EPIMORPHISMS IN CERTAIN VARIETIES OF PARTIALLY ORDERED SEMIGROUPS

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Dedicated to my school teacher, Mr. N.M. Kazmi.

1. Priliminaries

A partially ordered semigroup, briefly *posemigroup*, is a semigroup S endowed with a partial order $\leq$ which is compatible with the binary operation, i.e. for all $s_1, s_2, t_1, t_2 \in S$, $(s_1 \leq t_1,\ s_2 \leq t_2)$ implies $s_1 s_2 \leq t_1 t_2$. A posemigroup with identity is called a *pomonoid*. A posemigroup *homomorphism* $f : S \longrightarrow T$ is a monotone semigroup homomorphism, i.e. for all $s_1, s_2 \in S$, $f(s_1 s_2) = f(s_1)f(s_2)$ and $s_1 \leq s_2$ in S implies $f(s_1) \leq f(s_2)$ in T. If S and T are both pomonoids with identities 1_S and 1_T, then f is said to be a pomonoid homomorphism if we further have $f(1_S) = f(1_T)$. A posemigroup (pomonoid) homomorphism f is termed *epimorphism* if it is right cancelative (in the usual sense of category theory). We call $f : S \longrightarrow T$ an *order-embedding* if $f(s_1) \leq f(s_2)$ implies $s_1 \leq s_2$, $s_1, s_2 \in S$.

In what follows, we shall also treat a posemigroup (resp. pomonoid) S as a semigroup (resp. monoid) by simply disregarding the order. Let $\mathcal{A}$ be a class of posemigroups (pomonoids). Then by $\mathcal{A}'$ we shall denote the class obtained by disregarding the orders in $\mathcal{A}$. Clearly $\mathcal{A}'$ is a subclass of $\mathcal{A}$. Naturally, we shall be considering algebraic and order theoretic morphisms when speaking of $\mathcal{A}'$ and $\mathcal{A}$ respectively.

A class of posemigroups (pomonoids) is called a *variety* if it is closed under taking products (which are endowed with componentwise order), homomorphic images and subposemigroups (subpomonoids). It is also possible to alternatively define posemigroup (pomonoid) varieties with the help of inequalities using a Birkhoff type characterization; we refer to [8] for details. Trivially, a class of posemigroups (pomonoids), that is a variety in algebraic sense (if one disregards the orders) is also a variety of posemigroups (pomonoids). Also, every variety (whether algebraic or order theoretic) naturally gives rise to a category.

The author is a recipient of the Estonian Science Foundation's MOBILITAS research grant MJD 198. My research was also partially supported by the Estonian targeted financing project SF0180039s08.

One can easily observe that $f : S \longrightarrow T$ is necessarily an epimorphism in the category of all posemigroups if it is such in the category of all semigroups. Our aim is to show that the converse of this statement, which may not be true in general, holds in certain varieties of posemigroups (equivalently semigroups). As we shall be frequently using these concepts, it is worth recalling S-posets and their tensor products.

Let S be a pomonoid and X a poset. Then X is called a *left S-poset*, and we denote it by ${}_SX$, if it is a left S-act with the left action $S \times X \longrightarrow X$ of S being monotone, i.e. $(s_1, x_1) \leq (s_2, x_2)$ implies $s_1x_1 \leq s_2x_2$. Right S-posets are defined analogously. Let A_S and ${}_SB$ be respectively right and left S-posets. Then a poset $A\hat{\otimes}_S B$ is called the tensor product of A_S and ${}_SB$ (over S) if it satisfies the following conditions:

(1) there exists a balanced monotone map $\alpha : A \times B \longrightarrow A\hat{\otimes}_S B$ (where $A \times B$ is endowed with the Cartesian order), such that
(2) for any poset X admitting a monotone balanced map $\beta : A \times B \longrightarrow X$ there exists a unique monotone map $\varphi : A\hat{\otimes}_S B \longrightarrow X$ such that $\beta = \varphi \circ \alpha$.

Clearly S_S and ${}_SS$ are special S-posets. These are the only S-posets we shall be dealing with in the sequel.

