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Structure of weak suborders of a poset

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joint work with
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The lattice of all these subrelations is algebraic and we investigate its structure.
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In the second part, we deal with analogue notions and properties in the framework of lattice valued orderings.

As an application, we present an introduction to lattice valued ordered groupoids and groups.

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Equivalently, for a poset (P, ρ), we deal with *all* **weak suborderings of** ρ.

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Theorem

For a poset (P,ρ), $(\mathcal{O}_w(P,\rho),\subseteq)$ is a complete lattice.

Denote by $\mathcal{O}(P, \rho)$ the set of all suborderings of (P, ρ).
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- *For every $Q \subseteq P$,*

$$[\Delta_Q, Q^2 \cap \rho] = \mathcal{O}(Q, Q^2 \cap \rho).$$

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- *If* $\Delta \wedge \theta = \Delta \wedge \sigma$ *and* $\Delta \vee \theta = \Delta \vee \sigma$, *then* $\theta = \sigma$.

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The lattice $(\mathcal{O}_w(P,\rho),\subseteq)$ is algebraic.

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The maps $m_\Delta : \theta \mapsto \theta \wedge \Delta$ and $n_\Delta \theta \mapsto \theta \vee \Delta$ are lattice endomorphisms.

$$\ker m_\Delta \cong \mathcal{P}(P) \quad \textit{and} \quad \ker n_\Delta \cong \mathcal{O}(P, \rho).$$

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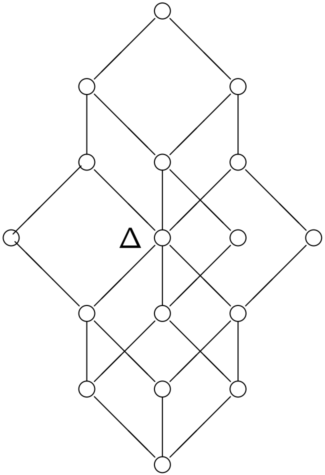

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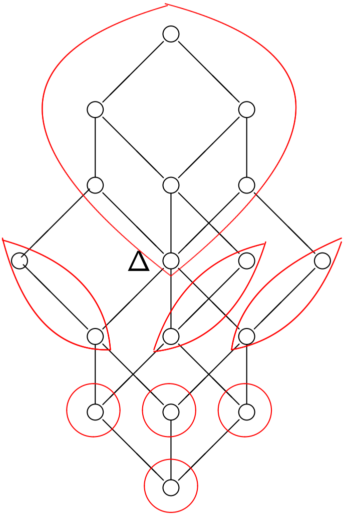

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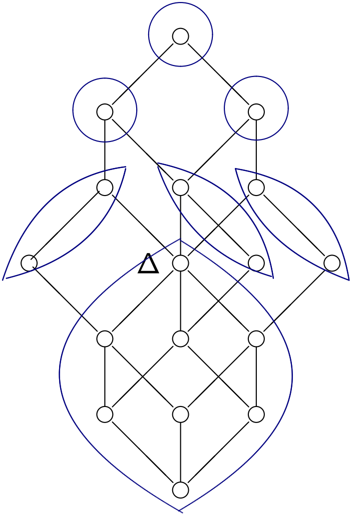

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A mapping $\rho : X^2 \to L$ is an *L*-**valued** (binary) **relation** on X.

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Lemma

Every cut of an L-valued sub–poset α of a poset P is a sub–poset of P.

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An L-valued sub–poset $\Upsilon : P \to L$ of $(P, \leqslant)$ is an **L-valued down–set** or an **L-valued semi–ideal** of P if for all $x, y \in P$

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Dually, $\digamma : P \to L$ is an **L-valued up–set** or an **L-valued semi–filter** of P if for all $x, y \in P$

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Proposition

Let $\alpha : P \to L$ be an L-valued sub–poset of $(P, \leqslant)$. Then α is an L-valued up(down)–set of P if and only if every cut of α is an up(down)–set in P.

