Invariance groups of finite functions and orbit equivalence of permutation groups

Tamás Waldhauser

University of Szeged

NSAC 2013
Novi Sad, 7th June 2013

Joint work with

- Eszter Horváth,
- Reinhard Pöschel,
- Géza Makay.

Joint work with

- Eszter Horváth,
- Reinhard Pöschel,
- Géza Makay.

We acknowledge helpful discussions with

- Erik Friese,
- Keith Kearnes,
- Erkko Lehtonen,
- P^3 (Péter Pál Pálfy),
- Sándor Radeleczki.

Invariance groups

Definition

The invariance group of a function $f\colon \mathbf{k}^n \to \mathbf{m}$ is

$$S(f) = \{\sigma \in S_n \mid f(x_1, \ldots, x_n) \equiv f(x_{1\sigma}, \ldots, x_{n\sigma})\}.$$

Invariance groups

Definition

The invariance group of a function $f\colon \mathbf{k}^n \to \mathbf{m}$ is

$$S(f) = \{\sigma \in S_n \mid f(x_1, \ldots, x_n) \equiv f(x_{1\sigma}, \ldots, x_{n\sigma})\}.$$

Definition

- A group G is (k, m)-representable if there is a function $f\colon \mathbf{k}^n \to \mathbf{m}$ such that $S(f) = G$.

Invariance groups

Definition

The invariance group of a function $f\colon \mathbf{k}^n \to \mathbf{m}$ is

$$S(f) = \{\sigma \in S_n \mid f(x_1, \ldots, x_n) \equiv f(x_{1\sigma}, \ldots, x_{n\sigma})\}.$$

Definition

- A group G is (k, m)-representable if there is a function $f\colon \mathbf{k}^n \to \mathbf{m}$ such that $S(f) = G$.
- A group G is (k, ∞)-representable if G is (k, m)-representable for some m.

Invariance groups

Definition

The invariance group of a function $f\colon \mathbf{k}^n \to \mathbf{m}$ is

$$S(f) = \{\sigma \in S_n \mid f(x_1, \ldots, x_n) \equiv f(x_{1\sigma}, \ldots, x_{n\sigma})\}.$$

Definition

- A group G is (k, m)-representable if there is a function $f\colon \mathbf{k}^n \to \mathbf{m}$ such that $S(f) = G$.
- A group G is (k, ∞)-representable if G is (k, m)-representable for some m.

Special cases:

- G is $(2, 2)$-representable iff G is the invariance group of a Boolean function $f\colon \mathbf{2}^n \to \mathbf{2}$.

Invariance groups

Definition

The invariance group of a function $f\colon \mathbf{k}^n \to \mathbf{m}$ is

$$S(f) = \{\sigma \in S_n \mid f(x_1, \ldots, x_n) \equiv f(x_{1\sigma}, \ldots, x_{n\sigma})\}.$$

Definition

- A group G is (k, m)-representable if there is a function $f\colon \mathbf{k}^n \to \mathbf{m}$ such that $S(f) = G$.
- A group G is (k, ∞)-representable if G is (k, m)-representable for some m.

Special cases:

- G is $(2, 2)$-representable iff G is the invariance group of a Boolean function $f\colon \mathbf{2}^n \to \mathbf{2}$.
- G is $(2, \infty)$-representable iff G is the invariance group of a pseudo-Boolean function $f\colon \mathbf{2}^n \to \mathbf{m}$.

Abstract representation

Frucht 1939:

Every group is isomorphic to the automorphism group of a graph.

Abstract representation

Frucht 1939:

Every group is isomorphic to the automorphism group of a graph.

Corollary

Every group is isomorphic to the invariance group of some Boolean function (i.e., $(2, 2)$*-representable).*

Abstract representation

Frucht 1939:

Every group is isomorphic to the automorphism group of a graph.

Corollary

Every group is isomorphic to the invariance group of some Boolean function (i.e., $(2,2)$-representable).

Proof.

$f\colon \mathbf{2}^n \to \mathbf{2} \;\leftrightsquigarrow\; \mathcal{H} = \big(\mathbf{n},\, \{E \subseteq \mathbf{n} \mid f\,(\chi_E) = 1\}\big)$ □

Abstract representation

Frucht 1939:

Every group is isomorphic to the automorphism group of a graph.

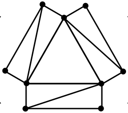
Corollary

Every group is isomorphic to the invariance group of some Boolean function (i.e., $(2,2)$-representable).

Proof.

$$f\colon \mathbf{2}^n \to \mathbf{2} \;\leftrightsquigarrow\; \mathcal{H} = \big(\mathbf{n},\, \{E \subseteq \mathbf{n} \mid f(\chi_E) = 1\}\big)$$ □

Example

$S\Bigg($  $\Bigg) \cong A_3$

Concrete representation

Example

Suppose that $S(f) = A_3$ for some $f\colon \mathbf{2}^3 \longrightarrow \mathbf{m}$.

