λ-semidirect products and inductive categories

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Based on joint work with Victoria Gould

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- Semidirect products, coverings and embeddings of monoids

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Now if $s \in S$, then

$$s = at \quad \text{where } a = a(t \cdot 1)$$

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$$\begin{array}{rrcl} & at & = & bs \\ \Leftrightarrow & a & = & b \quad \text{and } t \,\sigma_A\, s \end{array}$$

Consequently $t \,\sigma_A\, s$ implies that $(s \cdot 1)t = (t \cdot 1)s$.

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is a set of commuting idempotents.

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Then

- $f_a \in \mathcal{A}$ and $\{f_a : a \in A\} \approx A$
- $\theta : S \to \mathcal{A} \rtimes T/\sigma_A$ is defined by

$$(at)\theta = (f_a, [t]) \quad \text{where } a = a(t \cdot 1).$$

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S is called a λ-semidirect product of A and T.

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He proved that given a left ample semigroup S and a left ample congruence ρ on S, satisfying $\rho \cap \mathcal{R}^* = \imath$, S is isomorphic to a subsemigroup T of $A \rtimes^{\lambda} S/\rho$, with A as a semilattice.

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M. Branco, G. Gomes and V. Gould (2010) extended this result to the λ-semidirect product of a semilattice and a left restriction semigroup.

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Dually right restriction semigroups form a variety of unary semigroups. In this case the unary operation is denoted by *.

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Multiplication in $A \rtimes^{\lambda} T$ is defined by the rule:

$$(a,t)(b,u) \; = \; \Big(\big((tu)^{+} \cdot a\big)\big(t \cdot b\big), tu \Big)$$

Coverings and embeddings in the λ-semidirect product of left restriction monoids

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Two sided case

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and multiplication is defined by:

$$(a, t)(b, u) = \Big(\big((tu)^{+} \cdot a\big)\big(t \cdot b\big), tu \Big)$$

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Let $E = \{\mathbf{d}(x) : x \in C\}$. It follows from the axioms that $E = \{\mathbf{r}(x) : x \in C\}$ and $\mathbf{C}$ is a small category in standard sense with set of identities E and set of objects identified with E.

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We then say that $(C, \cdot, \mathbf{d}, \mathbf{r}, \leq)$ is an *inductive category*.

Theorem Let A and T be restriction semigroups and suppose that T acts on A on the left and right by morphisms preserving $(\cdot, +, *)$ such that for all $t \in T$ and for all $a \in S$, the following compatibility conditions hold:

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Inductive categories and λ-semidirect products

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where

$$\mathbf{d}(a, t) = (a^+, t^+), \quad \mathbf{r}(a, t) = (a^* \circ t, t^*)$$

The partial binary operation on V is defined by the rule

$$(a,t)\cdot(b,u)=\begin{cases}(a(t\cdot b),tu) & \text{if } \mathbf{r}(a,t)=\mathbf{d}(b,u)\\ \text{undefined otherwise}\end{cases}$$

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$$(a,t)\leq(b,u) \quad \text{if and only if} \quad a\leq t^{+}\cdot b\,,\ \ t\leq u.$$

Also for $(a,t)\in V$ and $(x\cdot e,x)\in E$, the restriction and co-restriction are defined as:

$$\begin{aligned}((x\cdot e,x)|(a,t)) &= (x\cdot ea,xt)\\ ((a,t)|(x\cdot e,x)) &= (((tx)^{+}\cdot a)(t\cdot(x\cdot e)),tx).\end{aligned}$$

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Then $\otimes$ coincides with λ-semidirect product

$$(a,t)(b,u) = \left(\left((tu)^{+} \cdot a\right)\left(t \cdot b\right), tu\right)$$