Automorphic equivalence of many-sorted algebras.

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1. Motivation. Universal algebraic geometry.

Θ - variety of algebras. Category Θ^0: $|X_0| = \aleph_0$,
$\mathrm{Ob}\Theta^0 = \{F(X) \in \Theta \mid X \subset X_0, |X| < \aleph_0\}$,
mophisms - homomorphisms.

Equations: $T \subset (F(X))^2$, $F(X) \in \mathrm{Ob}\Theta^0$.

$H \in \Theta$. Solutions in H:
$T'_H = \{\varphi \in \mathrm{Hom}(F(X), H) \mid T \subseteq \ker\varphi\}$.

Algebraic closure of T in H: $T''_H = \bigcap_{\varphi \in T'_H} \ker\varphi$.

$Cl_H(F(X)) = \left\{T \subset (F(X))^2 \mid T''_H = T\right\}$.

Definition: $H_1, H_2 \in \Theta$ are geometrically equivalent if $\forall F \in \mathrm{Ob}\Theta^0$ holds $Cl_{H_1}(F) = Cl_{H_2}(F)$.

B. Plotkin, *Algebras with the same (algebraic) geometry*, 2003.

Definition: $H_1, H_2 \in \Theta$ are automorphically equivalent if $\exists \Phi \in \mathrm{Aut}\Theta^0$, $\forall F \in \mathrm{Ob}\Theta^0$ exists bijection $\alpha(\Phi)_F : Cl_{H_1}(F) \to Cl_{H_2}(\Phi(F))$ such that if $F_1, F_2 \in \mathrm{Ob}\Theta^0$, $\mu_1, \mu_2 \in \mathrm{Hom}(F_1, F_2)$, $T \in Cl_{H_1}(F_2)$ and $\tau\mu_1 = \tau\mu_2$,

then $\tilde{\tau}\Phi(\mu_1) = \tilde{\tau}\Phi(\mu_2)$,

where $\tau : F_2 \to F_2/T$,

$\tilde{\tau} : \Phi(F_2) \to \Phi(F_2)/\alpha(\Phi)_{F_2}(T)$ - natural epimorphisms.

If this condition holds then $\alpha(\Phi)_F$ are uniquely defined by Φ.

Both these relations are really equivalences.

Definition: An automorphism Υ of a category $\mathfrak{K}$ is inner if $\forall A \in \mathrm{Ob}\mathfrak{K}\ \exists$ an isomorphism $s_A^{\Upsilon} : A \to \Upsilon(A)$ such that $\forall \alpha \in \mathrm{Mor}_{\mathfrak{K}}(A,B)$

$$\begin{array}{ccc} A & \overrightarrow{s_A^{\Upsilon}} & \Upsilon(A) \\ \downarrow \alpha & & \Upsilon(\alpha) \downarrow \\ B & \underrightarrow{s_B^{\Upsilon}} & \Upsilon(B) \end{array}$$

is a commutative diagram.

Proposition: If an inner automorphism Υ of Θ^0 provide an automorphic equivalence of $H_1, H_2 \in \Theta$ then H_1 and H_2 are geometrically equivalent.

Quotient group $\mathfrak{A}/\mathfrak{Y}$ measures the difference from the automorphic equivalence to the geometric equivalence. $\mathfrak{A}$ is a group of the all automorphisms of the category Θ^0, $\mathfrak{Y}$ is a group of the all inner automorphisms.

2. Automorphisms of the category Θ^0. Many-sorted case.

We need to consider one-generated free algebras.

Θ - variety of the all representations of group over linear spaces over the field k.

$$(G,V) \in \Theta, F(X,Y) = \left(G(X), \bigoplus_{y \in Y} kG(X)y \right),$$

$F(x) = (G(x), \{0\}), F(y) = (\{1\}, ky)$.

Θ - variety of the all actions of semigroups over sets. $(S,M) \in \Theta$,

$$F(X,Y) = \left(S(X), Y \cup \left(\bigcup_{y \in Y} S(X) \circ y \right) \right).$$

$F(x) = (S(x), \varnothing), F(y) = (\varnothing, \{y\})$. ??!!