Concrete representation

Example

Suppose that $S\left(f\right) = A_3$ for some $f \colon \mathbf{2}^3 \longrightarrow \mathbf{m}$. Then f must be constant on the orbits of A_3 acting on $\mathbf{2}^3$:

$$\begin{array}{lcl} 000 & \longmapsto & a \\ 100, 010, 001 & \longmapsto & b \\ 011, 101, 110 & \longmapsto & c \\ 111 & \longmapsto & d \end{array}$$

Concrete representation

Example

Suppose that $S(f) = A_3$ for some $f \colon \mathbf{2}^3 \to \mathbf{m}$. Then f must be constant on the orbits of A_3 acting on $\mathbf{2}^3$:

$$\begin{array}{lcl} 000 & \mapsto & a \\ 100, 010, 001 & \mapsto & b \\ 011, 101, 110 & \mapsto & c \\ 111 & \mapsto & d \end{array}$$

However, such a function is totally symmetric, i.e., $S(f) = S_3$.

Concrete representation

Example

Suppose that $S(f) = A_3$ for some $f\colon \mathbf{2}^3 \longrightarrow \mathbf{m}$. Then f must be constant on the orbits of A_3 acting on $\mathbf{2}^3$:

$$\begin{array}{lcl} 000 & \longmapsto & a \\ 100, 010, 001 & \longmapsto & b \\ 011, 101, 110 & \longmapsto & c \\ 111 & \longmapsto & d \end{array}$$

However, such a function is totally symmetric, i.e., $S(f) = S_3$. Thus A_3 is not $(2, \infty)$-representable.

Concrete representation

Example

Suppose that $S(f) = A_3$ for some $f\colon \mathbf{2}^3 \longrightarrow \mathbf{m}$. Then f must be constant on the orbits of A_3 acting on $\mathbf{2}^3$:

$$\begin{array}{lcl} 000 & \mapsto & a \\ 100, 010, 001 & \mapsto & b \\ 011, 101, 110 & \mapsto & c \\ 111 & \mapsto & d \end{array}$$

However, such a function is totally symmetric, i.e., $S(f) = S_3$. Thus A_3 is not $(2, \infty)$-representable.

Let $g\colon \mathbf{3}^3 \longrightarrow \mathbf{2}$ such that $g(0,1,2) = g(1,2,0) = g(2,1,0) = 1$ and $g = 0$ everywhere else.
Then $S(g) = A_3$, thus A_3 is $(3, 2)$-representable.

Ein Kleines Problem

Clote, Kranakis 1991:

If G is $(2, \infty)$-representable, then G is $(2, 2)$-representable.

Ein Kleines Problem

Clote, Kranakis 1991:

If G is $(2, \infty)$-representable, then G is $(2, 2)$-representable.

Kisielewicz 1998:

False!

Ein Kleines Problem

Clote, Kranakis 1991:
If G is $(2, \infty)$-representable, then G is $(2, 2)$-representable.

Kisielewicz 1998:
False! The Klein four-group

$$V = \{\text{id}, (12)\,(34)\,, (13)\,(24)\,, (14)\,(23)\} \leq S_4$$

is a counterexample;

Ein Kleines Problem

Clote, Kranakis 1991:
If G is $(2, \infty)$-representable, then G is $(2, 2)$-representable.

Kisielewicz 1998:
False! The Klein four-group

$$V = \{\text{id}, (12)\,(34)\,, (13)\,(24)\,, (14)\,(23)\} \leq S_4$$

is a counterexample; moreover, it is the only counterexample that one could "easily" find.

Ein Kleines Problem

Clote, Kranakis 1991:

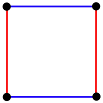
If G is $(2,\infty)$-representable, then G is $(2,2)$-representable.

Kisielewicz 1998:

False! The Klein four-group

$$V = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \leq S_4$$

is a counterexample; moreover, it is the only counterexample that one could "easily" find.

$V = S\Big($  $\Big)$

Ein Kleines Problem

Clote, Kranakis 1991:

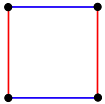
If G is $(2,\infty)$-representable, then G is $(2,2)$-representable.

Kisielewicz 1998:

False! The Klein four-group

$$V = \{\mathrm{id}, (12)\,(34)\,, (13)\,(24)\,, (14)\,(23)\} \leq S_4$$

is a counterexample; moreover, it is the only counterexample that one could “easily” find.

$V = S\Big($  $\Big)$

$\implies$ V is $(2,3)$-representable but not $(2,2)$-representable.

