Lattices of regular closed sets in closure spaces: semidistributivity and Dedekind-MacNeille completions

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NSAC 2013, Novi Sad, June 2013
Joint work with Luigi Santocanale

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- where we set

$$[n] \underset{\text{def.}}{=} \{1, 2, \dots, n\}\,,$$

$$\mathcal{I}_n \underset{\text{def.}}{=} \{(i, j) \in [n] \times [n] \mid i < j\}\,,$$

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- Alternate definition: $\mathsf{P}(n) = \{\mathsf{Inv}(\sigma) \mid \sigma \in \mathfrak{S}_n\}$, ordered by $\subseteq$.

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- Conversely, every subset $\mathbf{x} \subseteq \mathcal{I}_n$, such that both $\mathbf{x}$ and $\mathcal{I}_n \setminus \mathbf{x}$ are transitive, is $\mathrm{Inv}(\sigma)$ for a unique $\sigma \in \mathfrak{S}_n$ (Dushnik and Miller 1941, Guilbaud and Rosenstiehl 1963).

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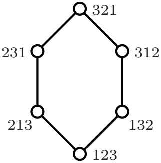
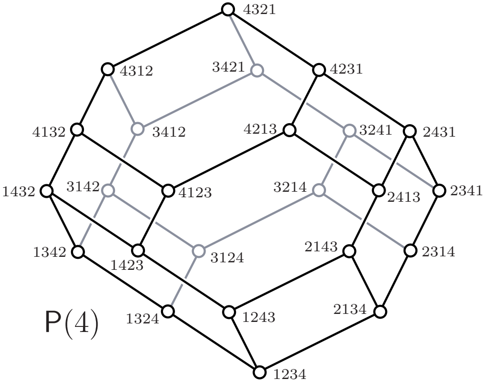
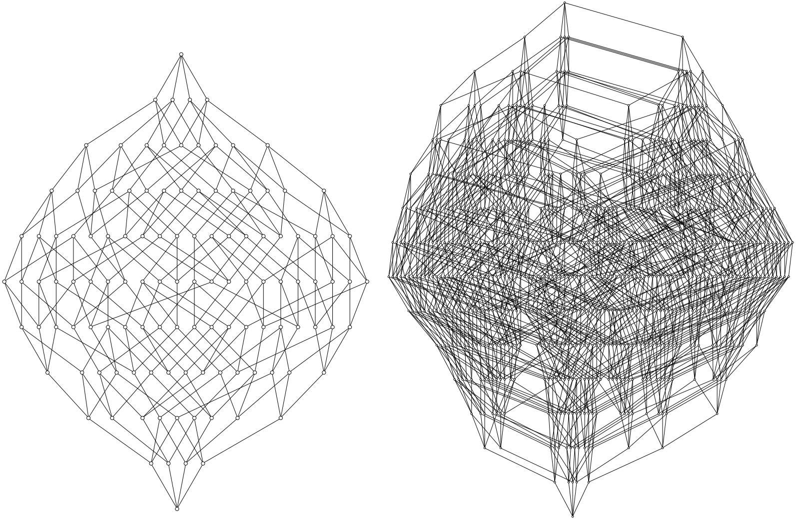
- Observe that each $\mathbf{x} \in \mathrm{P}(n)$ is a strict ordering. It can be proved (Dushnik and Miller 1941) that those are exactly the finite strict orderings of order-dimension 2.

The permutohedra P(2), P(3), and P(4).

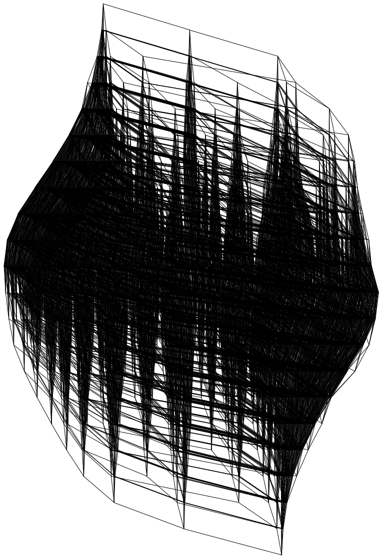




The permutohedra P(5) and P(6)



The permutohedron P(7)



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Hence $\mathrm{P}(n)$ is an ortholattice.

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The permutohedron $\mathrm{P}(n)$ is a bounded homomorphic image of a free lattice, for every positive integer n.

This means that there are a finitely generated free lattice F and a surjective lattice homomorphism $f \colon F \twoheadrightarrow \mathrm{P}(n)$ such that each $f^{-1}\{x\}$ has both a least and a largest element.

Regular closed sets

- Closure space: pair (Ω, φ), where $\varphi \colon \mathfrak{P}(\Omega) \to \mathfrak{P}(\Omega)$, with $\varphi(\varnothing) = \varnothing$, $X \subseteq Y \Rightarrow \varphi(X) \subseteq \varphi(Y)$, $X \subseteq \varphi(X)$, $\varphi \circ \varphi = \varphi$.

