On morphisms of lattice-valued formal contexts

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Outline

Formal Concept Analysis

- *Formal Concept Analysis (FCA)* has taken its origin as an attempt to restructure mathematics, e.g., lattice theory.
- Since then, FCA has been developed as a subfield of applied mathematics, based in mathematization of concept hierarchies.
- The aim of FCA is to support the rational communication of humans by mathematically developing appropriate conceptual structures, which can be logically activated.

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Formal contexts

One of the main building blocks of FCA provide *formal contexts*.

Definition 1

A *formal context* is a triple (G, M, I), which comprises a set of objects G, a set of attributes M, and a binary incidence relation I between G and M, where $g\,I\,m$ means "object g has attribute m".

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Formal context morphisms

There exist at least three (different) ways of defining a morphism between two formal contexts (G_1, M_1, I_1) and (G_2, M_2, I_2).

1. The theory of FCA employs pairs of maps $G_1 \xrightarrow{\alpha} G_2$, $M_1 \xrightarrow{\beta} M_2$ such that $g\, I_1\, m$ iff $\alpha(g)\, I_2\, \beta(m)$ for every $g \in G_1$, $m \in M_1$.
2. The theory of *Chu spaces* uses pairs of maps $G_1 \xrightarrow{\alpha} G_2$, $M_2 \xrightarrow{\beta} M_1$ such that $g\, I_1\, \beta(m)$ iff $\alpha(g)\, I_2\, m$ for every $g \in G_1$, $m \in M_2$.

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Formal context morphisms

3. The theory of *Galois connections* relies on the pairs of maps $\mathcal{P}(G_1) \xrightarrow{\alpha} \mathcal{P}(G_2)$, $\mathcal{P}(M_2) \xrightarrow{\beta} \mathcal{P}(M_1)$, where $\mathcal{P}(X)$ stands for the powerset of X, such that the diagrams

$$\begin{array}{ccc} \mathcal{P}(G_1) & \xrightarrow{\alpha} & \mathcal{P}(G_2) \\ {\scriptstyle H_1}\downarrow & & \downarrow{\scriptstyle H_2} \\ \mathcal{P}(M_1) & \xleftarrow[\beta]{} & \mathcal{P}(M_2) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{P}(M_1) & \xleftarrow{\beta} & \mathcal{P}(M_2) \\ {\scriptstyle K_1}\downarrow & & \downarrow{\scriptstyle K_2} \\ \mathcal{P}(G_1) & \xrightarrow[\alpha]{} & \mathcal{P}(G_2) \end{array}$$

commute, where $H_j(S) = \{m \in M_j \mid s\, I_j\, m \text{ for every } s \in S\}$ and $K_j(T) = \{g \in G_j \mid g\, I_j\, t \text{ for every } t \in T\}$ (*Birkhoff operators*).

Lattice-valued formal contexts

- J. T. Denniston, A. Melton, and S. E. Rodabaugh compared the approaches of items (2) and (3) by considering their respective categories of *lattice-valued formal contexts* (in the sense of R. Bělohlávek) over a fixed commutative quantale Q, and constructing an embedding of each category into its counterparts.
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- This talk compares all three of the above-mentioned approaches to morphisms in the framework of lattice-valued formal contexts over a category of not necessarily commutative quantales.
- We construct a number of embeddings between their respective categories of formal contexts, showing that the approach of item (3) falls out of the FCA setting in the lattice-valued case.

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$\bigvee$-semilattices

Definition 2

CSLat($\bigvee$) is the variety of *$\bigvee$-semilattices*, i.e., partially ordered sets (posets), which have arbitrary joins.

Every $\bigvee$-semilattice homomorphism $A_1 \xrightarrow{\varphi} A_2$ has the *upper adjoint map* $A_2 \xrightarrow{\varphi^{\vdash}} A_1$ given by $\varphi^{\vdash}(a_2) = \bigvee\{a_1 \in A_1 \mid \varphi(a_1) \leqslant a_2\}$.

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Quantales

Definition 3

1. **Quant** is the variety of *quantales*, i.e., triples $(Q, \bigvee, \otimes)$, where
   - $(Q, \bigvee)$ is a $\bigvee$-semilattice;
   - $(Q, \otimes)$ is a semigroup;
   - $\otimes$ distributes across $\bigvee$ from both sides.
2. **UQuant** is the variety of *unital quantales*, i.e., quantales Q, which have an element 1_Q such that $(Q, \otimes, 1_Q)$ is a monoid.