Ein Kleines Problem

Clote, Kranakis 1991:

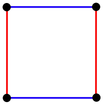
If G is $(2,\infty)$-representable, then G is $(2,2)$-representable.

Kisielewicz 1998:

False! The Klein four-group

$$V = \{\mathrm{id}, (12)\,(34)\,, (13)\,(24)\,, (14)\,(23)\} \leq S_4$$

is a counterexample; moreover, it is the only counterexample that one could "easily" find.

$V = S\Big($  $\Big)$

$\Longrightarrow$ V is $(2,3)$-representable but not $(2,2)$-representable.

Dalla Volta, Siemons 2012:

There are infinitely many groups that are $(2,\infty)$-representable but not $(2,2)$-representable. (?)

Ein Kleines Problem

Clote, Kranakis 1991:

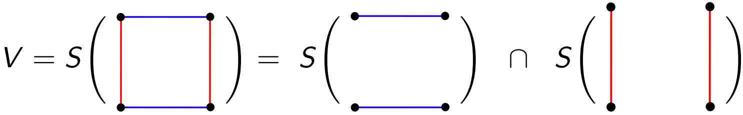
If G is $(2,\infty)$-representable, then G is $(2,2)$-representable.

Kisielewicz 1998:

False! The Klein four-group

$$V = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \leq S_4$$

is a counterexample; moreover, it is the only counterexample that one could "easily" find.



$\implies$ V is $(2,3)$-representable but not $(2,2)$-representable.

Dalla Volta, Siemons 2012:

There are infinitely many groups that are $(2,\infty)$-representable but not $(2,2)$-representable. (?)

Orbit closure

Clote, Kranakis 1991:

The following are equivalent for any group $G \leq S_n$:

(i) G is the invariance group of a pseudo-Boolean function (i.e., G is $(2, \infty)$-representable).

(ii) G is the intersection of invariance groups of Boolean functions (i.e., $(2, 2)$-representable groups).

Orbit closure

Clote, Kranakis 1991:

The following are equivalent for any group $G \leq S_n$:

(i) G is the invariance group of a pseudo-Boolean function (i.e., G is $(2, \infty)$-representable).

(ii) G is the intersection of invariance groups of Boolean functions (i.e., $(2, 2)$-representable groups).

(iii) G is orbit closed.

Orbit closure

Clote, Kranakis 1991:

The following are equivalent for any group $G \leq S_n$:

(i) G is the invariance group of a pseudo-Boolean function (i.e., G is $(2, \infty)$-representable).

(ii) G is the intersection of invariance groups of Boolean functions (i.e., $(2, 2)$-representable groups).

(iii) G is orbit closed.

Two subgroups of S_n are orbit equivalent if they have the same orbits on $\mathcal{P}(\mathbf{n})$

Orbit closure

Clote, Kranakis 1991:
The following are equivalent for any group $G \leq S_n$:

(i) G is the invariance group of a pseudo-Boolean function (i.e., G is $(2, \infty)$-representable).

(ii) G is the intersection of invariance groups of Boolean functions (i.e., $(2, 2)$-representable groups).

(iii) G is orbit closed.

Two subgroups of S_n are orbit equivalent if they have the same orbits on $\mathcal{P}(\mathbf{n}) \leftrightsquigarrow \mathbf{2}^n$.

Orbit closure

Clote, Kranakis 1991:
The following are equivalent for any group $G \leq S_n$:

(i) G is the invariance group of a pseudo-Boolean function (i.e., G is $(2, \infty)$-representable).

(ii) G is the intersection of invariance groups of Boolean functions (i.e., $(2, 2)$-representable groups).

(iii) G is orbit closed.

Two subgroups of S_n are orbit equivalent if they have the same orbits on $\mathcal{P}(\mathbf{n}) \leftrightsquigarrow \mathbf{2}^n$.

The orbit closure of G is the greatest element of its orbit equivalence class.

Primitive groups

Inglis; Cameron, Neumann, Saxl; Siemons, Wagner 1984–85:
Almost all primitive groups are orbit closed.

Primitive groups

Inglis; Cameron, Neumann, Saxl; Siemons, Wagner 1984–85:
Almost all primitive groups are orbit closed.

Seress 1997:
All primitive subgroups of S_n are orbit closed except for A_n and C_5, $\mathrm{AGL}\,(1,5)$, $\mathrm{PGL}\,(2,5)$, $\mathrm{AGL}\,(1,8)$, $\mathrm{A\Gamma L}(1,8)$, $\mathrm{AGL}\,(1,9)$, $\mathrm{ASL}\,(2,3)$, $\mathrm{PSL}\,(2,8)$, $\mathrm{P\Gamma L}(2,8)$ and $\mathrm{PGL}\,(2,9)$.