Set of names of sorts Γ,
set of operations (signature) Ω.
$\omega \in \Omega$, type of ω: $\tau_\omega = (i_1, \ldots, i_n; j)$,
$i_1, \ldots, i_n, j \in \Gamma$.

Many-sorted algebra is a set A with the "sorting" $\eta_A : A \to \Gamma$. $\eta_A^{-1}(i) = A^{(i)}, i \in \Gamma$.

If $a^{(i_1)} \in A^{(i_1)}, \ldots, a^{(i_n)} \in A^{(i_n)}$ then $\exists \omega(a^{(i_1)}, \ldots, a^{(i_n)}) \in A^{(j)}$.

It is possible $A^{(i)} = \varnothing$.

Homomorphism $\varphi : A \to B$ must be conforms with the "sorting": $\eta_A = \eta_B \varphi$ -
and with the operations: for every $a^{(i_1)} \in A^{(i_1)}, \ldots, a^{(i_n)} \in A^{(i_n)}$ the $\varphi(\omega(a^{(i_1)}, \ldots, a^{(i_n)})) = \omega(\varphi(a^{(i_1)}), \ldots, \varphi(a^{(i_n)}))$ holds.

If $A^{(i)} \neq \varnothing, B^{(i)} = \varnothing$ then $\mathrm{Hom}(A, B) = \varnothing$.

$X_0 = \bigcup_{i\in\Gamma} X_0^{(i)}, X_0^{(i)} = \left\{x_1^{(i)},\ldots,x_n^{(i)},\ldots\right\}$. $\widetilde{F}(X_0)$ - algebra of terms (in the signature Ω) in the alphabet X_0.

$w_1^{(i)}, w_2^{(i)} \in \left(\widetilde{F}(X_0)\right)^{(i)}$. $A \vdash w_1^{(i)} = w_2^{(i)}$ if $\forall \varphi \in \mathrm{Hom}\left(\widetilde{F}(X), A\right)$ the $\varphi\left(w_1^{(i)}\right) = \varphi\left(w_2^{(i)}\right)$ holds, where $\widetilde{F}(X)$ is an algebra of terms in the alphabet X, $X \subset X_0$ - set of symbols which really enter in $w_1^{(i)}$ or $w_2^{(i)}$. If $x_j^{(i)} \in X$ but $A^{(i)} = \varnothing$ then $A \vdash w_1^{(i)} = w_2^{(i)}$ by the principle of the empty set.

$\mathfrak{I} \subset \bigcup_{i\in\Gamma}\left(\left(\widetilde{F}(X_0)\right)^{(i)}\right)^2$, $\Theta(\mathfrak{I}) = \Theta$ - the variety of algebras defined by the identities $\mathfrak{I}$.

The first and the second theorems of homomorphisms, the projective propriety of free algebras fulfill according to this approach to the notions of many-sorted algebras, their homomorphisms and their varieties.

If $A \in \Theta$ and there exists $i \in \Gamma$ such that $A^{(i)} = \varnothing$ then there are free algebras $F(X)$ of the variety Θ, such that $\mathrm{Hom}(F(X), A) = \varnothing$. But for every $A \in \Theta$ there exists free algebra $F(X) \in \Theta$, such that $A \cong F(X)/T$, where T is a congruence.

The Birkhoff theorem about varieties can be proved according to this approach.

We define the notions of the universal algebraic geometry with minimal obvious changes:

equations $T \subset \bigcup_{i \in \Gamma} \left((F(X))^{(i)} \right)^2$,

also algebraic closure of T in H:

$$T''_H = \bigcap_{\varphi \in T'_H} \ker \varphi \subset \bigcup_{i \in \Gamma} \left((F(X))^{(i)} \right)^2.$$

If $\mathrm{Hom}(F(X), H) = \varnothing$ and $\mathrm{Hom}(F(X), H) \supset T'_H = \varnothing$ then $T''_H = \bigcap_{\varphi \in T'_H} \ker \varphi \subset \bigcup_{i \in \Gamma} \left((F(X))^{(i)} \right)^2$ by the principle of the empty set.