A quantale Q has two residuations, which are given by $q_1 \rightarrow_l q_2 = \bigvee\{q \in Q \mid q \otimes q_1 \leqslant q_2\}$ and $q_1 \rightarrow_r q_2 = \bigvee\{q \in Q \mid q_1 \otimes q \leqslant q_2\}$.

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Crisp forward powerset operator

Definition 4

Given a map $X_1 \xrightarrow{f} X_2$, the *forward powerset operator* w.r.t. f is the map $\mathcal{P}(X_1) \xrightarrow{f^{\rightarrow}} \mathcal{P}(X_2)$, which is defined by $f^{\rightarrow}(S) = \{f(s) \mid s \in S\}$.

Lattice-valued forward powerset operators I

Theorem 5

1. *Given a variety* $\mathbf{L}$*, which extends* $\mathbf{CSLat}(\bigvee)$*, every subcategory* $\mathbf{S}$ *of* $\mathbf{L}$ *provides a functor* $\mathbf{Set} \times \mathbf{S} \xrightarrow{(-)^{\rightarrow}} \mathbf{CSLat}(\bigvee)$*, which is defined by* $((X_1, L_1) \xrightarrow{(f,\varphi)} (X_2, L_2))^{\rightarrow} = L_1^{X_1} \xrightarrow{(f,\varphi)^{\rightarrow}} L_2^{X_2}$*, where* $((f, \varphi)^{\rightarrow}(\alpha))(x_2) = \varphi(\bigvee_{f(x_1)=x_2} \alpha(x_1))$.
2. *Let* $\mathbf{L}$ *be a variety, which extends* $\mathbf{CSLat}(\bigvee)$*, and let* $\mathbf{S}$ *be a subcategory of* $\mathbf{L}^{op}$ *such that for every* $\mathbf{S}$*-morphism* $L_1 \xrightarrow{\varphi} L_2$*, the map* $L_1 \xrightarrow{\varphi^{op\vdash}} L_2$ *is* $\bigvee$*-preserving. Then there exists a functor* $\mathbf{Set} \times \mathbf{S} \xrightarrow{(-)^{\vdash\dashrightarrow}} \mathbf{CSLat}(\bigvee)$ *defined by* $((X_1, L_1) \xrightarrow{(f,\varphi)} (X_2, L_2))^{\vdash\dashrightarrow} = L_1^{X_1} \xrightarrow{(f,\varphi)^{\vdash\dashrightarrow}} L_2^{X_2}$*, where* $((f, \varphi)^{\vdash\dashrightarrow}(\alpha))(x_2) = \varphi^{op\vdash}(\bigvee_{f(x_1)=x_2} \alpha(x_1))$.

Lattice-valued forward powerset operators II

Theorem 6

1. *Given a variety* $\mathbf{L}$*, which extends* $\mathbf{CSLat}(\bigvee)$*, every subcategory* $\mathbf{S}$ *of* $\mathbf{L}^{op}$ *provides a functor* $\mathbf{Set}^{op} \times \mathbf{S} \xrightarrow{(-)^{\rightarrow o}} (\mathbf{CSLat}(\bigvee))^{op}$ *with* $((X_1, L_1) \xrightarrow{(f,\varphi)} (X_2, L_2))^{\rightarrow o} = L_1^{X_1} \xrightarrow{((f,\varphi)^{\rightarrow o})^{op}} L_2^{X_2}$*, where* $((f,\varphi)^{\rightarrow o}(\alpha))(x_1) = \varphi^{op}(\bigvee_{f^{op}(x_2)=x_1} \alpha(x_2))$.
2. *Let* $\mathbf{L}$ *be a variety, which extends* $\mathbf{CSLat}(\bigvee)$*, and let* $\mathbf{S}$ *be a subcategory of* $\mathbf{L}$ *such that for every* $\mathbf{S}$*-morphism* $L_1 \xrightarrow{\varphi} L_2$*, the map* $L_2 \xrightarrow{\varphi^{\vdash}} L_1$ *is* $\bigvee$*-preserving. Then there exists a functor* $\mathbf{Set}^{op} \times \mathbf{S} \xrightarrow{(-)^{\vdash\dashrightarrow o}} (\mathbf{CSLat}(\bigvee))^{op}$ *defined by* $((X_1, L_1) \xrightarrow{(f,\varphi)} (X_2, L_2))^{\vdash\dashrightarrow o} = L_1^{X_1} \xrightarrow{((f,\varphi)^{\vdash\dashrightarrow o})^{op}} L_2^{X_2}$*, where* $((f,\varphi)^{\vdash\dashrightarrow o}(\alpha))(x_1) = \varphi^{\vdash}(\bigvee_{f^{op}(x_2)=x_1} \alpha(x_2))$.