Primitive groups

Inglis; Cameron, Neumann, Saxl; Siemons, Wagner 1984–85:
Almost all primitive groups are orbit closed.

Seress 1997:
All primitive subgroups of S_n are orbit closed except for A_n and C_5, $\mathrm{AGL}(1,5)$, $\mathrm{PGL}(2,5)$, $\mathrm{AGL}(1,8)$, $\mathrm{A\Gamma L}(1,8)$, $\mathrm{AGL}(1,9)$, $\mathrm{ASL}(2,3)$, $\mathrm{PSL}(2,8)$, $\mathrm{P\Gamma L}(2,8)$ and $\mathrm{PGL}(2,9)$.

Theorem
All primitive groups are $(3,\infty)$-representable except for the alternating groups.

A Galois connection

For $a = (a_1, \ldots, a_n) \in \mathbf{k}^n$ and $\sigma \in S_n$, let $a^\sigma = (a_{1\sigma}, \ldots, a_{n\sigma})$.

A Galois connection

For $a = (a_1, \ldots, a_n) \in \mathbf{k}^n$ and $\sigma \in S_n$, let $a^\sigma = (a_{1\sigma}, \ldots, a_{n\sigma})$.

If $f \colon \mathbf{k}^n \to \mathbf{k}$ and $\sigma \in S_n$, then we write

$$\sigma \vdash f :\Longleftrightarrow f\left(a^\sigma\right) = f\left(a\right) \text{ for all } a \in \mathbf{k}^n.$$

A Galois connection

For $a = (a_1, \ldots, a_n) \in \mathbf{k}^n$ and $\sigma \in S_n$, let $a^\sigma = (a_{1\sigma}, \ldots, a_{n\sigma})$.

If $f \colon \mathbf{k}^n \to \mathbf{k}$ and $\sigma \in S_n$, then we write

$$\sigma \vdash f :\iff f(a^\sigma) = f(a) \text{ for all } a \in \mathbf{k}^n.$$

Let $O_k^{(n)} = \{f \mid f \colon \mathbf{k}^n \to \mathbf{k}\}$, and for $F \subseteq O_k^{(n)}$ and $G \subseteq S_n$ define

$$F^\vdash := \{\sigma \in S_n \mid \forall f \in F : \sigma \vdash f\}, \qquad \overline{F}^{(k)} := (F^\vdash)^\vdash,$$
$$G^\vdash := \{f \in O_k^{(n)} \mid \forall \sigma \in G : \sigma \vdash f\}, \quad \overline{G}^{(k)} := (G^\vdash)^\vdash.$$

A Galois connection

For $a = (a_1, \ldots, a_n) \in \mathbf{k}^n$ and $\sigma \in S_n$, let $a^\sigma = (a_{1\sigma}, \ldots, a_{n\sigma})$.

If $f \colon \mathbf{k}^n \to \mathbf{k}$ and $\sigma \in S_n$, then we write

$$\sigma \vdash f :\iff f(a^\sigma) = f(a) \text{ for all } a \in \mathbf{k}^n.$$

Let $O_k^{(n)} = \{f \mid f \colon \mathbf{k}^n \to \mathbf{k}\}$, and for $F \subseteq O_k^{(n)}$ and $G \subseteq S_n$ define

$$F^\vdash := \{\sigma \in S_n \mid \forall f \in F : \sigma \vdash f\}, \qquad \overline{F}^{(k)} := (F^\vdash)^\vdash,$$
$$G^\vdash := \{f \in O_k^{(n)} \mid \forall \sigma \in G : \sigma \vdash f\}, \quad \overline{G}^{(k)} := (G^\vdash)^\vdash.$$

For $G \leq S_n$, we call $\overline{G}^{(k)}$ the Galois closure of G over $\mathbf{k}$.

Galois closed groups as invariance groups

Galois closed groups as invariance groups

Fact

The following are equivalent for any group $G \leq S_n$:

(i) *G is Galois closed over* $\mathbf{k}$*.*

(ii) *G is (k, ∞)-representable.*

Galois closed groups as invariance groups

Fact

The following are equivalent for any group $G \leq S_n$:

(i) *G is Galois closed over* $\mathbf{k}$.

(ii) *G is (k, ∞)-representable.*

(iii) *G is the invariance group of a function $f : \mathbf{k}^n \to \infty$.*

Galois closed groups as invariance groups

Fact

The following are equivalent for any group $G \leq S_n$:

(i) G *is Galois closed over* $\mathbf{k}$.

(ii) G *is* (k, ∞)*-representable.*

(iii) G *is the invariance group of a function* $f : \mathbf{k}^n \to \infty$.