$F(X) \in \mathrm{Ob}\Theta^0$ if $X^{(i)} \subset X_0^{(i)}$, $|X^{(i)}| < \aleph_0$, $i \in \Gamma$.

Assumption: $\forall \Phi \in \mathfrak{A}\ F(x) \cong \Phi(F(x)), x \in X_0$.

Theorem 1. $\forall \Phi \in \mathfrak{A}, \forall F \in \mathrm{Ob}\Theta^0$
$\exists s_A^\Phi : A \to \Phi(A)$ - bijection, such that

1) $\eta_A = \eta_{\Phi(A)} s_A^\Phi$

2) $\forall \alpha \in \mathrm{Mor}_{\Theta^0}(A, B)$ the $\Phi(\alpha) = s_B^\Phi \alpha (s_A^\Phi)^{-1}$

holds.

Proof: $a^{(i)} \in A^{(i)} \subset A$. $x^{(i)} \in X_0^{(i)}$,
$\sigma : F(x^{(i)}) \to \Phi(F(x^{(i)}))$ - isomorphism,
$\alpha : F(x^{(i)}) \to A$ - homomorphism such that
$\alpha(x^{(i)}) = a^{(i)}$. $s_A^\Phi(a^{(i)}) = \Phi(\alpha)\sigma(x^{(i)})$.■

Definition: An automorphism Ψ is called strongly stable if

1) $\forall F \in \mathrm{Ob}\Theta^0\ \Psi(F) = F$,

2) $\forall \alpha \in \mathrm{Mor}_{\Theta^0}(A, B)\ \Psi(\alpha) = s_B^\Psi \alpha (s_A^\Psi)^{-1}$,
$s_F^\Psi : F \to F$ -bijections such that

2.1) $\eta_F = \eta_F s_F^\Psi$

2.2) $\forall F(X) \in \mathrm{Ob}\Theta^0\ s_{F(X)|X}^\Psi = id_{|X}$.

Theorem 2. $\mathfrak{A} = \mathfrak{Y}\mathfrak{S}$.

2.1. Method of the verbal operations.

$w(x_1,\ldots,x_n) \in F(x_1,\ldots,x_n) = F \in \mathrm{Ob}\Theta^0$.

$H \in \Theta$, $h_1,\ldots,h_n \in H$ such that

$\eta_H(h_i) = \eta_F(x_i)$,

$w_H^*(h_1,\ldots,h_n) = w(h_1,\ldots,h_n)$.

w_H^* is the verbal operation defined by the word w and has the type $(\eta_F(x_1),\ldots,\eta_F(x_n);\eta_F(w))$.

$\omega \in \Omega$, $\tau_\omega = (i_1,\ldots,i_n;j)$,

$F(x^{(i_1)},\ldots,x^{(i_n)}) = F_\omega \in \mathrm{Ob}\Theta^0$, where

$x^{(i_j)} \in X_0^{(i_j)}$.

$\omega(x^{(i_1)},\ldots,x^{(i_n)}) \in F_\omega$.

$\Psi \in \mathfrak{S}$, $s_{F_\omega}^\Psi : F_\omega \to F_\omega$ - bijection, which fulfills conditions 2).

$s_{F_\omega}^\Psi(\omega(x^{(i_1)},\ldots,x^{(i_n)})) = w_\omega(x^{(i_1)},\ldots,x^{(i_n)}) \in F_\omega$.

H_W^* - set H with operations defined by the system of words $\{w_\omega \mid \omega \in \Omega\} = W$.

$\forall F \in \mathrm{Ob}\Theta^0$ $s_F^\Psi : F \to F_W^*$ is an isomorphism.

There is a bijection $\mathfrak{S} \leftrightarrow$ set of system of words $\{w_\omega \mid \omega \in \Omega\} = W$ such that

W1) $w_\omega \in F_\omega$,

W2) $\forall F = F(X) \in \mathrm{Ob}\Theta^0 \; \exists$ isomorphism $s_F : F \to F_W^*$ such that $s_F \mid_X = id_X$.

