Relation between pentagonal and GS-quasigroups

Stipe Vidak

Faculty of Science
Department of Mathematics
University of Zagreb
Croatia

June 8, 2013
Contents

1. Definitions and basic examples
2. Geometry
3. Relation between pentagonal and GS-quasigroups
4. Future work
Definition

A quasigroup \((Q, \cdot)\) is a grupoid in which for every \(a, b \in Q\) there exist unique \(x, y \in Q\) such that \(a \cdot x = b\) and \(y \cdot a = b\).

To make some expressions shorter and more readable we use abbreviations. For example, instead of writing \(a \cdot ((b \cdot c) \cdot d)\) we write \(a(bc \cdot d)\).
Definitions and basic examples

Geometry

Relation between pentagonal and GS-quasigroups

Future work

**Definition**

A **quasigroup** \((Q, \cdot)\) is a grupoid in which for every \(a, b \in Q\) there exist unique \(x, y \in Q\) such that \(a \cdot x = b\) and \(y \cdot a = b\).

To make some expressions shorter and more readable we use abbreviations. For example, instead of writing \(a \cdot ((b \cdot c) \cdot d)\) we write \(a(bc \cdot d)\).

**Definition**

An **IM-quasigroup** is a quasigroup \((Q, \cdot)\) in which following properties hold:

- \(a \cdot a = a\) \(\forall a \in Q\) ➔ idempotency
- \(ab \cdot cd = ac \cdot bd\) \(\forall a, b, c, d \in Q\) ➔ mediality
Along with idempotency and mediality, in IM-quasigroups next three properties are valid:

- \( ab \cdot a = a \cdot ba \quad \forall a, b \in Q \) \hspace{1cm} \text{elasticity}
- \( ab \cdot c = ac \cdot bc \quad \forall a, b, c \in Q \) \hspace{1cm} \text{right distributivity}
- \( a \cdot bc = ab \cdot ac \quad \forall a, b, c \in Q \) \hspace{1cm} \text{left distributivity}
Along with idempotency and mediality, in IM-quasigroups next three properties are valid:

- \( ab \cdot a = a \cdot ba \quad \forall a, b \in Q \) \hspace{1cm} \text{elasticity}
- \( ab \cdot c = ac \cdot bc \quad \forall a, b, c \in Q \) \hspace{1cm} \text{right distributivity}
- \( a \cdot bc = ab \cdot ac \quad \forall a, b, c \in Q \) \hspace{1cm} \text{left distributivity}

**Example**

\[ C(q) = (\mathbb{C}, \ast), \text{ where } \ast \text{ is defined with} \]

\[ a \ast b = (1 - q)a + qb, \]

and \( q \in \mathbb{C}, q \neq 0, 1. \)
A **GS-quasigroup** is a quasigroup \((Q, \cdot)\) in which following properties hold:

- \(a \cdot a = a\) \(\forall a \in Q\) \hspace{2cm} \text{idempotency}
- \(a(ab \cdot c) \cdot c = b\) \(\forall a, b, c \in Q\)

- every GS-quasigroup is an IM-quasigroup
**Definition**

A **GS-quasigroup** is a quasigroup \((Q, \cdot)\) in which following properties hold:

- \(a \cdot a = a\) \(\forall a \in Q\) \hspace{1cm} \text{idempotency}
- \(a(ab \cdot c) \cdot c = b\) \(\forall a, b, c \in Q\)

- every GS-quasigroup is an IM-quasigroup

**Example**

\(C(q) = (\mathbb{C}, \ast)\), where \(\ast\) is defined with

\[ a \ast b = (1 - q)a + qb, \]

and \(q\) is a solution of the equation \(q^2 - q - 1 = 0\).
Solutions of the equation $q^2 - q - 1 = 0$ are

$$q_1 = \frac{1 + \sqrt{5}}{2} \text{ and } q_2 = \frac{1 - \sqrt{5}}{2}.$$
Solutions of the equation \( q^2 - q - 1 = 0 \) are

\[
q_1 = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad q_2 = \frac{1 - \sqrt{5}}{2}.
\]