(iv) G *is the intersection of invariance groups of functions* $\mathbf{k}^n \to \mathbf{2}$.

(v) G *is the intersection of invariance groups of functions* $\mathbf{k}^n \to \mathbf{k}$.

Galois closed groups as invariance groups

Fact

The following are equivalent for any group $G \leq S_n$*:*

(i) G *is Galois closed over* $\mathbf{k}$.

(ii) G *is* (k, ∞)*-representable.*

(iii) G *is the invariance group of a function* $f : \mathbf{k}^n \longrightarrow \infty$.

(iv) G *is the intersection of invariance groups of functions* $\mathbf{k}^n \longrightarrow \mathbf{2}$.

(v) G *is the intersection of invariance groups of functions* $\mathbf{k}^n \longrightarrow \mathbf{k}$.

(vi) G *is orbit closed with respect to the action of* S_n *on* $\mathbf{k}^n$.

Orbits and closures

For $a = (a_1, \ldots, a_n) \in \mathbf{k}^n$ and $G \leq S_n$, define

$$a^G = \left\{ a^\sigma \mid \sigma \in G \right\}, \qquad \mathrm{Orb}^{(k)}(G) = \left\{ a^G \mid a \in \mathbf{k}^n \right\}.$$

Orbits and closures

For $a = (a_1, \ldots, a_n) \in \mathbf{k}^n$ and $G \leq S_n$, define

$$a^G = \{a^\sigma \mid \sigma \in G\}, \qquad \mathrm{Orb}^{(k)}(G) = \{a^G \mid a \in \mathbf{k}^n\}.$$

For all $G, H \leq S_n$ we have

$$\overline{G}^{(k)} = \overline{H}^{(k)} \iff G^{\vdash} = H^{\vdash}$$

Orbits and closures

For $a = (a_1, \ldots, a_n) \in \mathbf{k}^n$ and $G \leq S_n$, define

$$a^G = \{a^\sigma \mid \sigma \in G\}, \qquad \mathrm{Orb}^{(k)}(G) = \{a^G \mid a \in \mathbf{k}^n\}.$$

For all $G, H \leq S_n$ we have

$$\overline{G}^{(k)} = \overline{H}^{(k)} \iff G^\vdash = H^\vdash \iff \mathrm{Orb}^{(k)}(G) = \mathrm{Orb}^{(k)}(H);$$

Orbits and closures

For $a = (a_1, \ldots, a_n) \in \mathbf{k}^n$ and $G \leq S_n$, define

$$a^G = \{a^\sigma \mid \sigma \in G\}, \qquad \mathrm{Orb}^{(k)}(G) = \left\{a^G \mid a \in \mathbf{k}^n\right\}.$$

For all $G, H \leq S_n$ we have

$$\overline{G}^{(k)} = \overline{H}^{(k)} \iff G^{\vdash} = H^{\vdash} \iff \mathrm{Orb}^{(k)}(G) = \mathrm{Orb}^{(k)}(H);$$

$$\overline{G}^{(k)} = \left\{\sigma \in S_n \mid \forall a \in \mathbf{k}^n : \ a^\sigma \in a^G\right\}.$$

Orbits and closures

For $a = (a_1, \ldots, a_n) \in \mathbf{k}^n$ and $G \leq S_n$, define

$$a^G = \{a^\sigma \mid \sigma \in G\}, \qquad \mathrm{Orb}^{(k)}(G) = \left\{a^G \mid a \in \mathbf{k}^n\right\}.$$

For all $G, H \leq S_n$ we have

$$\overline{G}^{(k)} = \overline{H}^{(k)} \iff G^\vdash = H^\vdash \iff \mathrm{Orb}^{(k)}(G) = \mathrm{Orb}^{(k)}(H);$$

$$\overline{G}^{(k)} = \left\{\sigma \in S_n \mid \forall a \in \mathbf{k}^n : \ a^\sigma \in a^G\right\}.$$

The case $k = 2$ corresponds to orbit equivalence and orbit closure.

Orbits and closures

For $a = (a_1, \ldots, a_n) \in \mathbf{k}^n$ and $G \leq S_n$, define

$$a^G = \{a^\sigma \mid \sigma \in G\}, \qquad \mathrm{Orb}^{(k)}(G) = \{a^G \mid a \in \mathbf{k}^n\}.$$

For all $G, H \leq S_n$ we have

$$\overline{G}^{(k)} = \overline{H}^{(k)} \iff G^{\vdash} = H^{\vdash} \iff \mathrm{Orb}^{(k)}(G) = \mathrm{Orb}^{(k)}(H);$$

$$\overline{G}^{(k)} = \left\{\sigma \in S_n \mid \forall a \in \mathbf{k}^n: \; a^\sigma \in a^G\right\}.$$

The case $k = 2$ corresponds to orbit equivalence and orbit closure.

