On Zariski topologies of Abelian groups with operations

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5 June 2013, Novi Sad

The 4th Novi Sad Algebraic Conference
& Semigroups and Applications 2013

Topologizing algebras

Topologizing groups

In 1944 Markov posed the following question:

Does every infinite group admit a non-discrete Hausdorff topology in which its multiplication and inversion are continuous?

He (implicitly) defined a T_1 topology on a group, called now its Zariski topology, and proved: *For countable groups, the answer is positive iff the Zariski topology is non-discrete.*

It was proved that the answer is affirmative for Abelian groups (Kertész and Szele, 1953) and negative in general (for uncountable groups: Shelah, 1976 (under CH), Hesse, 1979 (without CH); for countable groups: Olshanski based on Adian's construction, 1980).

Remark. Any infinite group admits a non-discrete Hausdorff topology in which all left and right shifts and inversion are continuous (Zelenyuk, 2006).

Topologizing rings

The same question can be posed for rings (and other algebras):

Does every infinite ring admit some non-discrete Hausdorff topology in which its operations are continuous?

Similarly to the case of groups, Markov proved: *For countable rings, the answer is positive iff the Zariski topology is non-discrete.*

In 1970s Arnautov obtained the negative answer for uncountable rings. On the other hand, he shown: *The Zariski topology of every infinite ring is non-discrete*, thus giving the affirmative answer for countable rings.

In 1997 Protasov gave a short proof of Arnautov's result by using Hindman's Finite Sums Theorem, a famous statement in Ramsey-theoretic algebra obtained via ultrafilter extensions of semigroups.

Following close ideas, we prove non-discreteness of Zariski topologies for a wider class of universal algebras, called here *polyrings*, which includes various classical algebras besides rings.

Actually, we state a much stronger fact: If K is a polyring, then K^n considered as a subspace of K^{n+1} with its Zariski topology is closed nowhere dense in it. Our proof uses a multidimensional generalization of Hindman's theorem (Bergelson–Hindman, 1996).

Zariski topologies of polyrings

Polyrings

Definition. $(K, 0, +, \Omega)$ is a *polyring* iff $(K, 0, +)$ is an Abelian group and any operation $F \in \Omega$ (of arbitrary arity) is distributive w.r.t. the addition, i.e. the shifts

$$x \mapsto F(a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_{n-1})$$

are endomorphisms of $(K, 0, +)$, for all $i < n$ and $a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-1} \in K$.

Examples. Various classical algebras: Abelian groups with operators, modules, rings, differential rings, linear algebras, etc.

Fact. For any Abelian group $(K, 0, +)$ there is the *largest* polyring $(K, 0, +, \Omega)$.

Zariski topologies

Let K be a polyring and $n < \omega$. If $F \in K[x_1, \ldots, x_n]$ is a term of n variables, let

$$S_F = \Big\{(a_1, \ldots, a_n) \in K^n : F(a_1, \ldots, a_n) = 0\Big\}$$

denote the set of solutions of the equation $F(x_1, \ldots, x_n) = 0$ in K.

Definition. A set $S \subseteq K^n$ is *closed in the Zariski topology on* K^n iff S is an intersection of finite unions of sets S_F.

Facts. 1. The Zariski topology on K is a T_1 topology in which all shifts are continuous.
2. The Zariski topology on K^{n+1} includes the product of the Zariski topologies on K^n and K, and can be stronger.
3. K^n is homeomorphic to $K^n \times \{0\} \subseteq K^{n+1}$ (and will be identified with it below).

The main result

Theorem. *Let K be an infinite polyring. For any term $F \in K[x_1, \ldots, x_n]$ the mapping of K^n into K defined by F is closed nowhere dense in K^{n+1}. In particular, so is K^n.*

Roughly speaking, this shows that such spaces, although can be not Hausdorff, allow a reasonable notion of topological dimension.

Corollary. *If K is an infinite polyring,* $0 < n < \omega$, *then K^n is non-discrete.*

Remark. If $\Omega \subseteq \Omega'$ then the Zariski topology of $(K, 0, +, \Omega')$ is stronger than one of $(K, 0, +, \Omega)$. Since there is the largest polyring with a given $(K, 0, +)$, Theorem gives the best possible result in this direction.

Questions

We mention only a few questions. All algebras below are considered with their Zariski topologies.

1. *Is every non-discrete group K nowhere dense in K^2?*

2. *If E is an endomorphism of a non-discrete group K, is $(K, \cdot, E)$ also non-discrete?*

By our result, both answers are affirmative for Abelian groups.

3. *Is every (non-commutative) field connected?*

This fails for some rings.

4. *Given a polyring K, classify closed subsets of K^n up to: (i) homeomorphisms; (ii) local homeomorphisms.*

This may be unclear even for fields.