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Proposition

An L-valued set $\mu : P \to L$ is an L-valued down–set in P if and only if for all $x, y \in P$ the following holds:

$$\mu(x) \wedge k_{\leqslant}(y, x) \leq \mu(y).$$

Dually, μ is an L-valued up–set on P if and only if for all $x, y \in P$

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For $a \in P$, the mapping $f_{\downarrow a} : P \to L$ is an **L-valued principal ideal** generated by a, if it is an L-valued down–set of P satisfying: for every $x \in P$

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Consequently, for $a, b \in P$, we define an **L-valued interval** $f_{[a,b]}$ on P as an L-valued set on P, such that for every $x \in P$

$$f_{[a,b]}(x) := (f_{\uparrow a} \cap f_{\downarrow b})(x),$$

for some $f_{\uparrow a}$ and $f_{\downarrow b}$ on P.

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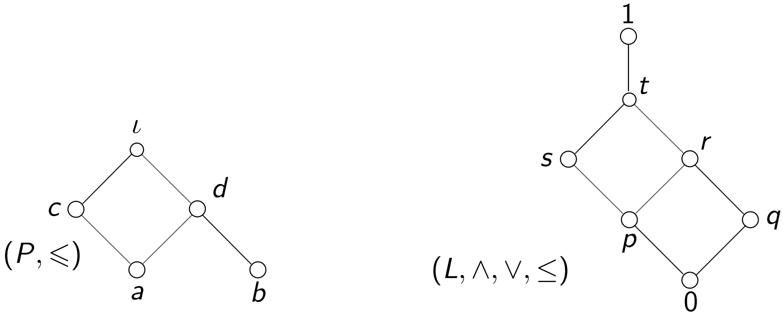

Figure 1

The functions

$$f_{\downarrow\iota} = \begin{pmatrix} a & b & c & d & \iota \\ t & t & s & r & p \end{pmatrix} \quad \text{and} \quad f_{\uparrow a} = \begin{pmatrix} a & b & c & d & \iota \\ p & 0 & s & r & t \end{pmatrix},$$

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$$f_{[a,\iota]} = \begin{pmatrix} a & b & c & d & \iota \\ p & 0 & s & r & p \end{pmatrix}$$

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Further, the functions

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Consequently,

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All the cuts of an L-valued convex sub–poset of P *are (ordinary) convex sub–posets of* P.

Convexity for lattice valued sub–posets

An L-valued set $\mu : P \to L$ is said to be a **convex** L-valued sub–poset of P if for all $x, y, z \in P$ the following holds:

$$\mu(x) \wedge \mu(z) \wedge k_{\leqslant}(x, y) \wedge k_{\leqslant}(y, z) \leq \mu(y).$$

Proposition

An L-valued subset $\zeta : P \to L$ of P is convex if

$$\zeta = F \cap \Upsilon,$$

for some L-valued up-set F and L-valued down-set Υ on P.

All the cuts of an L-valued convex sub–poset of P are (ordinary) convex sub–posets of P.
Any L-valued interval on P is an L-valued convex sub–poset of P.

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  A lattice valued relation ρ on X is a **lattice valued ordering relation** (**lattice valued order**) on X if it is reflexive, antisymmetric and transitive.

A lattice valued relation $\rho : X^2 \to L$ on a set X is **weakly reflexive** if

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Proposition

Let $\rho : X^2 \to L$ be an L-valued ordering relation, such that L is a complete lattice without zero divisors under $\wedge$. Then, $\operatorname{supp} \rho$ *is an ordering relation on X.*

Proposition

If $\rho : X^2 \to L$ *is a weak L-valued ordering relation on* X*, and* $\delta(\rho) : X \to L$*, defined by* $\delta(\rho)(x) := \rho(x, x)$*. Then for each non-zero* $p \in L$*, the cut-relation* ρ_p *is an order on the cut-subset* $\delta(\rho)_p$ *of* X*.*

Lattice valued poset with lattice valued order

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An L-valued relation $\rho : P \to L$ on a set P is an **L-valued relation on an L-valued subset** $\mu : P \to L$ of P if for all $x, y \in P$

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Every L-valued relation $\rho : P \to L$ which is reflexive on $\mu : P \to L$ is weakly reflexive on P.

We say that an L-valued relation ρ on an L-valued subset μ of P is an **L-valued ordering on μ**, if it is reflexive (in the above sense), antisymmetric as defined by (a) and transitive in the sense of (t).