Galois connections

Definition 7

A tuple $((X_1, \leqslant), f, g, (X_2, \leqslant))$ is an *order-reversing Galois connection* provided that $(X_1, \leqslant)$, $(X_2, \leqslant)$ are posets, and $X_1 \underset{g}{\overset{f}{\rightleftarrows}} X_2$ are maps with $x_1 \leqslant g(x_2)$ iff $x_2 \leqslant f(x_1)$ for every $x_1 \in X_1$, $x_2 \in X_2$.

Formal contexts as Chu spaces

Definition 8

Let $\mathbf{L}$ be a variety, which extends **Quant**, and let $\mathbf{S}$ be a subcategory of $\mathbf{L}^{op}$. $\mathbf{S}\text{-}\mathbf{FC}^{C}$ is the category, which comprises the following data.

Objects: tuples $\mathcal{K} = (G, M, L, I)$ (*(lattice-valued) formal contexts*), where G is the set of context *objects*, M is the set of context *attributes*, L is an $\mathbf{S}$-object, and $G \times M \xrightarrow{I} L$ is a map, which is called the context *incidence relation*.

Morphisms: $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ (*(lattice-valued) formal context morphisms*) are triples $(G_1, M_1, L_1) \xrightarrow{f=(\alpha,\beta,\varphi)} (G_2, M_2, L_2)$ in $\mathbf{Set} \times \mathbf{Set}^{op} \times \mathbf{S}$ with $I_1(g, \beta^{op}(m)) = \varphi^{op} \circ I_2(\alpha(g), m)$ for every $g \in G_1$, $m \in M_2$.

Modified formal contexts as Chu spaces

Definition 9

Let **L** be a variety, which extends **Quant**, and let **S** be a subcategory of **L**. $\mathbf{S}\text{-}\mathbf{FC}_m^C$ is the category, which comprises the following data.

Objects: (lattice-valued) formal contexts.

Morphisms: $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ are triples $(G_1, M_1, L_1) \xrightarrow{f=(\alpha,\beta,\varphi)} (G_2, M_2, L_2)$ in $\mathbf{Set} \times \mathbf{Set}^{op} \times \mathbf{S}$ with $\varphi \circ I_1(g, \beta^{op}(m)) = I_2(\alpha(g), m)$ for every $g \in G_1$, $m \in M_2$.

Formal contexts of B. Ganter and R. Wille

Definition 10

Let **L** be a variety, which extends **Quant**, and let **S** be a subcategory of $\mathbf{L}^{op}$. $\mathbf{S}\text{-}\mathbf{FC}^{GW}$ is the category, which comprises the following data.

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Definition 11

Let $\mathbf{L}$ be a variety, which extends **Quant**, and let $\mathbf{S}$ be a subcategory of $\mathbf{L}$. $\mathbf{S}\text{-}\mathbf{FC}_m^{GW}$ is the category, which comprises the following data.

Objects: (lattice-valued) formal contexts.

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Lattice-valued Birkhoff operators

Definition 12

Every lattice-valued formal context $\mathcal{K}$ provides the following *(lattice-valued) Birkhoff operators*:

1. $L^G \xrightarrow{H} L^M$ given by $(H(s))(m) = \bigwedge_{g \in G}(s(g) \rightarrow_l I(g, m))$;
2. $L^M \xrightarrow{K} L^G$ given by $(K(t))(g) = \bigwedge_{m \in M}(t(m) \rightarrow_r I(g, m))$.

Theorem 13

For every lattice-valued context $\mathcal{K}$, (L^G, H, K, L^M) is an order-reversing Galois connection.

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Crisp Birkhoff operators

Example 14

Every crisp context $\mathcal{K}$ provides the maps

1. $\mathcal{P}(G) \xrightarrow{H} \mathcal{P}(M)$, $H(S) = \{m \in M \mid s\, I\, m \text{ for every } s \in S\}$;
2. $\mathcal{P}(M) \xrightarrow{K} \mathcal{P}(G)$, $K(T) = \{g \in G \mid g\, I\, t \text{ for every } t \in T\}$;

which are the classical Birkhoff operators of a binary relation.