Proposition

For all $G \leq S_n$ we have $\overline{G}^{(2)} \geq \overline{G}^{(3)} \geq \cdots \geq \overline{G}^{(n)} = \cdots = G$.

A formula for the closure

Proposition

For every $G \leq S_n$ and $k \geq 2$, we have

$$\overline{G}^{(k)} = \bigcap_{a \in \mathbf{k}^n} \operatorname{Stab}(a) \cdot G.$$

A formula for the closure

Proposition

For every $G \leq S_n$ and $k \geq 2$, we have

$$\overline{G}^{(k)} = \bigcap_{a \in \mathbf{k}^n} \operatorname{Stab}(a) \cdot G.$$

Proof.

$$\overline{G}^{(k)} = \left\{ \sigma \in S_n \mid \forall a \in \mathbf{k}^n : \; a^\sigma \in a^G \right\}$$

A formula for the closure

Proposition

For every $G \leq S_n$ and $k \geq 2$, we have

$$\overline{G}^{(k)} = \bigcap_{a \in \mathbf{k}^n} \operatorname{Stab}(a) \cdot G.$$

Proof.

$$\begin{aligned} \overline{G}^{(k)} &= \left\{ \sigma \in S_n \mid \forall a \in \mathbf{k}^n : \; a^\sigma \in a^G \right\} \\ &= \left\{ \sigma \in S_n \mid \forall a \in \mathbf{k}^n \; \exists \pi \in G : \; a^\sigma = a^\pi \right\} \end{aligned}$$

A formula for the closure

Proposition

For every $G \leq S_n$ and $k \geq 2$, we have

$$\overline{G}^{(k)} = \bigcap_{a \in \mathbf{k}^n} \mathrm{Stab}\,(a) \cdot G.$$

Proof.

$$\begin{aligned}
\overline{G}^{(k)} &= \left\{\sigma \in S_n \mid \forall a \in \mathbf{k}^n : \ a^\sigma \in a^G\right\} \\
&= \left\{\sigma \in S_n \mid \forall a \in \mathbf{k}^n \ \exists \pi \in G : \ a^\sigma = a^\pi\right\} \\
&= \left\{\sigma \in S_n \mid \forall a \in \mathbf{k}^n \ \exists \pi \in G : \ \sigma\pi^{-1} \in \mathrm{Stab}\,(a)\right\}
\end{aligned}$$

A formula for the closure

Proposition

For every $G \leq S_n$ and $k \geq 2$, we have

$$\overline{G}^{(k)} = \bigcap_{a \in \mathbf{k}^n} \mathrm{Stab}\,(a) \cdot G.$$

Proof.

$$\begin{aligned}
\overline{G}^{(k)} &= \left\{\sigma \in S_n \mid \forall a \in \mathbf{k}^n : \; a^\sigma \in a^G\right\} \\
&= \left\{\sigma \in S_n \mid \forall a \in \mathbf{k}^n \; \exists \pi \in G : \; a^\sigma = a^\pi\right\} \\
&= \left\{\sigma \in S_n \mid \forall a \in \mathbf{k}^n \; \exists \pi \in G : \; \sigma\pi^{-1} \in \mathrm{Stab}\,(a)\right\} \\
&= \left\{\sigma \in S_n \mid \forall a \in \mathbf{k}^n : \; \sigma \in \mathrm{Stab}\,(a) \cdot G\right\}
\end{aligned}$$

A formula for the closure

Proposition

For every $G \leq S_n$ and $k \geq 2$, we have

$$\overline{G}^{(k)} = \bigcap_{a \in \mathbf{k}^n} \mathrm{Stab}\,(a) \cdot G.$$

Proof.

$$\begin{aligned}
\overline{G}^{(k)} &= \left\{\sigma \in S_n \mid \forall a \in \mathbf{k}^n : \; a^\sigma \in a^G\right\} \\
&= \left\{\sigma \in S_n \mid \forall a \in \mathbf{k}^n \; \exists \pi \in G : \; a^\sigma = a^\pi\right\} \\
&= \left\{\sigma \in S_n \mid \forall a \in \mathbf{k}^n \; \exists \pi \in G : \; \sigma\pi^{-1} \in \mathrm{Stab}\,(a)\right\} \\
&= \left\{\sigma \in S_n \mid \forall a \in \mathbf{k}^n : \; \sigma \in \mathrm{Stab}\,(a) \cdot G\right\} \\
&= \bigcap_{a \in \mathbf{k}^n} \mathrm{Stab}\,(a) \cdot G. \qquad \square
\end{aligned}$$

The case $k = n - 1$

Theorem

If $k = n - 1 \geq 2$, then all subgroups of S_n except A_n are Galois closed over $\mathbf{k}$.