If we regard the complex numbers as the points of the Euclidean plane and if we rewrite \( a \ast b = (1 - q)a + qb \) as

\[
\frac{a \ast b - a}{b - a} = q,
\]

we notice that the point \( a \ast b \) divides the pair \( a, b \) in the ratio \( q \), i.e. golden-section ratio.
A **pentagonal quasigroup** is an IM-quasigroup \((Q, \cdot)\) in which following property holds

\[(ab \cdot a)b \cdot a = b \quad \forall a, b \in Q\]
Definition

A pentagonal quasigroup \textit{is an IM-quasigroup} \((Q, \cdot)\) \textit{in which following property holds}

\[
(ab \cdot a)b \cdot a = b \quad \forall a, b \in Q
\]

\textbf{pentagonality}

All calculations in pentagonal quasigroups are done using properties of idempotency, mediality, elasticity, left and right distributivity and following properties (which all arise from pentagonality):
Definitions and basic examples

Geometry

Relation between pentagonal and GS-quasigroups

Future work

Theorem

In an IM-quasigroup \((Q, \cdot)\) identities (1), (2), (3) and (4) are all mutually equivalent and they imply identity (5).
Example

$C(q) = (\mathbb{C}, \ast)$, where $\ast$ is defined with

$$a \ast b = (1 - q)a + qb,$$

and $q$ is a solution of the equation $q^4 - 3q^3 + 4q^2 - 2q + 1 = 0$.

This equation arises from the property of pentagonality.
Solutions of the equation $q^4 - 3q^3 + 4q^2 - 2q + 1 = 0$ are:

$q_{1,2} = \frac{1}{4}(3 + \sqrt{5} \pm i\sqrt{10 + 2\sqrt{5}}) \approx 1.31 \pm 0.95i$

$q_{3,4} = \frac{1}{4}(3 - \sqrt{5} \pm i\sqrt{10 - 2\sqrt{5}}) \approx 0.19 \pm 0.59i$
Solutions of the equation \( q^4 - 3q^3 + 4q^2 - 2q + 1 = 0 \) are:

\[
q_{1,2} = \frac{1}{4}(3 + \sqrt{5} \pm i\sqrt{10 + 2\sqrt{5}}) \approx 1.31 \pm 0.95i
\]

\[
q_{3,4} = \frac{1}{4}(3 - \sqrt{5} \pm i\sqrt{10 - 2\sqrt{5}}) \approx 0.19 \pm 0.59i
\]

If we regard the complex numbers as the points of the Euclidean plane and if we rewrite \( a \ast b = (1 - q)a + qb \) as

\[
\frac{a \ast b - a}{b - a} = \frac{q - 0}{1 - 0},
\]

we notice that points \( a, b \) and \( a \ast b \) are the vertices of a triangle directly similar to the triangle with the vertices 0, 1 and \( q \).
Solutions of the equation $q^4 - 3q^3 + 4q^2 - 2q + 1 = 0$ are:

$$q_{1,2} = \frac{1}{4}(3 + \sqrt{5} \pm i\sqrt{10 + 2\sqrt{5}}) \approx 1.31 \pm 0.95i$$

$$q_{3,4} = \frac{1}{4}(3 - \sqrt{5} \pm i\sqrt{10 - 2\sqrt{5}}) \approx 0.19 \pm 0.59i$$

If we regard the complex numbers as the points of the Euclidean plane and if we rewrite $a \ast b = (1 - q)a + qb$ as

$$\frac{a \ast b - a}{b - a} = \frac{q - 0}{1 - 0},$$

we notice that points $a$, $b$ and $a \ast b$ are the vertices of a triangle directly similar to the triangle with the vertices 0, 1 and $q$. We get a characteristic triangle for each of $q_i$, $i = 1, 2, 3, 4$. 

Stipe Vidak

Relation between pentagonal and GS-quasigroups
Definitions and basic examples

Relation between pentagonal and GS-quasigroups

Future work

Stipe Vidak

Relation between pentagonal and GS-quasigroups
A more general example of GS / pentagonal quasigroups is \((Q, \ast)\),

\[ a \ast b = a + \varphi(b - a), \]

where \((Q, +)\) is an abelian group and \(\varphi\) is its automorphism which satisfies \(\varphi^2 - \varphi - 1 = 0 / \varphi^4 - 3\varphi^3 + 4\varphi^2 - 2\varphi + 1 = 0\).
A more general example of GS / pentagonal quasigroups is \((Q, \cdot)\),

\[ a \cdot b = a + \varphi(b - a), \]

where \((Q, +)\) is an abelian group and \(\varphi\) is its automorphism which satisfies \(\varphi^2 - \varphi - 1 = 0 / \varphi^4 - 3\varphi^3 + 4\varphi^2 - 2\varphi + 1 = 0\).