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- Every orthoposet appears as some $\mathrm{Clop}(\Omega, \varphi)$ (Mayet 1982, Katrnoška 1982)

What happens for convex geometries?

Convex geometry: closure space (Ω, φ) such that ($\mathbf{x}$ closed, $p, q \in \Omega \setminus \mathbf{x}$, and $\varphi(\mathbf{x} \cup \{p\}) = \varphi(\mathbf{x} \cup \{q\})) \Rightarrow p = q$.

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Theorem (Santocanale and W. 2012)

For (more general spaces than) finite convex geometries, the lattice $\operatorname{Reg}(\Omega, \varphi)$ is always pseudocomplemented.

Transitive binary relations

- For a transitive binary relation $\mathbf{e} \subseteq P \times P$, set $\Omega = \mathbf{e}$, $\varphi(\mathbf{a}) = \mathrm{cl}(\mathbf{a})$ =transitive closure of $\mathbf{a}$ $(\forall \mathbf{a} \subseteq \mathbf{e})$.

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- For $\mathbf{e} = [n] \times [n]$, $\mathrm{Reg}(\mathbf{e}, \mathrm{cl}) = \mathrm{Clop}(\mathbf{e}, \mathrm{cl}) = \mathrm{Bip}(n)$, the bipartition lattice on $[n]$ (Foata and Zeilberger 1996, Han 1996, Hetyei and Krattenthaler 2011).

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- $\mathrm{Bip}(n)$ contains an M_3 whenever $n \geq 3$.

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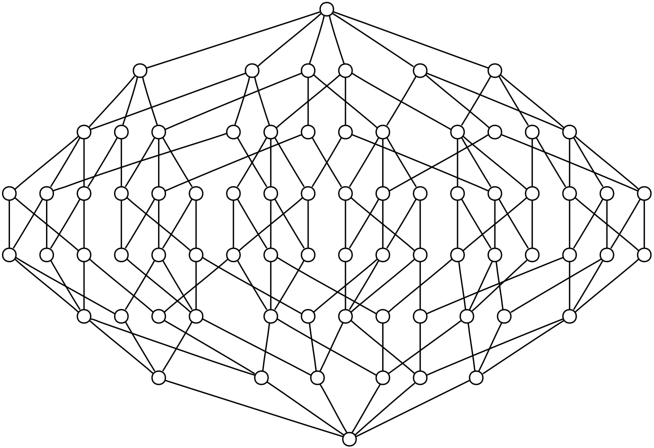
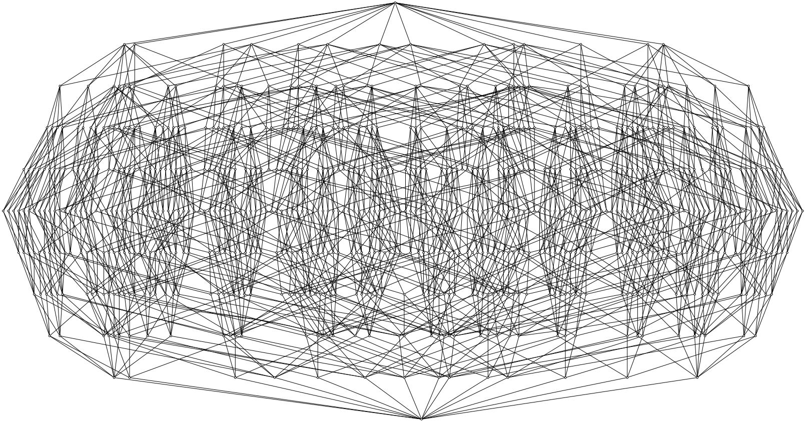
3. For $\mathbf{e}$ finite, $\mathrm{Reg}(\mathbf{e}, \mathrm{cl})$ is semidistributive iff it is a bounded homomorphic image of a free lattice, iff every connected component of $\mathbf{e}$ is either antisymmetric or $E \times E$ with $\mathrm{card}\, E = 2$.

The lattice Bip(3)



The lattice Bip(4)



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- Strongly bi-convex $\Rightarrow$ bi-convex $\Rightarrow$ relatively convex.
- $\operatorname{Clop}^*(E, \operatorname{conv}_E) = \{X \subseteq E \mid X \text{ is strongly bi-convex}\}$.

Convex sets and Dedekind-MacNeille completion

Theorem (Santocanale and W. 2013)

Let E be a subset in a real affine space Δ. Then $\operatorname{Reg}(E, \operatorname{conv}_E)$ is the Dedekind-MacNeille completion of $\operatorname{Clop}^*(E, \operatorname{conv}_E)$ (thus of $\operatorname{Clop}(E, \operatorname{conv}_E)$).