The case $k = n - 1$

Theorem

If $k = n - 1 \geq 2$, then all subgroups of S_n except A_n are Galois closed over $\mathbf{k}$*.*

Definition (Clote, Kranakis 1991)

A group $G \leq S_n$ is weakly representable, if G is (k, ∞)-representable for some $k < n$.

The case $k = n - 1$

Theorem

If $k = n - 1 \geq 2$, then all subgroups of S_n except A_n are Galois closed over $\mathbf{k}$.

Definition (Clote, Kranakis 1991)

A group $G \leq S_n$ is weakly representable, if G is (k, ∞)-representable for some $k < n$.

Corollary

All subgroups of $G \leq S_n$ except for A_n are weakly representable.

The case $k = n - 1$

Theorem

If $k = n - 1 \geq 2$, then all subgroups of S_n except A_n are Galois closed over $\mathbf{k}$.

Definition (Clote, Kranakis 1991)

A group $G \leq S_n$ is weakly representable, if G is (k, ∞)-representable for some $k < n$.

Corollary

All subgroups of $G \leq S_n$ except for A_n are weakly representable.

Proof.

$$G \leq S_n \text{ is weakly representable} \iff \exists k < n : \overline{G}^{(k)} = G$$

The case $k = n - 1$

Theorem

If $k = n - 1 \geq 2$, then all subgroups of S_n except A_n are Galois closed over **k**.

Definition (Clote, Kranakis 1991)

A group $G \leq S_n$ is weakly representable, if G is (k, ∞)-representable for some $k < n$.

Corollary

All subgroups of $G \leq S_n$ except for A_n are weakly representable.

Proof.

$$G \leq S_n \text{ is weakly representable} \iff \exists k < n: \ \overline{G}^{(k)} = G$$
$$\iff \overline{G}^{(n-1)} = G$$

The case $k = n - 1$

Theorem

If $k = n - 1 \geq 2$, then all subgroups of S_n except A_n are Galois closed over $\mathbf{k}$*.*

Definition (Clote, Kranakis 1991)

A group $G \leq S_n$ is weakly representable, if G is (k, ∞)-representable for some $k < n$.

Corollary

All subgroups of $G \leq S_n$ except for A_n are weakly representable.

Proof.

$$
\begin{aligned}
G \leq S_n \text{ is weakly representable} \quad &\iff \quad \exists k < n : \ \overline{G}^{(k)} = G \\
&\iff \quad \overline{G}^{(n-1)} = G \\
&\iff \quad G \neq A_n
\end{aligned}
$$

□

The case $k = n - 2$

Theorem

If $k = n - 2 \geq 2$, then the Galois closures of subgroups of S_n are:

- $\overline{A_n}^{(k)} = S_n$;

The case $k = n - 2$

Theorem

If $k = n - 2 \geq 2$, then the Galois closures of subgroups of S_n are:

- $\overline{A_n}^{(k)} = S_n$;
- $\overline{A_{n-1}}^{(k)} = S_{n-1}$;

The case $k = n - 2$

Theorem

If $k = n - 2 \geq 2$, then the Galois closures of subgroups of S_n are:

- $\overline{A_n}^{(k)} = S_n$;
- $\overline{A_{n-1}}^{(k)} = S_{n-1}$;
- $\overline{C_4}^{(k)} = D_4$ (for $n = 4$);

The case $k = n - 2$

Theorem

If $k = n - 2 \geq 2$, then the Galois closures of subgroups of S_n are:

- $\overline{A_n}^{(k)} = S_n$;
- $\overline{A_{n-1}}^{(k)} = S_{n-1}$;
- $\overline{C_4}^{(k)} = D_4$ *(for $n = 4$)*;
- *all other subgroups of S_n are closed.*

The case $k = n - d$

Theorem

Let $n > \max\left(2^d, d^2 + d\right)$ and $G \leq S_n$. Then G is not Galois closed over $\mathbf{k}$ if and only if

1. $G \leq_{\mathrm{sd}} A_L \times \Delta$ *or*
2. $G <_{\mathrm{sd}} S_L \times \Delta$,

where $\mathbf{n} = L \dot\cup D$ with $|L| > d,\ |D| < d$ and $\Delta \leq S_D$.
The closure of these groups is $\overline{G}^{(k)} = S_L \times \Delta$.

The case $k = n - d$

Theorem

Let $n > \max\left(2^d, d^2 + d\right)$ and $G \leq S_n$. Then G is not Galois closed over $\mathbf{k}$ *if and only if*

1. $G \leq_{\mathrm{sd}} A_L \times \Delta$ *or*
2. $G <_{\mathrm{sd}} S_L \times \Delta$,

where $\mathbf{n} = L \dot\cup D$ *with* $|L| > d$, $|D| < d$ *and* $\Delta \leq S_D$.

