ldots, a_q] \), \( B = [b_1, b_2, \ldots, b_k] \) and \( C = [c_1, c_2, \ldots, c_r] \) be the lists of non-negative integers in a non-decreasing order. Then \( A, B \) and \( C \) are score lists of some oriented tripartite graph if and only if

\[
\sum_{i=1}^{t} a_i + \sum_{j=1}^{m} b_j + \sum_{k=1}^{n} c_k \geq 2(tm + mn + nr)
\]

for \( 1 \leq t \leq p, 1 \leq m \leq q \) and \( 1 \leq n \leq r \), with the equality when \( t = p, m = q \) and \( n = r \).

**Proof.** The necessity of the condition follows from the fact that the oriented subtripartite graph induced by \( t \) vertices from the first part, \( m \) vertices from the second part, and \( n \) vertices from the third part has a sum of scores \( 2(tm + mn + nr) \).
For sufficiency, let $A, B$ and $C$ be the lists of non-negative integers in non-decreasing order satisfying the conditions (1) but $A, B$ and $C$ do not form the score lists of any oriented tripartite graph. Let these lists be chosen such that $p, q$ and $r$ are the smallest possible and $a_k$ is the least for the choice of $p, q$, and $r$. We have two cases to consider.

Case (1) Suppose the equality in (1) holds for some $\ell < p$, $m \leq q$ and $n \leq r$. That is

$$
\sum_{i=1}^{\ell} a_i + \sum_{j=1}^{m} b_j + \sum_{k=1}^{n} c_k = 2(\ell m + mn + n\ell)
$$

Let $A_1 = [a_1, a_2, ..., a_{\ell}]$, $B_1 = [b_1, b_2, ..., b_m]$ and $C_1 = [c_1, c_2, ..., c_n]$. Clearly, $A_1, B_1$ and $C_1$ satisfy the conditions (1) and by the minimality of $p, q$ and $r$ the lists $A_1, B_1$ and $C_1$ are the score lists of some oriented tripartite graph $D_1$. Define

$$
A_2 = [a_{\ell+1} - 2(m + n), a_{\ell+2} - 2(m + n), ..., a_p - 2(m + n)],
$$

$$
B_2 = [b_{m+1} - 2(n + \ell), b_{m+2} - 2(n + \ell), ..., b_q - 2(n + \ell)],
$$

and

$$
C_2 = [c_{n+1} - 2(\ell + m), c_{n+2} - 2(\ell + m), ..., c_r - 2(\ell + m)].
$$

Consider the sum

$$
\sum_{i=1}^{\ell} (a_{\ell+i} - 2(m + n)) + \sum_{j=1}^{M} (b_{m+j} - 2(n + \ell)) + \sum_{k=1}^{N} (c_{n+k} - 2(\ell + m))
$$

$$
= \sum_{i=1}^{\ell} a_i + \sum_{j=1}^{m} b_j + \sum_{k=1}^{n} c_k - \left(\sum_{i=1}^{\ell} a_i + \sum_{j=1}^{m} b_j + \sum_{k=1}^{n} c_k\right)
$$

$$
- 2(\ell (m + n) - 2M (n + \ell) - 2N (\ell + m))
$$

$$
\geq 2((\ell + L)(m + M) + (m + M)(n + N) + (\ell + L)(n + N) - 2(\ell m + mn + n\ell)
$$

$$
- 2(\ell (m + n) - 2M (n + \ell) - 2N (\ell + m))
$$

$$
= 2(LM + MN + NL)
$$

for $1 \leq L \leq p - \ell, 1 \leq M \leq q - m$ and $1 \leq N \leq r - n$, with equality for $L = p - \ell, M = q - m$ and $N = r - n$. Thus the lists $A_2, B_2$ and $C_2$ satisfy
conditions (1) and by the minimality of \( p, q \) and \( r \), the lists \( A_2, B_2 \) and \( C_2 \) for the score lists of some oriented tripartite graph \( D_2 \).

Let \( X_i, Y_i \) and \( Z_i \) be the parts of the oriented tripartite graph \( D_1 \) with the score lists \( A_i, B_i \) and \( C_i \) for \( i = 1, 2 \). Suppose \( X = X_1 \cup X_2, Y = Y_1 \cup Y_2 \) and \( Z = Z_1 \cup Z_2 \). Define an oriented tripartite graph \( D \) with parts \( X, Y \) and \( Z \) and such that arcs of \( D \) are the arcs of \( D_1 \) and each vertex of \( X_2 \) is adjacent to every vertex of \( Y_1 \) and \( Z_1 \), each vertex of \( Y_2 \) is adjacent to every vertex of \( X_1 \) and \( Z_1 \) and each vertex of \( Z_2 \) is adjacent to every vertex of \( X_1 \) and \( Y_1 \). Thus we get an oriented tripartite graph \( D \) with the score lists \( A, B \) and \( C \), which is a contradiction.

Case (ii) Suppose the strict inequality holds in (1) for \( n \neq p, m \neq q \) and \( n \neq r \). Also assume that \( a_1 \neq 0 \). Let \( A_1 = [a_1 - 1, a_2, \ldots, a_p + 1], B_1 = [b_1, b_2, \ldots, b_q] \) and \( C_1 = [c_1, c_2, \ldots, c_r] \). Obviously, \( A_1, B_1 \) and \( C_1 \) satisfy (1) and by the minimality of \( a_1 \), the lists \( A_1, B_1 \) and \( C_1 \) are the score lists of some oriented tripartite graph \( D_1 \) with parts \( X_1, Y_1 \) and \( Z_1 \) (say). Let \( s_{x_1} = a_1 - 1 \) and \( s_{x_p} = a_p + 1 \). Since \( s_{x_p} > s_{x_1} \), there exists a vertex \( u \) either in \( Y_1 \), or in \( Z_1 \) such that either \( x_p \rightarrow u \rightarrow x_1 \), or \( x_p \rightarrow N_u \rightarrow x_1 \), or \( x_p \rightarrow uN_1 \rightarrow x_1 \), or \( x_p \rightarrow uN_2 \rightarrow x_1 \), or \( x_p \rightarrow N_1N_2 \rightarrow x_1 \), or \( x_p \rightarrow N_N \rightarrow x_1 \), or \( x_p \rightarrow N_N \rightarrow x_1 \), or \( x_p \rightarrow N_N \rightarrow x_1 \), or \( x_p \rightarrow N_N \rightarrow x_1 \) respectively, the result is an oriented tripartite graph \( D \) with the score lists \( A, B \) and \( C \). This is again a contradiction, which completes the proof. \( \Box \)

In case of tripartite tournaments [2], the lists \( A = [a_1, a_2, \ldots, a_p] \), \( B = [b_1, b_2, \ldots, b_q] \) and \( C = [c_1, c_2, \ldots, c_r] \) of non-negative integers in non-decreasing order are realizable as scores if and only if

\[
\sum_{i=1}^{\ell} a_i + \sum_{j=1}^{m} b_j + \sum_{k=1}^{n} c_k \geq \ell m + mn + n\ell
\]

for \( 1 \leq \ell \leq p, 1 \leq m \leq q \) and \( 1 \leq n \leq r \), with equality when \( \ell = p, m = q, \) and \( n = r \).

This result now follows from Theorem 4.

References


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