The Persistence Lattice

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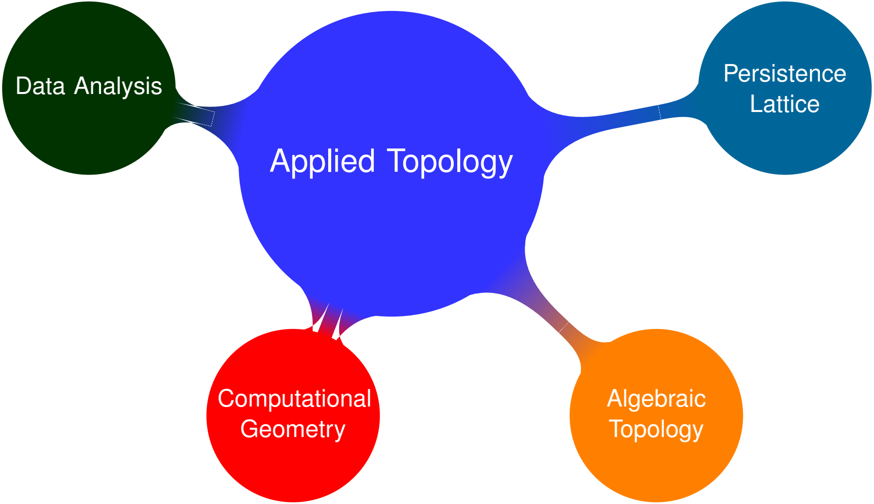
(in a joint work with Primož Škraba)

Jožef Stefan Institute
Ljubljana, Slovenia

Novi Sad Algebra Conference,
June 8, 2013

Motivations



Topological Data Analysis

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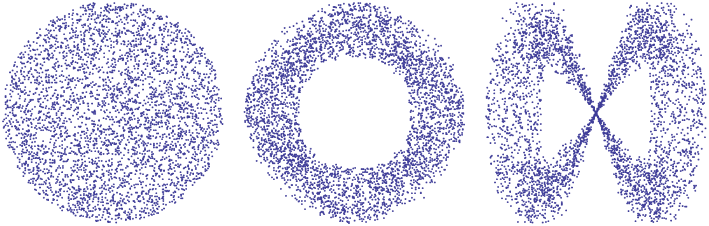

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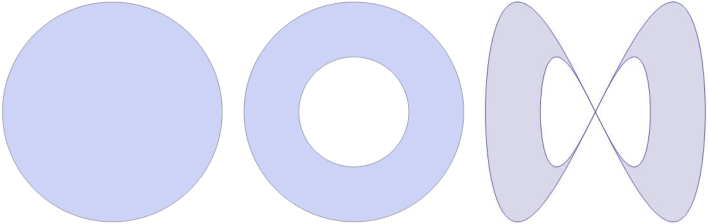

Recovering the Space

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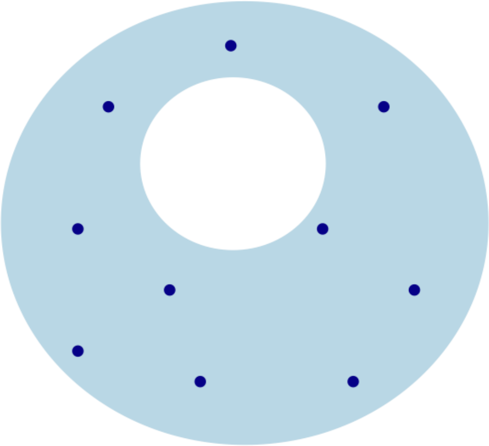

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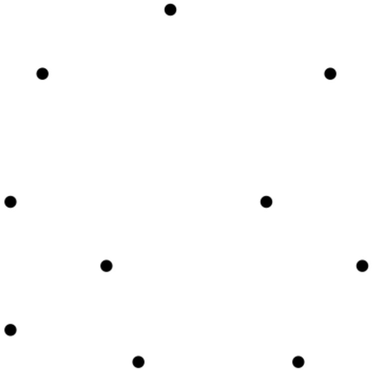