It can be shown that these are in fact the most general examples of GS / pentagonal quasigroups. We get Toyoda-like representation theorems for them.
**Theorem**

*GS-quasigroup on the set $Q$ exists if and only if exists an abelian group on the set $Q$ with an automorphism $\varphi$ which satisfies*

$$\varphi^2 - \varphi - 1 = 0.$$  

**Theorem**

*Pentagonal quasigroup on the set $Q$ exists if and only if exists an abelian group on the set $Q$ with an automorphism $\varphi$ which satisfies*

$$\varphi^4 - 3\varphi^3 + 4\varphi^2 - 2\varphi + 1 = 0.$$
Let’s first introduce some basic geometric concepts.

**Definition**

A **point** in the quasigroup \((Q, \cdot)\) is an element of the set \(Q\).

A **segment** in the quasigroup \((Q, \cdot)\) is a pair of points \(\{a, b\}\).

A **\(n\)-gon** in the quasigroup \((Q, \cdot)\) is an ordered \(n\)-tuple of points \((a_1, a_2, \ldots, a_n)\) up to a cyclic permutation.
Geometry of pentagonal quasigroups

- parallelogram, midpoint of the segment, center of the parallelogram
Geometry of pentagonal quasigroups

- parallelogram, midpoint of the segment, center of the parallelogram
- midpoint doesn’t have to be unique: quasigroup \( Q_{16} \) with 16 elements
Geometry of pentagonal quasigroups

- parallelogram, midpoint of the segment, center of the parallelogram
- midpoint doesn’t have to be unique: quasigroup $Q_{16}$ with 16 elements
- regular pentagon, center of the regular pentagon
Definitions and basic examples

Geometry

Relation between pentagonal and GS-quasigroups

Future work

Geometry of pentagonal quasigroups

- parallelogram, midpoint of the segment, center of the parallelogram
- midpoint doesn’t have to be unique: quasigroup $Q_{16}$ with 16 elements
- regular pentagon, center of the regular pentagon

Definition

Let $a$, $b$, $c$, $d$ and $e$ be points of a pentagonal quasigroup $(Q, \cdot)$. Pentagon $(a, b, c, d, e)$ is called regular pentagon if $ab = c$, $bc = d$ and $cd = e$. This is denoted by $RP(a, b, c, d, e)$. 

Stipe Vidak  

Relation between pentagonal and GS-quasigroups
A regular pentagon \((a, b, c, d, e)\) is uniquely determined by the ordered pair of points \((a, b)\).
Definition

Let $a$, $b$, $c$, $d$ and $e$ be points in a pentagonal quasigroup $(Q, \cdot)$ such that $RP(a, b, c, d, e)$. The center of the regular pentagon $(a, b, c, d, e)$ is the point $o$ such that $o = oa \cdot b$. 

[Diagram of a regular pentagon with labels for the points $a$, $b$, $o$, $oa$, and $o$]
If we rewrite \( o = oa \cdot b \) using theorem of characterization, we get

\[
(2 \cdot 1 - \varphi)(o) = (1 - \varphi)(a) + b.
\]
If we rewrite \( o = oa \cdot b \) using theorem of characterization, we get
\[
(2 \cdot 1 - \varphi)(o) = (1 - \varphi)(a) + b.
\]

**Example**

\((Q_5, \cdot), \ RP(0, 1, 2, 3, 4)\)

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

\(00 \cdot 1 = 2, \ 10 \cdot 1 = 3, \ 20 \cdot 1 = 4, \ 30 \cdot 1 = 0, \ 40 \cdot 1 = 1\)

*There is no o such that o = oa \cdot b.*

*Quasigroup \((Q_5, \cdot)\) is generated by the automorphism \(\varphi(x) = 2x\).*
Geometry of GS-quasigroups

- geometry of GS-quasigroups is much more developed
Geometry of GS-quasigroups

- geometry of GS-quasigroups is much more developed
- parallelogram, midpoint of the segment, center of the parallelogram
Geometry of GS-quasigroups

