Direction cones for the representation of tomonoids

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Background
(Lotfi Zadeh)

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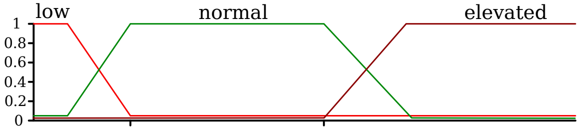

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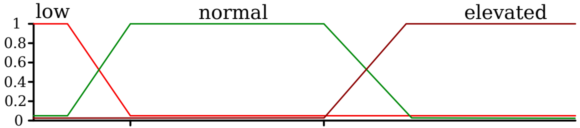

Fuzzy sets to model the result of a blood test

The collection of vague propositions
gives rise (is supposed to give rise)
to a residuated ℓ-monoid $(L; \wedge, \vee, \odot, \rightarrow, 0, 1)$
(Petr Hájek).

Algebras for fuzzy logic

We frequently deal with certain residuated ℓ-monoids called MTL-algebras (Ll. Godo, F. Esteva):

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(Ll. Godo, F. Esteva):

$$(L; \wedge, \vee, \odot, \rightarrow, 0, 1)$$

strength of propositions

(strong) conjunction

implication

truth, falsity

The finite case

Theorem (A. Ciabattoni, G. Metcalfe, F. Montagna)

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One of the big issues of many-valued logics:

How can totally ordered finite MTL-algebras be described?

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Definition

$(L; \leqslant, +, 0)$ is a totally ordered monoid, or tomonoid, if:

(T1) $(L; +, 0)$ is a monoid;

(T2) $\leqslant$ is a translation-invariant total order: $a \leqslant b$ implies $a + c \leqslant b + c$ and $c + a \leqslant c + b$.

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A tomonoid is called

commutative if $+$ is commutative;

positive if 0 is the bottom element.

finitely generated if L is so as a monoid.

Congruences and orders on free monoids

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However: Not all tomonoids are formally integral.

Preorders

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Question:
Can we describe $\preccurlyeq$ by means of something like a positive cone?

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Moreover, the direction cone of $\preccurlyeq$ is

$$C_{\preccurlyeq} \; = \; \{z \in \mathbb{Z}^n \colon a \preccurlyeq b \text{ for any } a, b \in \mathbb{N}^n \text{ such that } z = b - a\}.$$

Direction cones

Theorem

$C \subseteq \mathbb{Z}^n$ is the direction cone of a monomial preorder iff:

(C1) Let $z \in \mathbb{N}^n$. Then $z \in C$ and, if $z \neq 0$, $-z \notin C$.

(C2) Let $(x_1, \ldots, x_k)$, $k \geqslant 2$, be an addable k-tuple of elements of C. Then $x_1 + \ldots + x_k \in C$.

(C3) For each $z \in \mathbb{Z}^n$, either $z \in C$ or $-z \in C$.

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$(x_1, \ldots, x_k)$ is *addable* if for $i = 1, \ldots, k$

$$x_i + \ldots + x_k \trianglelefteq (x_1 + \ldots + x_k) \vee 0.$$

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Let $C \subseteq \mathbb{Z}^n$ be a direction cone. Then the monomial preorder induced by C is the smallest preorder $\preccurlyeq_C$ such that

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$\preccurlyeq_{C_{\preccurlyeq}}$ is contained in $\preccurlyeq$, hence:

Theorem

Any c.p.f. tomonoid is the quotient of a cone tomonoid.

Example

Let L be a tomonoid generated by a and b:

$$\begin{aligned} 0 \ < \ a \ < \ b \ < \ 2a \ < \ a+b \ < \ 2b \ < \ 3a \\ < \ 2a+b \ < \ a+2b \ = \ 4a \ < \ 1. \end{aligned}$$

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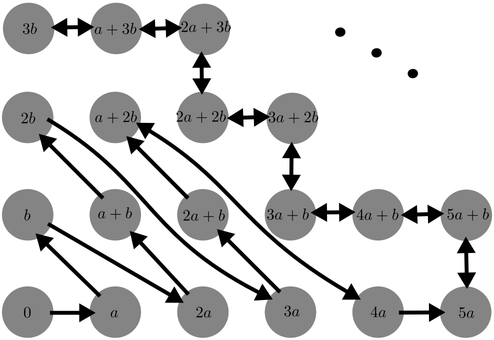
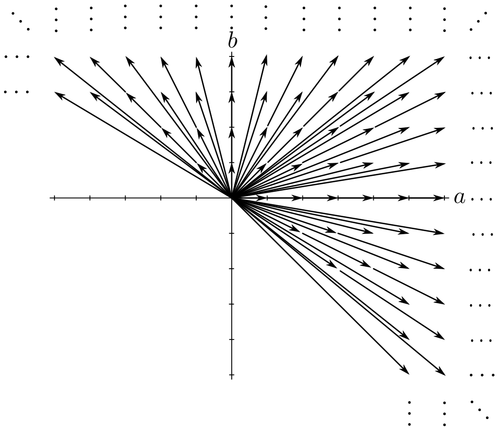


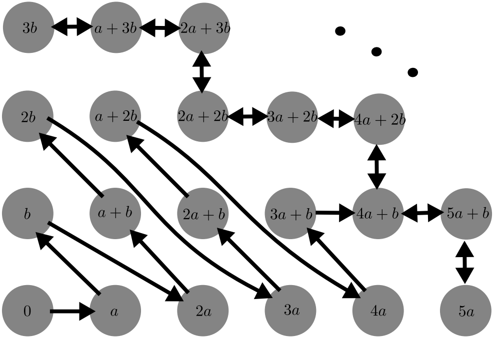
The monomial preorder $\preccurlyeq$ representing L.

Example, ctd.



The direction cone of $\preccurlyeq$.

Example, ctd.



The cone tomonoid whose quotient is L.

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- A cone tomonoid is specified by a direction cone, which is a subset of a $\mathbb{Z}^n$ subject to conditions similar to the case of positive group cones.

The finite case

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Let $\approx$ be the congruence on $\mathbb{N}^n$ inducing the finite tomonoid L.

Then we choose S ("support"), a finite subset of $\mathbb{N}^n$ having a non-empty intersection with each $\approx$-class.

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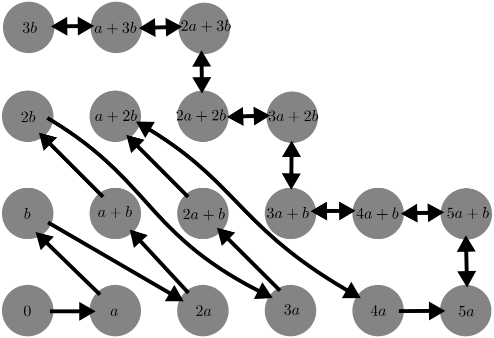
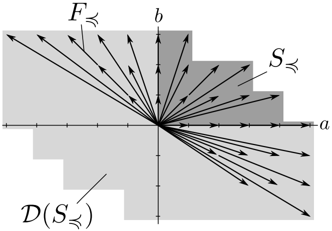
We include into the direction cone only differences of elements of S.

Example, again



The support of $\preccurlyeq$.

Example, ctd.



The direction f-cone of $\preccurlyeq$.

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- An f-cone tomonoid is specified by the pair (S, C), where
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- An f-cone tomonoid is specified by the pair (S, C), where S, the support, is a finite $\trianglelefteq$-ideal of $\mathbb{N}^n$; C, the f-cone, is a subset of the set of differences of elements of S.
- The pairs (S, C) subject to certain conditions lead to an f-cone tomonoid, and all f-cone tomonoids arise in this way.