$\Phi \in \mathfrak{S} \cap \mathfrak{Y} \Leftrightarrow \exists \{\mu_F : F \to F_W^* \mid F \in \mathrm{Ob}\Theta^0\}$ - system of isomorphisms such that $\forall \alpha \in \mathrm{Mor}_{\Theta^0}(A, B)$ the $\mu_B \alpha = \alpha \mu_A$ holds $(\Phi \leftrightarrow W)$.

Lemma 1. If W fulfills conditions W1), W2), $H \in \Theta$ then $H_W^* \in \Theta$.

Lemma 2. W fulfills conditions W1), W2), $H_1, H_2 \in \Theta$. $\varphi : H_1 \to H_2$ is a homomorphism $\Leftrightarrow$ $\varphi : (H_1)_W^* \to (H_2)_W^*$ is a homomorphism.

3. Application to the universal algebraic geometry of many-sorted algebras.

Proposition 1. $\Phi \in \mathfrak{A}$ provides automorphic equivalence of the algebras $H_1, H_2 \in \Theta$ if and only if $\forall F \in \mathrm{Ob}\Theta^0$

$Cl_{H_1}(F) \ni T \to s_F^{\Phi}(T) \in Cl_{H_2}(\Phi(F))$ is a bijection. These bijections not depend on choice of system $\{s_F^{\Phi} \mid F \in \mathrm{Ob}\Theta^0\}$, only on automorpism Φ.

Proposition 2. If $\Phi \in \mathfrak{Y}$ provides automorphic equivalence of the algebras $H_1, H_2 \in \Theta$, then H_1, H_2 are geometrically equivalent.

Theorem 1. If $H \in \Theta$, $\Phi \in \mathfrak{S}$, $\Phi \leftrightarrow W$ then Φ provides the automorphic equivalence of algebras H_W^* and H.

Proof: By consideration of diagrams:

$$\begin{array}{ccc} F & \xrightarrow{(s_F^{\Phi})^{-1}} & F \\ \downarrow \varphi & & \psi \downarrow \\ H & & H_W^* \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{s_F^{\Phi}} & F \\ \downarrow \psi & & \varphi \downarrow \\ H_W^* & & H \end{array} .$$

$\psi \in \mathrm{Hom}(F, H_W^*) \Rightarrow$
$\psi (s_F^{\Phi})^{-1} \in \mathrm{Hom}(F_W^*, H_W^*) = \mathrm{Hom}(F, H)$,
$\varphi \in \mathrm{Hom}(F, H) = \mathrm{Hom}(F_W^*, H_W^*) \Rightarrow$
$\varphi s_F^{\Phi} \in \mathrm{Hom}(F, H_W^*)$.■

Theorem 2. $H_1, H_2 \in \Theta$. They are automorphically equivalent $\Leftrightarrow$ H_1 geometrically equivalent to the $(H_2)_W^*$, where $W = \{w_\omega \mid \omega \in \Omega\}$ is a system of words which fulfills conditions W1), W2).

4. Examples.
4.1. Variety of the all actions of semigroups on sets.

$|\Gamma| = 2$: 1 - sort of elements of semigroups, 2 - sort of elements of sets.

$\Omega = \{\bullet, \circ\}$. $F(X) = F$. $F^{(1)} = S(X^{(1)}) = S$,

$F^{(2)} = (S(X^{(1)}) \circ X^{(2)}) \cup X^{(2)} = M$.

IBN propriety? $|X^{(1)}| + 1 = |S/S^2|$.

$R_0 = \{(a,b) \in M^2 \mid \exists s \in S(s \circ a = b)\}$.

R - minimal equivalence such that $R_0 \subset R$.

$|M/R| = |X^{(2)}|$.

So $F(X) \cong F(Y) \Leftrightarrow |X^{(1)}| = |Y^{(1)}|, |X^{(2)}| = |Y^{(2)}|$.

$F(x^{(i)}) \cong \Phi(F(x^{(i)})), i \in \Gamma?$

$\forall\Theta$ Φ transforms isomorphisms to the isomorphisms, because isomorphisms defined "algebraically".