Formal contexts of J. T. Denniston *et al.*

Definition 15

Given a variety **L**, which extends **Quant**, and a subcategory **S** of **L**, **S**-$\mathbf{FC}^{DMR}$ is the category, concrete over the product category $\mathbf{Set} \times \mathbf{Set}^{op}$, which comprises the following data.

Objects: lattice-valued formal contexts $\mathcal{K}$ with L an object of **S**.

Morphisms: $\mathcal{K}_1 \xrightarrow{f=(\alpha,\beta)} \mathcal{K}_2$ are $\mathbf{Set} \times \mathbf{Set}^{op}$-morphisms $(L_1^{G_1}, L_1^{M_1}) \xrightarrow{(\alpha,\beta)} (L_2^{G_2}, L_2^{M_2})$, making the next diagrams commute

$$\begin{array}{ccc} L_1^{G_1} & \xrightarrow{\alpha} & L_2^{G_2} \\ {\scriptstyle H_1}\downarrow & & \downarrow{\scriptstyle H_2} \\ L_1^{M_1} & \xleftarrow[\beta^{op}]{} & L_2^{M_2} \end{array} \qquad \begin{array}{ccc} L_1^{M_1} & \xleftarrow{\beta^{op}} & L_2^{M_2} \\ {\scriptstyle K_1}\downarrow & & \downarrow{\scriptstyle K_2} \\ L_1^{G_1} & \xrightarrow[\alpha]{} & L_2^{G_2}. \end{array}$$

Relations versus Birkhoff operators

- There is a one-to-one correspondence between relations $I \subseteq G \times M$ and order-reversing Galois connections on $(\mathcal{P}(G), \mathcal{P}(M))$.
- What about the lattice-valued case?

Definition 16

Given a $\bigvee$-semilattice L and a set X, every $S \subseteq X$ and every $a \in L$ provide the map $X \xrightarrow{\chi_S^a} L$, which is defined by

$$\chi_S^a(x) = \begin{cases} a, & x \in S \\ \bot_L, & \text{otherwise.} \end{cases}$$

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Lattice-valued relations versus Birkhoff operators

Theorem 17

Let G, M be sets and let L be a unital quantale. For every order-reversing Galois connection $(L^G, \alpha, \beta, L^M)$, equivalent are:

1. *There exists a map $G \times M \xrightarrow{I} L$ such that $\alpha = H$ and $\beta = K$.*
2. *For every $g \in G$, $m \in M$, $a \in L$, it follows that*
   - (a) $(\alpha(\chi^{1_L}_{\{g\}}))(m) = (\beta(\chi^{1_L}_{\{m\}}))(g)$;
   - (b) $(\alpha(\underline{a} \otimes \chi^{1_L}_{\{g\}}))(m) = a \to_l (\alpha(\chi^{1_L}_{\{g\}}))(m)$;
   - (c) $(\beta(\chi^{1_L}_{\{m\}} \otimes \underline{a}))(g) = a \to_r (\beta(\chi^{1_L}_{\{m\}}))(g)$.
3. *For every $g \in G$, $m \in M$, $a \in L$, it follows that*
   - (a) $(\alpha(\underline{a} \otimes \chi^{1_L}_{\{g\}}))(m) = a \to_l (\beta(\chi^{1_L}_{\{m\}}))(g)$;
   - (b) $(\beta(\chi^{1_L}_{\{m\}} \otimes \underline{a}))(g) = a \to_r (\alpha(\chi^{1_L}_{\{g\}}))(m)$.

Consequences

Every map $G \times M \xrightarrow{I} L$ gives rise to an order-reversing Galois connection, but the converse way needs additional requirements.

Counterexample

Let L be the unit interval $\mathbb{I} = ([0,1], \bigvee, \wedge, 1)$, and let both G and M be singletons. One can assume that both $\mathbb{I}^G$ and $\mathbb{I}^M$ is $\mathbb{I}$. The order-reversing involution map $\mathbb{I} \xrightarrow{\alpha} \mathbb{I}$, $\alpha(a) = 1 - a$ is a part of the order-reversing Galois connection $(\mathbb{I}, \alpha, \alpha, \mathbb{I})$. The condition of, e.g., Theorem 17(3)(a) gives $\alpha(a) = a \to \alpha(1)$ for every $a \in \mathbb{I}$. However, for $a = \frac{1}{2}$, one obtains that $\alpha(\frac{1}{2}) = \frac{1}{2} \neq 0 = \frac{1}{2} \to 0 = \frac{1}{2} \to \alpha(1)$.