Poset of regions of a central hyperplane arrangement

- Central hyperplane arrangement in $\mathbb{R}^d$: finite set $\mathcal{H}$ of hyperplanes through 0. Regions (set $\mathcal{R}$): connected components of $\mathbb{R}^d \setminus \bigcup \mathcal{H}$ (necessarily open). Base region $B \in \mathcal{R}$.

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Theorem (Santocanale and W. 2013)

$\operatorname{Pos}(\mathcal{H}, B) \cong \operatorname{Clop}^*(E, \operatorname{conv}_E)$, for a suitably defined finite $E \subseteq \mathbb{R}^d$.

Partitions in graphs

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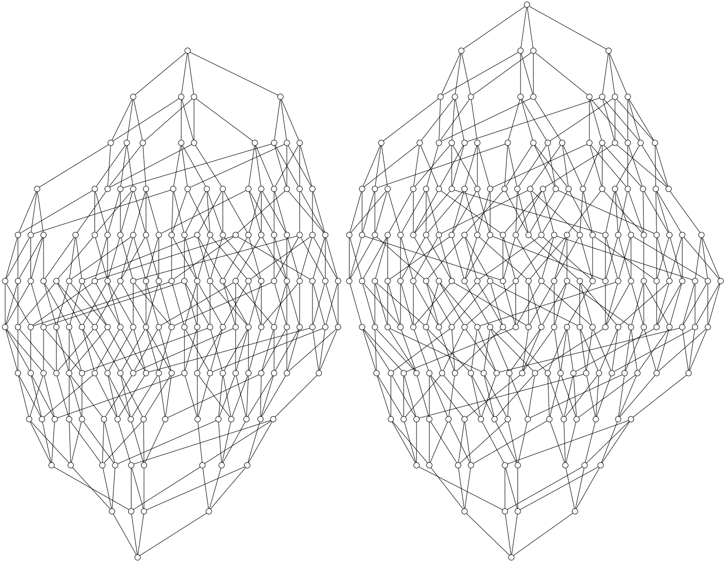
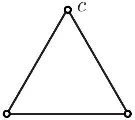
- For G the underlying graph of a Dynkin diagram $\mathcal{G}$, $\operatorname{Clop}(\delta_G, \mathrm{cl}) = \operatorname{Reg}(\delta_G, \mathrm{cl})$ and this lattice bears mysterious connections with the Coxeter lattice of type $\mathcal{G}$ (thus with hyperplane arrangements).

The extended permutohedron on $\mathcal{D}_4$, and the corresponding Coxeter lattice



The extended permutohedron on $\mathcal{K}_3$



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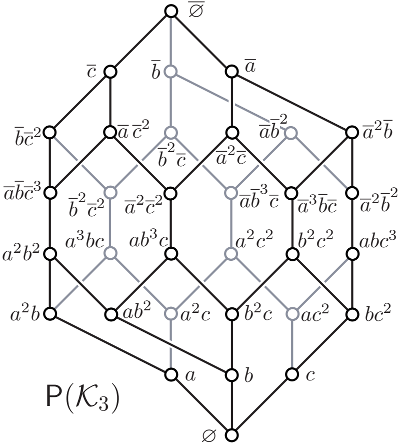

$\mathsf{P}(\mathcal{K}_3)$

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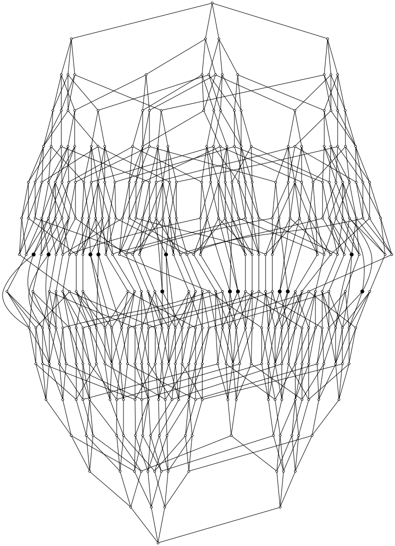

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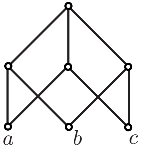
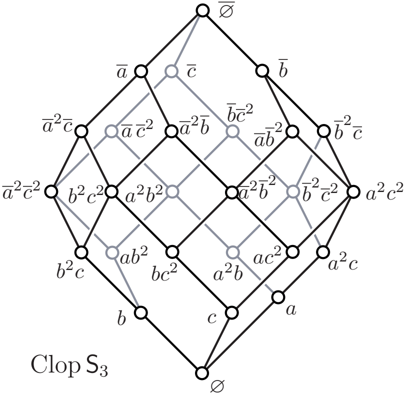
However, $\mathrm{Reg}(S, \mathrm{cl})$ may not be spatial.

The extended permutohedron on S_3



S_3



$\operatorname{Clop}\mathsf{S}_3$