- geometry of GS-quasigroups is much more developed
- parallelogram, midpoint of the segment, center of the parallelogram
- golden section ratio
Geometry of GS-quasigroups

- geometry of GS-quasigroups is much more developed
- parallelogram, midpoint of the segment, center of the parallelogram
- golden section ratio
- GS-trapezoids, affine regular pentagons
Geometry of GS-quasigroups

- geometry of GS-quasigroups is much more developed
- parallelogram, midpoint of the segment, center of the parallelogram
- golden section ratio
- GS-trapezoids, affine regular pentagons
- DGS-trapezoids, GS-deltoids, affine regular dodecachedron, affine regular icosahedron...
Theorem

Let \((Q, \cdot)\) be a pentagonal quasigroup and let \(*: Q \times Q \rightarrow Q\) be a binary operation defined with

\[ a * b = (ba \cdot a)a \cdot b. \]

Then \((Q, *)\) is GS-quasigroup.
Previous theorem tells that pentagonal quasigroup ”inherits” entire geometry of GS-quasigroups.
Previous theorem tells that pentagonal quasigroup ”inherits” entire geometry of GS-quasigroups. **GS-trapezoid** \((a, b, c, d)\) is defined in GS-quasigroup and it is completely determined with its three vertices \(a, b\) and \(c\). Previous theorem enables definition of GS-trapezoid in any pentagonal quasigroup.

**Definition**

Let \((Q, \cdot)\) be a pentagonal quasigroup and \(a, b, c, d \in Q\). We say that quadrangle \((a, b, c, d)\) is **GS-trapezoid**, denoted by \(GST(a, b, c, d)\), if \(d = (ca \cdot b)a \cdot c\).
Concept of **affine regular pentagon** \((a, b, c, d, e)\) is defined in GS-quasigroup if \((a, b, c, d)\) and \((b, c, d, e)\) are GS-trapezoids. It is completely determined with its three vertices \(a, b\) and \(c\). Previous theorem enables definition of affine regular pentagon in any pentagonal quasigroup.

**Definition**

Let \((Q, \cdot)\) be a pentagonal quasigroup and \(a, b, c, d, e \in Q\). We say that pentagon \((a, b, c, d, e)\) is **affine regular pentagon**, denoted by \(\text{ARP}(a, b, c, d, e)\), if \(d = (ca \cdot b)a \cdot c\) and \(e = (ac \cdot b)c \cdot a\).
Definitions and basic examples

Geometry

Relation between pentagonal and GS-quasigroups

Future work

Stipe Vidak
Barlotti’s theorem in pentagonal quasigroups:

**Theorem**

Let \((Q, \cdot)\) be a pentagonal quasigroup and \(ARP(a, b, c, d, e)\), \(RP(b, a, a_1, a_2, a_3)\) with center \(o_a\), \(RP(c, b, b_1, b_2, b_3)\) with center \(o_b\), \(RP(d, c, c_1, c_2, c_3)\), \(RP(e, d, d_1, d_2, d_3)\) and \(RP(a, e, e_1, e_2, e_3)\). If \(RP(o_a, o_b, o_c, o_d, o_e)\), then \(o_c\), \(o_d\) and \(o_e\) are centers of regular pentagons \((d, c, c_1, c_2, c_3)\), \((e, d, d_1, d_2, d_3)\) and \((a, e, e_1, e_2, e_3)\), respectively.
Contents

1 Definitions and basic examples
2 Geometry
3 Relation between pentagonal and GS-quasigroups
4 Future work
• develop more geometry of pentagonal quasigroups
• develop more geometry of pentagonal quasigroups
• determine the set of possible orders of finite pentagonal quasigroups
- develop more geometry of pentagonal quasigroups
- determine the set of possible orders of finite pentagonal quasigroups
- study similarities with some known subclasses of IM-quasigroups (quadratical, hexagonal, Napoleon’s...) and make some generalizations
• develop more geometry of pentagonal quasigroups
• determine the set of possible orders of finite pentagonal quasigroups
• study similarities with some known subclasses of IM-quasigroups (quadratical, hexagonal, Napoleon’s...) and make some generalizations
• plane tilings in pentagonal quasigroups
Definitions and basic examples
Geometry
Relation between pentagonal and GS-quasigroups
Future work

Relation between pentagonal and GS-quasigroups