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From $\mathbf{S}\text{-}\mathbf{FC}^{C}$ to $\mathbf{S}\text{-}\mathbf{FC}^{DMR}$

Definition 18

- $\mathbf{S}\text{-}\mathbf{FC}^{C}_{*}$ is a subcategory of $\mathbf{S}\text{-}\mathbf{FC}^{C}$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have surjective maps $G_1 \xrightarrow{\alpha} G_2$, $M_2 \xrightarrow{\beta^{op}} M_1$, and an $\mathbf{S}$-isomorphism $L_1 \xrightarrow{\varphi} L_2$.
- Let $\mathbf{L}$ extend $\mathbf{UQuant}$. $\mathbf{S}\text{-}\mathbf{FC}^{C}_{**}$ (resp. $\mathbf{S}\text{-}\mathbf{FC}^{C}_{*\bullet}$) is a full subcategory of $\mathbf{S}\text{-}\mathbf{FC}^{C}_{*}$, whose objects $\mathcal{K} = (G, M, L, I)$ have non-empty G (resp. M) and, moreover, $1_L \neq \perp_L$.

Theorem 19

There exists a functor $\mathbf{S}\text{-}\mathbf{FC}^{C}_{*} \xrightarrow{\mathrm{H}_{CD}} \mathbf{S}\text{-}\mathbf{FC}^{DMR}$, *which is given by* $\mathrm{H}_{CD}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = \mathcal{K}_1 \xrightarrow{((\alpha,\varphi)^{\vdash\dashrightarrow},((\beta,\varphi)^{\rightarrow\circ})^{op})} \mathcal{K}_2$. *Its restriction to* $\mathbf{S}\text{-}\mathbf{FC}^{C}_{**}$ *(resp.* $\mathbf{S}\text{-}\mathbf{FC}^{C}_{*\bullet}$*) is a (non-full) embedding.*

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- Let $\mathbf{L}$ extend $\mathbf{UQuant}$. $\mathbf{S\text{-}FC}^{C}_{**}$ (resp. $\mathbf{S\text{-}FC}^{C}_{*\bullet}$) is a full subcategory of $\mathbf{S\text{-}FC}^{C}_{*}$, whose objects $\mathcal{K} = (G, M, L, I)$ have non-empty G (resp. M) and, moreover, $1_L \neq \bot_L$.

Theorem 19

There exists a functor $\mathbf{S\text{-}FC}^{C}_{*} \xrightarrow{\mathrm{H}_{CD}} \mathbf{S\text{-}FC}^{DMR}$, *which is given by* $\mathrm{H}_{CD}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = \mathcal{K}_1 \xrightarrow{((\alpha,\varphi)^{\vdash\dashrightarrow},((\beta,\varphi)^{\rightarrow o})^{op})} \mathcal{K}_2$. *Its restriction to* $\mathbf{S\text{-}FC}^{C}_{**}$ *(resp.* $\mathbf{S\text{-}FC}^{C}_{*\bullet}$*) is a (non-full) embedding.*

From $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m}$ to $\mathbf{S}\text{-}\mathbf{FC}^{DMR}$

Definition 20

- $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m*}$ is a subcategory of $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m}$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have surjective maps $G_1 \xrightarrow{\alpha} G_2$, $M_2 \xrightarrow{\beta^{op}} M_1$, and an $\mathbf{S}$-isomorphism $L_1 \xrightarrow{\varphi} L_2$.
- Let $\mathbf{L}$ extend $\mathbf{UQuant}$. $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m**}$ (resp. $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m*\bullet}$) is a full subcategory of $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m*}$, whose objects $\mathcal{K} = (G, M, L, I)$ have non-empty G (resp. M) and, moreover, $1_L \neq \bot_L$.