The closure of these groups is $\overline{G}^{(k)} = S_L \times \Delta$.

Remark

Using the simplicity of alternating groups, one can show that these subdirect products are of the following form:

1. $G = A_L \times \Delta$;

The case $k = n - d$

Theorem

Let $n > \max\left(2^d, d^2 + d\right)$ and $G \leq S_n$. Then G is not Galois closed over $\mathbf{k}$ if and only if

1. $G \leq_{\mathrm{sd}} A_L \times \Delta$ *or*
2. $G <_{\mathrm{sd}} S_L \times \Delta$,

where $\mathbf{n} = L \dot{\cup} D$ with $|L| > d$, $|D| < d$ and $\Delta \leq S_D$.
The closure of these groups is $\overline{G}^{(k)} = S_L \times \Delta$.

Remark

Using the simplicity of alternating groups, one can show that these subdirect products are of the following form:

1. $G = A_L \times \Delta$;
2. $G = (A_L \times \Delta_0) \cup \left((S_L \setminus A_L) \times (\Delta \setminus \Delta_0)\right)$,
   where $\Delta_0 \leq \Delta$ is a subgroup of index 2.

Interesting subgroups of S_4, S_5 and S_6

Interesting subgroups of S_4, S_5 and S_6

$G \leq S_n$	$\overline{G}^{(2)}$	$\overline{G}^{(3)}$	$\overline{G}^{(4)}$
C_4	D_4	C_4	C_4

Interesting subgroups of S_4, S_5 and S_6

$G \leq S_n$	$\overline{G}^{(2)}$	$\overline{G}^{(3)}$	$\overline{G}^{(4)}$
C_4	D_4	C_4	C_4
C_5	D_5	C_5	C_5
$\mathrm{AGL}(1,5)$	S_5	$\mathrm{AGL}(1,5)$	$\mathrm{AGL}(1,5)$

Interesting subgroups of S_4, S_5 and S_6

$G \leq S_n$	$\overline{G}^{(2)}$	$\overline{G}^{(3)}$	$\overline{G}^{(4)}$
C_4	D_4	C_4	C_4
C_5	D_5	C_5	C_5
$\mathrm{AGL}(1,5)$	S_5	$\mathrm{AGL}(1,5)$	$\mathrm{AGL}(1,5)$
$C_4 \times S_2$	$D_4 \times S_2$	$C_4 \times S_2$	$C_4 \times S_2$
$D_4 \times_{\mathrm{sd}} S_2$	$D_4 \times S_2$	$D_4 \times_{\mathrm{sd}} S_2$	$D_4 \times_{\mathrm{sd}} S_2$
$A_3 \wr A_2$	$S_3 \wr S_2$	$A_3 \wr A_2$	$A_3 \wr A_2$
$S_3 \wr_{\mathrm{sd}} S_2$	$S_3 \wr S_2$	$S_3 \wr_{\mathrm{sd}} S_2$	$S_3 \wr_{\mathrm{sd}} S_2$
$(S_3 \wr S_2) \cap A_6$	$S_3 \wr S_2$	$S_3 \wr S_2$	$(S_3 \wr S_2) \cap A_6$
$\mathrm{PGL}(2,5)$	S_6	$\mathrm{PGL}(2,5)$	$\mathrm{PGL}(2,5)$
Rot ()	Sym ()	Rot ()	Rot ()

References

- P.J. Cameron, P.M. Neumann, J. Saxl, *On groups with no regular orbits on the set of subsets,* Arch. Math. **43** (1984), 295–296.
- P. Clote, E. Kranakis, *Boolean functions, invariance groups, and parallel complexity,* SIAM J. Comput. **20** (1991), 553–590.
- F. Dalla Volta, J. Siemons, *Orbit equivalence and permutation groups defined by unordered relations,* J. Algebr. Comb. **35** (2012), 547–564.
- E. K. Horváth, G. Makay, R. Pöschel, T. Waldhauser, *Invariance groups of finite functions and orbit equivalence of permutation groups,* arXiv:1210.1015.
- N.F.J. Inglis, *On orbit equivalent permutation groups,* Arch. Math. **43** (1984), 297–300.
- A. Kisielewicz, *Symmetry groups of Boolean functions and constructions of permutation groups,* J. Algebra **199** (1998), 379–403.
- Á. Seress, *Primitive groups with no regular orbits on the set of subsets,* Bull. Lond. Math. Soc., **29** (1997), 697–704.
- J. Siemons, A. Wagner, *On finite permutation groups with the same orbits on unordered sets,* Arch. Math. **45** (1985), 492–500.