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Theorem

The function ρ defined above is an L-valued order on L-valued set μ on P.

If $(P, \leqslant)$ is a poset, then a pair (μ, ρ) is an ***L*-valued poset with *L*-valued ordering** if $\mu : P \to L$ is an L-valued subset of P and $\rho : P^2 \to L$ is the L-valued ordering on P defined above:

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Theorem

Let $(P, \leqslant)$ be a poset, $\mu : P \to L$ an L-valued subset of P, and $\rho : P^2 \to L$ an L-valued relation on μ. Then (μ, ρ) is an L-valued poset with L-valued order on $(P, \leqslant)$, if and only if for every $p \in L, p \neq 0$, pair (μ_p, ρ_p) is a sub–poset of $(P, \leqslant)$.

Theorem

Let $\mathcal{F}$ be a collection of sub–posets of a poset $(P, \leqslant)$, closed under set intersections and containing P as a member. Then there is a lattice L and an L-valued sub–poset (M, ρ) of P so that the collection of its cuts coincides with $\mathcal{F}$. Moreover, the order on each cut is the corresponding cut of ρ.

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$\Delta = \{(x, x) \mid x \in P\}$, and
$\uparrow\Delta$, $\downarrow\Delta$ respectively the filter and the ideal in the poset $(\mathcal{FP}, \subseteq)$, generated by Δ.

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  $$\underline{\rho(\mu)}(x, y) = \begin{cases} \mu(x) & \text{if } x = y, \\ 0 & \text{if } x \neq y \end{cases},$$

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Let (μ, ρ) be an L-valued sub-poset of a poset $(M, \leqslant)$. Then (μ, ρ) is a L-valued chain if and only if for all $x, y \in M$ such that x is not comparable with y, we have

$$\mu(x) \wedge \mu(y) = 0.$$

Proposition

Let (μ, ρ) be a L-valued sub-poset of a poset $(M, \leqslant)$. Then (μ, ρ) is an L-valued chain if and only if every its non-zero cut μ_p is a chain in $(M, \leqslant)$, with respect to ρ_p.

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Let $(M, \leqslant)$ be a poset which is a lattice, and (μ, ρ) an L-valued sub-poset of M.
Then, (μ, ρ) is an **L-valued sublattice** of M, if μ is an L-valued sublattice as an L-valued algebra, i.e. , if for all $x, y \in M$,

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Proposition

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L-valued subgroup

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If $(G, \cdot, ^{-1}, e)$ is a group and $(L, \wedge, \vee, \leq)$ a complete lattice, then the mapping $\mu : G \to L$ is an **L-valued subgroup** of G if the following holds: for all $x, y \in G$

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- $\mu(e) = 1.$

Compatibility

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Let $(G, \cdot)$ be a groupoid and $\rho : G^2 \to L$ an L-valued relation on G. We say that ρ is **compatible with operation "$\cdot$" on G**, if for all $x, y, z \in G$ the following holds:

$$\rho(x, y) \leq \rho(x \cdot z, y \cdot z) \wedge \rho(z \cdot x, z \cdot y).$$

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Let $(G, \cdot)$ be a groupoid and $\mu : G \to L$ its L-valued subgrupoid. We say that an L-valued relation $\rho : G^2 \to L$ on μ is **compatible with operation "$\cdot$" on μ**, if for all $x, y, z \in G$ the following holds:

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Let $\left(G, \cdot, {}^{-1}, e, \leqslant\right)$ be an ordered group. Let also $\mu : G \to L$ and $\rho : G^2 \to L$ be an L-valued set on G and an L-valued relation on μ, respectively. The pair (μ, ρ) is an **L-valued ordered subgroup of G** if the following hold:

1. μ is an L-valued subgroup of G;
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Theorem

Let G be an ordered group, $\mu : G \to L$ an L-valued subset of G and $\rho : G^2 \to L$ an L-valued relation on μ. Then (μ, ρ) is an L-valued ordered subgroup of G if and only if for every $p \in L$, the cut μ_p is an ordered subgroup of G.