$\Phi\left(\coprod_{j\in J} F_i\right) \cong \coprod_{j\in J} \Phi(F_j)$ $(F_j \in \mathrm{Ob}\Theta^0)$, because coproduct defined "algebraically".

$\coprod_{j\in J} F_j(X_j) \cong F(X)$ such that $|X^{(i)}| = \sum_{j\in J} \left|X_j^{(i)}\right|$, $i \in \Gamma$.

So Φ provide automorphism $\varphi : \mathbb{N}^{|\Gamma|} \to \mathbb{N}^{|\Gamma|}$ of the additive monoid:

$\varphi\left((|X^{(i)}|)_{i\in\Gamma}\right) = (|Y^{(i)}|)_{i\in\Gamma}$, where $\Phi(F(X)) = F(Y)$.

φ transforms the minimal set of generators to the minimal set of generators.

In our case it is impossible that $\varphi(1,0)=(0,1)$.
If $\Phi(F(x^{(1)}))\cong F(x^{(2)})$ then
$\Phi\left(F\left(x_1^{(1)},x_2^{(1)}\right)\right)\cong\Phi\left(F\left(x_1^{(1)}\right)\sqcup F\left(x_2^{(1)}\right)\right)\cong$
$F\left(x_1^{(2)},x_2^{(2)}\right)$.
$X=X^{(1)}$, $|X|<\infty$. $F(X)\hookrightarrow F\left(x_1^{(1)},x_2^{(1)}\right)$.
$\forall\Theta^0$ injection also defined "algebraically":
$\forall\alpha\in\mathrm{Mor}_{\Theta^0}(A,B)$ $\ker\alpha=\Delta$ if and only if
$\forall C\in\mathrm{Ob}\Theta^0$, $\forall\beta,\gamma\in\mathrm{Mor}_{\Theta^0}(C,A)$ $\alpha\beta=\alpha\gamma\Rightarrow$
$\beta=\gamma$.
So Φ preserves injections.
$\Phi(F(X))\hookrightarrow\Phi\left(F\left(x_1^{(1)},x_2^{(1)}\right)\right)$
$F(X^{(2)})\hookrightarrow F\left(x_1^{(2)},x_2^{(2)}\right)$, where $|X^{(2)}|=|X|$.$\bowtie$
If $\varphi(1,0)=(1,0)$, $\varphi(0,1)=(0,1)$ then
$\Phi(F(x^{(1)}))\cong F(x^{(1)})$, $\Phi(F(x^{(2)}))\cong F(x^{(2)})$.

Two system of words which fulfill conditions W1), W2):

W_1: $w_\cdot\left(x_1^{(1)},x_2^{(1)}\right) = x_1^{(1)} \cdot x_2^{(1)}$,

$w_\circ(x^{(1)},x^{(2)}) = x^{(1)} \circ x^{(2)}$;

W_2: $w_\cdot\left(x_1^{(1)},x_2^{(1)}\right) = x_2^{(1)} \cdot x_1^{(1)}$,

$w_\circ(x^{(1)},x^{(2)}) = x^{(1)} \circ x^{(2)}$.

Otherwise $s_F : F \to F_W^*$ such that $s_F \mid_X = id_X$ will not be isomorphisms.

$\Phi_2 \leftrightarrow W_2$ is not inner: $F = F\left(x_1^{(1)},x_2^{(1)}\right)$, $\alpha \in \mathrm{End}F$ such that $\alpha\left(x_1^{(1)}\right) = x_1^{(1)}x_2^{(1)}$, $\alpha\left(x_2^{(1)}\right) = x_2^{(1)}$. $\mu : F \to F_{W_2}^*$ isomorphism.

$\alpha\mu \neq \mu\alpha$. $|\mathfrak{A}/\mathfrak{Y}| = 2$.

Example of actions which are automorphically equivalent but not geometrically equivalent.