2. Closure and saturation for pomonoids

The primry aim of this section is to put together some results, concerning closure of pomonoids, that we have recently proved. We also pose a couple of questions concerning the epimorphisms and saturation for pomonoids. We begin by recalling dominions.

Definition 1 (Definition 1 of [7]). Let U be a subpomonoid of a pomonoid S. Then the subpomonoid $\hat{dom}_S(U) = \{x \in S :$ for all pairs of pomonoid homomorphisms $\alpha, \beta : S \longrightarrow T$ with $\alpha|_U = \beta|_U$, we have $x\alpha = x\beta\}$ is called the *dominion* of U (in S).

The following zigzag theorem for pomonoids provides a criterion to check if an element $d \in S$ falls in $\hat{dom}_S(U)$.

Theorem 1 (Sohail Nasir). *Let U be a subpomonoid of a pomonoid S. Then $d \in \hat{dom}_S(U)$ if and only if $d\hat{\otimes}1 = 1\hat{\otimes}d$ in $S\hat{\otimes}_U S$.*

While ignoring the orders, one may also consider the (algebraic) dominion $dom_S(U)$—for instance, see [2]—of U in S. In the unordered scenario we have the following celebrated zigzag theorem, originally due to J.R. Isbell.

Theorem 2 (Jim Renshaw)**.** *Let U be a submonoid of a monoid S. Then $d \in dom_S(U)$ if and only if $d \otimes 1 = 1 \otimes d$ in $S \otimes_U S$.*

Recall, for example from [6], that $d \otimes 1 = 1 \otimes d$ in $S \otimes_U S$ implies $d \hat{\otimes} 1 = 1 \hat{\otimes} d$ in $S \hat{\otimes}_U S$. We therefore have:

$$U \subseteq dom_S(U) \subseteq d\hat{o}m_S(U) \subseteq S. \tag{1}$$

By analogy with [1], a subpomonoid U of S will be termed *closed* (in S) if $d\hat{o}m_S(U) \subseteq U$ (whence indeed $d\hat{o}m_S(U) = U$). We shall call U *absolutely closed* if it is closed in all of its pomonoid extensions. Also, again by analogy with [1], U will be called *saturated* if $d\hat{o}m_S(U) \subsetneqq S$ for all pomonoids $S \supsetneqq U$. One can easily observe that

i) a pomonoid homomorphism $f : S \longrightarrow T$ is an epimorphism if and only if $d\hat{o}m_T(\mathrm{Im}\, f) = T$, and (consequently)
ii) a saturated monoid can never be epimorphically embedded in any other monoid (i.e. $S \hookrightarrow T$, with $S \subsetneqq T$ cannot be an epimorphism). Thus
iii) all epimorphisms in a variety of saturated pomonoids are surjective. The converse statement, viz.
iv) a variety is saturated if all of its epis are onto, also holds.

The next theorem, taken from [7], tells that U being closed in S is not affected by the introduction of orders.

Theorem 3 (Sohail Nasir)**.** *Let U be a subpomonoid of a pomonoid S. Then U is closed in S as a pomonoid if and only if it is such as a monoid.*

Also, it is clear from (1) that U is saturated as a monoid if it is such as a pomonoid. Nontheless, we don't have any answer to the following (converse) question.

Problem 1. *What information can be extracted about $d\hat{o}m_S(U)$ if U is algebraically saturated (i.e. can we say anything about $d\hat{o}m_S(U)$ if $dom_S(U) \subsetneqq S$ for every $S \supsetneqq U$)?*

One may also ask the following question (which is perhaps more complicated).

Problem 2. *Given that U, as monoid, cannot be properly epimorphically embedded in any monoid, is it possible to embed U epimorphically (in order theoretic sense) in some pomonoid $S \supsetneqq U$?*

3. The case of posemigroups

We first adapt the notions of previous section and few more to the setting of posemigroups. We define dominions for posemigroups by just replacing posemigroups for pomonoids in Definition 1. Similarly, posemigroup amalgams (and their embeddings) are also defined by substituting, in the corresponding definitions, posemigroups for pomonoids (and posemigroup order-embeddings for monoid order-embeddings), see [7].