Theorem 21

There exists a functor $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m*} \xrightarrow{H_{CmD}} \mathbf{S}\text{-}\mathbf{FC}^{DMR}$, *which is given by* $H_{CmD}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = \mathcal{K}_1 \xrightarrow{((\alpha,\varphi)^{\rightarrow},((\beta,\varphi)^{\vdash\dashrightarrow o})^{op})} \mathcal{K}_2$. *Its restriction to* $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m**}$ *(resp.* $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m*\bullet}$*) is a (non-full) embedding.*

From $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m}$ to $\mathbf{S}\text{-}\mathbf{FC}^{DMR}$

Definition 20

- $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m*}$ is a subcategory of $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m}$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have surjective maps $G_1 \xrightarrow{\alpha} G_2$, $M_2 \xrightarrow{\beta^{op}} M_1$, and an $\mathbf{S}$-isomorphism $L_1 \xrightarrow{\varphi} L_2$.
- Let $\mathbf{L}$ extend $\mathbf{UQuant}$. $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m**}$ (resp. $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m*\bullet}$) is a full subcategory of $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m*}$, whose objects $\mathcal{K} = (G, M, L, I)$ have non-empty G (resp. M) and, moreover, $1_L \neq \bot_L$.

Theorem 21

There exists a functor $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m*} \xrightarrow{\mathrm{H}_{CmD}} \mathbf{S}\text{-}\mathbf{FC}^{DMR}$, *which is given by* $\mathrm{H}_{CmD}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = \mathcal{K}_1 \xrightarrow{((\alpha,\varphi)^{\rightarrow},(((\beta,\varphi)^{\vdash\dashrightarrow o})^{op})} \mathcal{K}_2$. *Its restriction to* $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m**}$ *(resp.* $\mathbf{S}\text{-}\mathbf{FC}^{C}_{m*\bullet}$*) is a (non-full) embedding.*

Formal concepts, protoconcepts, and preconcepts

Definition 22

Let $\mathcal{K}$ be a lattice-valued formal context, and let $s \in L^G$, $t \in L^M$. The pair (s, t) is called a

- *(lattice-valued) formal concept of* $\mathcal{K}$ provided that $H(s) = t$ and $K(t) = s$;
- *(lattice-valued) formal protoconcept of* $\mathcal{K}$ provided that $K \circ H(s) = K(t)$ (equivalently, $H \circ K(t) = H(s)$);
- *(lattice-valued) formal preconcept of* $\mathcal{K}$ provided that $s \leqslant K(t)$ (equivalently, $t \leqslant H(s)$).

From $\mathbf{S}$-$\mathbf{FC}^{DMR}$ to $\mathbf{S}$-$\mathbf{FC}^{C}$

Definition 23

- Given an $\mathbf{L}$-algebra L, L-$\mathbf{FC}_{\mathrm{i}}^{DMR}$ is a subcategory of L-$\mathbf{FC}^{DMR}$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have injective maps $L^{G_1} \xrightarrow{\alpha} L^{G_2}$, $L^{M_2} \xrightarrow{\beta^{op}} L^{M_1}$.
- An $\mathbf{L}$-algebra L is called *quasi-strictly right-sided (qsrs-algebra)* provided that $a \leqslant (\top_L \rightarrow_l a) \otimes \top_L$ for every $a \in L$.

Theorem 24

There exists a functor L-$\mathbf{FC}_{\mathrm{i}}^{DMR} \xrightarrow{\mathrm{H}_{DC}^{\mathrm{i}}} \mathbf{S}$-$\mathbf{FC}^{C}$, which is given by $\mathrm{H}_{DC}^{\mathrm{i}}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = (L^{G_1}, L^{M_1}, L, \hat{I}_1) \xrightarrow{(\alpha,\beta,1_L)} (L^{G_2}, L^{M_2}, L, \hat{I}_2)$, where $\hat{I}_j(s,t) = \top_L$ if (s,t) is a formal concept of $\mathcal{K}_j$, and $\bot_L$ otherwise. If L is a qsrs-algebra, then $\mathrm{H}_{DC}^{\mathrm{i}}$ is a (non-full) embedding.

From $\mathbf{S}\text{-}\mathbf{FC}^{DMR}$ to $\mathbf{S}\text{-}\mathbf{FC}^{C}$

Definition 23

- Given an $\mathbf{L}$-algebra L, $L\text{-}\mathbf{FC}_{\mathrm{i}}^{DMR}$ is a subcategory of $L\text{-}\mathbf{FC}^{DMR}$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have injective maps $L^{G_1} \xrightarrow{\alpha} L^{G_2}$, $L^{M_2} \xrightarrow{\beta^{op}} L^{M_1}$.
- An $\mathbf{L}$-algebra L is called *quasi-strictly right-sided (qsrs-algebra)* provided that $a \leqslant (\top_L \to_I a) \otimes \top_L$ for every $a \in L$.

