Automorphism groups of free Steiner triple systems

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A *Steiner triple system* is an incidence structure consisting of points and blocks such that:

- every two distinct points are contained in precisely one block,
- any block has precisely three points.

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STS $\longleftrightarrow$ SL

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Ganter, Pfüller (1985):

The variety of all diassociative loops of exponent 2 is precisely the variety of all Steiner loops, which are in a one-to-one correspondence with Steiner triple systems.

Multiplication group

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The stabilizer of the unit element is called the *inner mapping group* of L.

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Strambach, S. (2009):

Theorem

If the product of any two distinct translations of the Steiner quasigroup has odd order, then the multiplication group of the Steiner loop of order n is the alternating group A_n or the symmetric group S_n depending whether n is divisible by 4 or not.

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Which groups can occur in the remaining cases?

Multiplication group

Grishkov, Rasskazova, S. (2012):

Theorem

Let $Mult(X)$ be the group of the multiplications of the free Steiner loop $D(X)$. Then

1. $Mult(X) = *_{v \in D(X)^*} C_v$ *is a free product of cyclic groups of order 2;*
2. $Mult(X)$ *acts on* $D(X)$ *and* $Mult(X) = \{R_v | v \in D(X)\} Stab_G(\emptyset)$. *Moreover,* $Stab_G(\emptyset)$ *is a free subgroup generated by* $R_v R_w R_{vw}$, $v, w \in D(X)$.

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Automorphisms

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$Aut(STS) \cong Aut(SL)$

$\varphi : X \longrightarrow D(X) : (x_1, ..., x_n) \mapsto (x_1, ..., x_i \cdot v, ..., x_n)$, with $v \in D(X \setminus x_i)$ is an automorphism of $D(X)$, called *elementary automorphism*.

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Problem 1. Which relations exist between X-elementary automorphisms of the loop $D(X)$?

3-generated

Theorem

Let $D(X)$ be a free Steiner loop with free generators $X = \{x_1, x_2, x_3\}$. Then the group of automorphisms $AutD(X)$ of the loop $D(X)$ is generated by the symmetric group S_3 and by the elementary automorphism $\varphi = e_1(x_2)$.

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$$(ij) = e_i(x_j)e_j(x_i)e_i(x_j).$$

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$$(i-1,i)(i,i+1)(i-1,i) = (i,i+1)(i-1,i)(i,i+1),$$

yields

$$(e_i(x_j)e_j(x_i))^3 = 1$$

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Conjecture

The group $Aut(D(x_1, x_2, x_3))$ *is generated by three involutions* $(12), (13)$ *and* $\varphi = e_1(x_2)$ *with relations*

$$(12)(13)(12) = (13)(12)(13),$$

$$(\varphi(12))^3 = (\varphi(13))^4 = 1.$$

Corollary

Let $D(X)$ be the free Steiner loop with free generators $X = \{a, b, c\}$ and let Q be the stabilizer $Stab_{AutD(X)}(c)$ of c in the automorphism group of $D(X)$. Then

$$Q =< \varphi, \tau, \xi >$$

with $\varphi(a, b, c) = (ab, b, c)$, $\xi(a, b, c) = (ac, b, c)$, $\tau(a, b, c) = (b, a, c)$.

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Theorem

If the Conjecture 1 is true then the Conjecture 2 is also true.

n-generated

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Theorem

The automorphism group $AutD(X)$ *of the free loop* $D(X)$ *is not finite generated if* $|X| > 3$.

ISL

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Let $a \in \mathfrak{S}$ be some fixed element and $IS = (\mathfrak{S}, a, \cdot)$ be a main isotope of the quasigroup associated to $\mathfrak{S}$ with multiplication

$$x \cdot y = y \cdot x = (ax)(ay).$$

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Then $x^2 = x \cdot x = (ax)(ax) = ax$, hence $x^2 \cdot y^2 = xy$, $x^3 = x(ax) = a$, and $(xy)y = (x^2 \cdot y^2)^2 \cdot y^2 = x$.

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Inversely, from a commutative loop S with identities $x^3 = a, (x^2y^2)^2y^2 = x$, can be recovered a Steiner triple system with the blocks:

- $\{x, y, x^2y^2\}$
- $\{a, x, x^2\}$

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A loop obtained in this way is called an *interior Steiner loop*.

Theorem

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Then

$$Aut(S(X)) = Aut(ES(X))$$

and

$$Aut(IS(X)) \simeq Stab_{AutES(X)}(a),$$

where $a \in IS(X)$ is the unit element of the loop $IS(X)$.

Thank you for your attention!