$H \in \Theta$, H free action with generators $\left\{x_1^{(1)}, x_2^{(1)}, x^{(2)}\right\}$ in the subvariety of Θ defined by identity $x_1^{(1)} x_2^{(1)} \left(x_1^{(1)}\right)^2 \circ x^{(2)} = x^{(2)}$. $H_{W_2}^*$ and H are automorphically equivalent.

$Cl_{H_{W_2}^*}(F) \ni T \to s_F T \in Cl_H(F)$ - monotonic bijection.

$\Delta_H'' = Id\left(H, \left\{x_1^{(1)}, x_2^{(1)}, x^{(2)}\right\}\right) = I$ - minimal H-closed set. If $H_{W_2}^*$ and H are geometrically equivalent, then I is a minimal $H_{W_2}^*$-closed set. So $s_F I = I$ and $s_F\left(x_1^{(1)} x_2^{(1)} \left(x_1^{(1)}\right)^2 \circ x^{(2)}, x^{(2)}\right) = \left(\left(x_1^{(1)}\right)^2 x_2^{(1)} x_1^{(1)} \circ x^{(2)}, x^{(2)}\right) \in I$. But identity $\left(x_1^{(1)}\right)^2 x_2^{(1)} x_1^{(1)} \circ x^{(2)} = x^{(2)}$ not fulfills in H.$\bowtie$

4.2. Variety of the all automatons (ASM's).

$|\Gamma| = 3$: 1 - sort of input signals, 2 - sort of states, 3 - sort of output signals.

$\Omega = \{\circ, *\}, \tau_\circ = (1,2;2), \tau_* = (1,2;3)$.

$\mathfrak{A}/\mathfrak{Y} = \{1\}$. Automorphic equivalence coincide with geometric equivalence.

4.3. Variety of the all representations of the Lie algebras over the field k.

We assume that $char(k) = 0$.

$|\Gamma| = 2$: 1 - sort of elements of the Lie algebras, 2 - sort of elements of the Lie algebras modules.

$\Omega = \{0_L, 0_M, \lambda_L, \lambda_M, +_L, +_M, [,], \circ \mid \lambda \in k\}$,

$\tau_{\lambda_L} = (1;1)$, $\tau_{\lambda_M} = (2;2)$, $\tau_\circ = (1,2;2)$.

$\mathfrak{A}/\mathfrak{Y} \cong \mathrm{Aut}k$, because the system of words which fulfill conditions W1), W2) and corresponds to automorphisms which are not inner are:

$w_{0_L} = 0$, $w_{0_M} = 0$, $w_{\lambda_L}(x^{(1)}) = \varphi(\lambda)x^{(1)}$,

$w_{\lambda_M}(x^{(2)}) = \varphi(\lambda)x^{(2)}$,

$w_{+_L}\left(x_1^{(1)}, x_2^{(1)}\right) = x_1^{(1)} + x_2^{(1)}$,

$w_{+_M}\left(x_1^{(2)}, x_2^{(2)}\right) = x_1^{(2)} + x_2^{(2)}$,

$w_{[,]}\left(x_1^{(1)}, x_2^{(1)}\right) = \left[x_1^{(1)}, x_2^{(1)}\right]$,

$w_\circ(x^{(1)}, x^{(2)}) = x^{(1)} \circ x^{(2)}$, where $\varphi \in \mathrm{Aut}k$.

Example of representations which are automorphically equivalent but not geometrically equivalent.

$k = \mathbb{Q}(\theta_1, \theta_2)$ - transcendent extension of $\mathbb{Q}$ of the degree 2, $H \in \Theta$, H free representation with generators $\left\{x_1^{(1)}, x_2^{(1)}, x^{(2)}\right\}$ in the subvariety of Θ defined by identity

$$(\theta_1\left[x_1^{(1)}, \left[x_1^{(1)}, x_2^{(1)}\right]\right] + \theta_2\left[\left[x_1^{(1)}, x_2^{(1)}\right], x_2^{(1)}\right] \circ x^{(2)}) = 0.$$

W corresponds to φ such that

$\varphi(\theta_1) = \theta_2, \varphi(\theta_2) = \theta_1.$

H_W^* and H are automorphically equivalent but not geometrically equivalent.