Theorem 24

There exists a functor $L\text{-}\mathbf{FC}_{\mathrm{i}}^{DMR} \xrightarrow{\mathrm{H}^{\mathrm{i}}_{DC}} \mathbf{S}\text{-}\mathbf{FC}^{C}$*, which is given by* $\mathrm{H}^{\mathrm{i}}_{DC}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = (L^{G_1}, L^{M_1}, L, \hat{I}_1) \xrightarrow{(\alpha,\beta,1_L)} (L^{G_2}, L^{M_2}, L, \hat{I}_2)$*, where* $\hat{I}_j(s,t) = \top_L$ *if* (s,t) *is a formal concept of* $\mathcal{K}_j$*, and* $\bot_L$ *otherwise. If* L *is a qsrs-algebra, then* $\mathrm{H}^{\mathrm{i}}_{DC}$ *is a (non-full) embedding.*

From $\mathbf{S}\text{-}\mathbf{FC}^{DMR}$ to $\mathbf{S}\text{-}\mathbf{FC}^{C}$

Definition 25

Given an $\mathbf{L}$-algebra L, $L\text{-}\mathbf{FC}^{DMR}_{rfp}$ is a subcategory of $L\text{-}\mathbf{FC}^{DMR}$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have maps $L^{G_1} \xrightarrow{\alpha} L^{G_2}$, $L^{M_2} \xrightarrow{\beta^{op}} L^{M_1}$ such that $K_2 \circ H_2 \circ \alpha(s) = \alpha(s)$ implies $K_1 \circ H_1(s) = s$, and $H_1 \circ K_1 \circ \beta^{op}(t) = \beta^{op}(t)$ implies $H_2 \circ K_2(t) = t$, for every $s \in L_1^{G_1}$, $t \in L_2^{M_2}$.

Theorem 26

There exists a functor $L\text{-}\mathbf{FC}^{DMR}_{rfp} \xrightarrow{\mathrm{H}^{rfp}_{DC}} \mathbf{S}\text{-}\mathbf{FC}^{C}$*, which is given by* $\mathrm{H}^{rfp}_{DC}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = (L^{G_1}, L^{M_1}, L, \hat{I}_1) \xrightarrow{(\alpha,\beta,1_L)} (L^{G_2}, L^{M_2}, L, \hat{I}_2)$*, where* $\hat{I}_j(s,t) = \top_L$ *if* (s,t) *is a formal concept of* $\mathcal{K}_j$*, and* $\bot_L$ *otherwise. If* L *is a qsrs-algebra, then the functor is a (non-full) embedding.*

From $\mathbf{S}$-$\mathbf{FC}^{DMR}$ to $\mathbf{S}$-$\mathbf{FC}^{C}$

Definition 25

Given an $\mathbf{L}$-algebra L, L-$\mathbf{FC}^{DMR}_{rfp}$ is a subcategory of L-$\mathbf{FC}^{DMR}$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have maps $L^{G_1} \xrightarrow{\alpha} L^{G_2}$, $L^{M_2} \xrightarrow{\beta^{op}} L^{M_1}$ such that $K_2 \circ H_2 \circ \alpha(s) = \alpha(s)$ implies $K_1 \circ H_1(s) = s$, and $H_1 \circ K_1 \circ \beta^{op}(t) = \beta^{op}(t)$ implies $H_2 \circ K_2(t) = t$, for every $s \in L_1^{G_1}$, $t \in L_2^{M_2}$.

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There exists a functor L-$\mathbf{FC}^{DMR}_{rfp} \xrightarrow{\mathrm{H}^{rfp}_{DC}} \mathbf{S}$-$\mathbf{FC}^{C}$, *which is given by* $\mathrm{H}^{rfp}_{DC}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = (L^{G_1}, L^{M_1}, L, \hat{I}_1) \xrightarrow{(\alpha,\beta,1_L)} (L^{G_2}, L^{M_2}, L, \hat{I}_2)$, *where* $\hat{I}_j(s,t) = \top_L$ *if* (s,t) *is a formal concept of* $\mathcal{K}_j$, *and* $\bot_L$ *otherwise. If* L *is a qsrs-algebra, then the functor is a (non-full) embedding.*

From $\mathbf{S}\text{-}\mathbf{FC}^{DMR}$ to $\mathbf{S}\text{-}\mathbf{FC}^{C}$

Definition 27

Given an $\mathbf{L}$-algebra L, $L\text{-}\mathbf{FC}^{DMR}_{orp}$ is a subcategory of $L\text{-}\mathbf{FC}^{DMR}$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have order-preserving maps $L^{G_1} \xrightarrow{\alpha} L^{G_2}$, $L^{M_2} \xrightarrow{\beta^{op}} L^{M_1}$.