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Theorem

Let $\mathcal{F}$ be a collection of subgroups of an ordered group $(G, \cdot, ^{-1}, e, \leqslant)$ which is closed under set intersections and contains G. Then there is a complete lattice L and an ordered L-valued subgroup (μ, ρ) of G, such that for every subgroup $H \in \mathcal{F}$, the cut μ_H coincides with H and it is ordered by ρ_H.

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L-valued cones

If (μ, ρ) is an L-valued-ordered subgroup of G, then the ***L*-valued positive cone** on μ, is an L-valued set $\pi_\mu : G \to L$, such that:

$$\pi_\mu(x) := \rho(e, x),$$

where e is the neutral element of G.
Obviously,

$$\pi_\mu(x) = \begin{cases} \mu(x) & \text{if } x \geqslant e, \\ 0 & \text{otherwise} . \end{cases}$$

Analogously, the ***L*-valued negative cone** is a function $\nu_\mu : G \to L$, such that

$$\nu_\mu(x) := \rho(x, e)$$

It follows that

$$\nu_\mu(x) = \begin{cases} \mu(x) & \text{if } x \leqslant e, \\ 0 & \text{otherwise} . \end{cases}$$

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Proposition

Let $\left(G, \cdot, ^{-1}, e, \leqslant\right)$ be an ordered group and (μ, ρ) its L-valued ordered subgroup. Then for all $x, y \in G$

$$\pi_\mu\left(x^{-1} \cdot y\right) \geq \rho(x, y) .$$

Denote by P_G and N_G the positive and the negative cone of G, as well as their characteristic functions, respectively.

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Let $(G, \cdot, {}^{-1}, e, \leqslant)$ be an ordered group and (μ, ρ) an L-valued-ordered subgroup of G. The following holds:

$$\pi_\mu = \mu \cap P_G.$$

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An L-valued-ordered subgroup (μ, ρ) of an ordered group $\left(G, \cdot, {}^{-1}, e, \leqslant\right)$ is an ***L*-valued-convex subgroup** of G if μ is an L-valued-convex subset on the poset $(G, \leqslant)$.

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Theorem

Let (μ, ρ) be an L-valued ordered subgroup of $(G, \cdot, ^{-1}, e, \leqslant)$. Then, the following are equivalent:

(i) (μ, ρ) is an L-valued convex subgroup of G.

(ii) The restriction of π_μ to P_G is an L-valued down-set in P_G.

(iii) The restriction of ν_μ to N_G is an L-valued up-set in N_G.

L-valued lattice ordered group

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Let $(G, \cdot, ^{-1}, e, \leqslant)$ be a lattice ordered group, L a complete lattice and (μ, ρ) an L-valued ordered subgroup of G. We say that (μ, ρ) is an ***L*-valued lattice ordered subgroup** of G, or an ***L*-valued ℓ-subgroup** of G if for every $x \in G$

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Theorem

Let μ be an L-valued subgroup of a lattice ordered group G. Then, (μ, ρ) is an L-valued ℓ-subgroup of G if and only if, for every $p \in L$, the cut μ_p is an ℓ-subgroup of G.

Theorem

Let $(G, \cdot, ^{-1}, e, \leqslant)$ *be a lattice ordered group, and* $L =$*SubG, i.e., L is the lattice of all subgroups of* G*, ordered dually to the set inclusion. Further, let* $H \subseteq L$ *consist of all convex* ℓ*-subgroups of* G*. Then, the mapping* $\mu : G \to L$*, such that for every* $x \in G$*,* $\mu(x) := \langle x \rangle_H$*, is an L-valued* ℓ*-subgroup of* G*.*

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Theorem

Let G *be an ordered group and L a complete lattice. Then* G *is totaly ordered if and only if every L-valued subgroup* μ *of* G *is an L-valued* ℓ*-subgroup of* G *under the order* $\rho : G \to L$*,* $\rho(x, y) = \mu(x) \wedge \mu(y) \wedge k_{\leqslant}(x, y)$.

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Proposition

An L-valued subgroup (μ, ρ) *of a lattice ordered group G is is an L-chain under* ρ *if for every pair of non-comparable elements* $x, y \in G$*,* $\mu(x) \wedge \mu(y) = 0$*.*

The end

Thank you!