Theorem 4 (Zigzag theorem for posemigroups)**.** *Let U be a subposemigroup of a posemigroup S. Then an element d of S is in $\hat{dom}_S(U)$, the dominion of U in S, if and only if*

$$d\hat{\otimes}1 = 1\hat{\otimes}d \text{ in } S^1\hat{\otimes}_{U^1}S^1,$$

where U^1 and S^1 are the pomonoids obtained from U and S (respectively) by adjoining a common identity, whether or not they already have one.

Proof. Denote by $\mathcal{S}$ the category of all posemigroups. Let $\mathcal{S}^1$ denote the category of pomonoids obtained by adjoining an identity to every object of $\mathcal{S}$; the morphisms in $\mathcal{S}^1$ are the natural extensions of those of $\mathcal{S}$. Now it suffices to observe that $d \in \hat{dom}_S(U)$ in $\mathcal{S}$ if and only if $d \in \hat{dom}_S(U)$ in $\mathcal{S}^1$. □

The following results are obtained using amalgamation of pomonoids and posemigroups (and the interplay between them), which I don't intend to discuss here.

Corollary 1. *A subposemigroup U is closed in a posemigroup S if and only if the special posemigroup amalgam $(U; S_1, S_2)$ is embeddable.*

It is now straightforward to verify the following.

Proposition 1. *A subposemigroup U is closed in a posemigroup S if and only if it is such as a semigroup.*

We can now make the following observations in the setting of posemigroups.

(i) It follows from Proposition 1 that a posemigroup U is absolutely closed iff it is such as a semigroup (within the class of semigroups that also qualify as posemigroup extensions of U).
(ii) Because inclusions of type (1) also hold for posemigroups, we can further assert that U is saturated as a posemigroup then it is such as a semigroup (within the class of posemigroup extensions of U).

Moreover, because in the case of posemigroups (resp. semigroups) also, $f : U \longrightarrow S$ is an epimorphism iff $\hat{dom}_S(\operatorname{Im} f) = S$ (resp. $dom_S(\operatorname{Im} f) = S$), we can further state the following.

(iii) All epimorphisms in a variety of saturated posemigroups (semigroups) are surjective (note that this observation concerns varieties because there may exist non-surjective epimorphisms from saturated semigroups as is shown in Example 3.4 of [3]).

Also U is clearly saturated if it is absolutely closed. So it follows that

(iv) epis of absolutely closed (in whatever sense) posemigroups are onto.

However Example 3.3 of [3] shows that saturated semigroups need not be absolutely closed.

4. Epimorphisms in certain categories of posemigroups

In the unordered scenario all varieties of absolutely closed semigroups have been determined.

Theorem 5 (Theorem 2 of [4]). *The absolutely closed varieties of semigroups are exactly the varieties consisting entirely of semilattices of groups, or entirely of right groups or entirely of left groups.*

Proposition 2. *Let $\mathcal{V}'$ be a variety of absolutely closed semigroups. Let $\mathcal{V}'$ be the corresponding variety of posemigroups. Then a posemigroup homomorphism f is epi in $\mathcal{V}$ iff it is such in $\mathcal{V}'$.*

Proof. If f is surjective then it is clearly epi in both the varieties and there is nothing to prove. So let f be a non-surjective epimorphism.

($\Longleftarrow$) This part is straightforward.

($\Longrightarrow$) Let $f : U \longrightarrow T$ be non-epi in $\mathcal{V}'$. This implies that $dom_T(\operatorname{Im} f) \subsetneq T$. But then by Theorem 4 of [7], $\operatorname{Im} f = dom_T(\operatorname{Im} f) = d\hat{o}m_S(\operatorname{Im} f)$. This implies that $d\hat{o}m_T(\operatorname{Im} f) \subsetneq T$. So f is non-epi in $\mathcal{V}$. □

Problem 3. *Are there any other order theoretic varieties of absolutely closed posemigroups?*

Problem 4. *Can we find a posemigroup epimorphism f that is not epi in the category of semigroups (of course we have to exclude the categories of the above theorem).*

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