Theorem 28

There exists a functor $L\text{-}\mathbf{FC}^{DMR}_{orp} \xrightarrow{H^{orp}_{DC}} \mathbf{S}\text{-}\mathbf{FC}^{C}$, *which is given by* $H^{orp}_{DC}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = (L^{G_1}, L^{M_1}, L, \hat{I}_1) \xrightarrow{(\alpha,\beta,1_L)} (L^{G_2}, L^{M_2}, L, \hat{I}_2)$, *where* $\hat{I}_j(s,t) = \top_L$ *if* (s,t) *is a formal preconcept of* $\mathcal{K}_j$, *and* $\bot_L$ *otherwise. If* L *is a qsrs-algebra, then the functor is a (non-full) embedding.*

From $\mathbf{S}$-$\mathbf{FC}^{DMR}$ to $\mathbf{S}$-$\mathbf{FC}^{C}$

Definition 27

Given an $\mathbf{L}$-algebra L, L-$\mathbf{FC}^{DMR}_{orp}$ is a subcategory of L-$\mathbf{FC}^{DMR}$, with the same objects, and whose morphisms $\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2$ have order-preserving maps $L^{G_1} \xrightarrow{\alpha} L^{G_2}$, $L^{M_2} \xrightarrow{\beta^{op}} L^{M_1}$.

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There exists a functor L-$\mathbf{FC}^{DMR}_{orp} \xrightarrow{\mathrm{H}^{orp}_{DC}} \mathbf{S}$-$\mathbf{FC}^{C}$, *which is given by* $\mathrm{H}^{orp}_{DC}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = (L^{G_1}, L^{M_1}, L, \hat{I}_1) \xrightarrow{(\alpha,\beta,1_L)} (L^{G_2}, L^{M_2}, L, \hat{I}_2)$, *where* $\hat{I}_j(s,t) = \top_L$ *if* (s,t) *is a formal preconcept of* $\mathcal{K}_j$, *and* $\bot_L$ *otherwise. If* L *is a qsrs-algebra, then the functor is a (non-full) embedding.*

From **S-FC**^{DMR} to **S-FC**^{C}

Theorem 29

There exists a functor $L\text{-}\mathbf{FC}^{DMR} \xrightarrow{\mathrm{H}_{DC}} \mathbf{S}\text{-}\mathbf{FC}^{C}$, *which is given by* $\mathrm{H}_{DC}(\mathcal{K}_1 \xrightarrow{f} \mathcal{K}_2) = (L^{G_1}, L^{M_1}, L, \hat{I}_1) \xrightarrow{(\alpha,\beta,1_L)} (L^{G_2}, L^{M_2}, L, \hat{I}_2)$, *where* $\hat{I}_j(s,t) = \top_L$ *if* (s,t) *is a formal protoconcept of* $\mathcal{K}_j$, *and* $\bot_L$ *otherwise. If* L *is a qsrs-algebra, then* H_{DC} *is a (non-full) embedding.*

Final remarks

- This talk considered some approaches to morphisms of lattice-valued formal contexts of Formal Context Analysis (FCA).
- We constructed several categories, whose objects are lattice-valued analogues of formal contexts of FCA, and whose morphisms reflect the crisp setting of Chu spaces, the lattice-valued setting of J. T. Denniston, A. Melton, and S. E. Rodabaugh, as well as the many-valued setting of B. Ganter and R. Wille.
- We considered a number of functors between the categories of formal contexts, embedding each of them into its counterparts.

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FCA without relations

The difference between the settings of relations and Galois connections in the lattice-valued case, motivates the following problem.

Problem 30

Is it possible to build a lattice-valued approach to FCA, which is based in order-reversing Galois connections on lattice-valued powersets, which are not generated by lattice-valued relations on their respective sets of objects and their attributes?

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Thank you for your attention!