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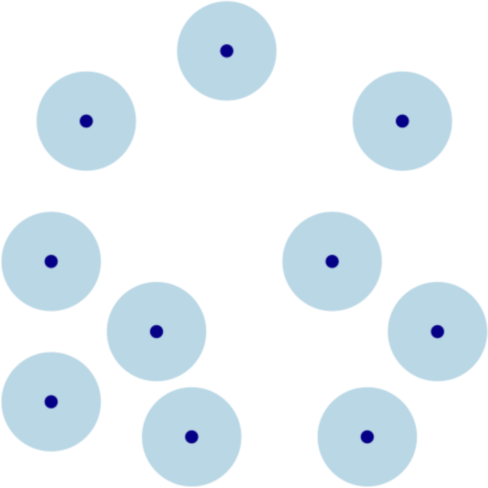

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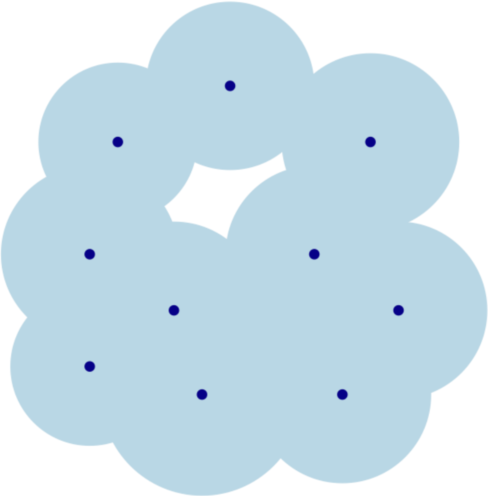

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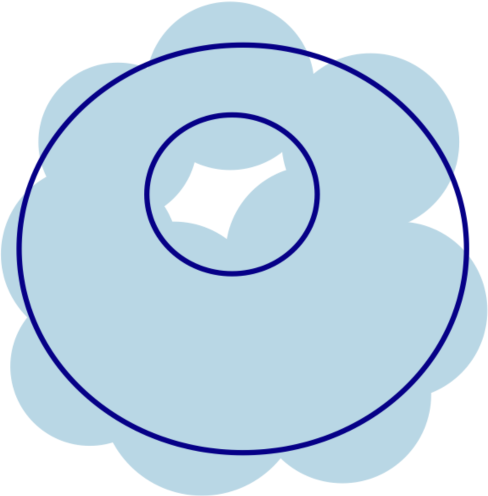
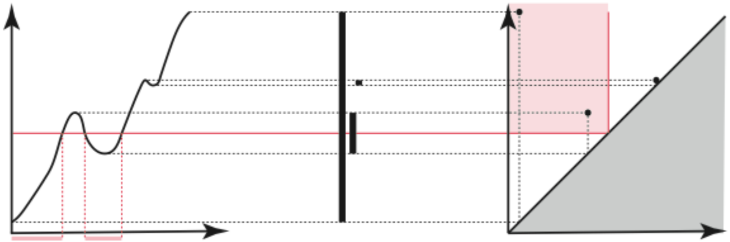


Persistent Homology

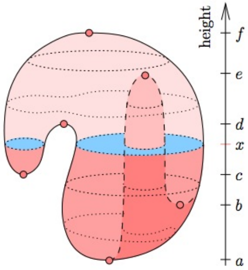
Persistence of H_0 of sublevel-sets of a real function.

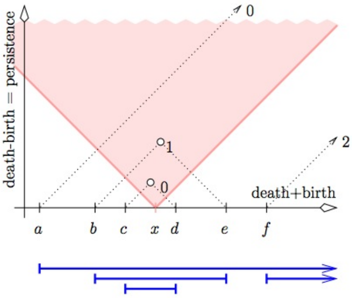


Mikael Vejdemo-Johansson, Sketches of a platypus: persistence homology and its foundations. arXiv:1212.5398v1 (2013)

Persistent Homology

Persistence of H_0 of sublevel-sets by the height function with six critical points on a topological sphere.





H. Edelsbrunner and Dmitry Morozov, Persistent Homology: theory and practice. 6th European Congress of Mathematics (2012), to appear.

Filtrations & Barcodes

General Setting: $\mathbb{X}$ space and $f : \mathbb{X} \to \mathbb{R}$.

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$$\mathbb{X}_\alpha \subseteq \mathbb{X}_\beta$$

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We get: a diagram of vector spaces and linear maps.

$$\mathrm{H}(\mathbb{X}_0) \longrightarrow \mathrm{H}(\mathbb{X}_1) \longrightarrow \mathrm{H}(\mathbb{X}_2) \longrightarrow \mathrm{H}(\mathbb{X}_3) \longrightarrow \mathrm{H}(\mathbb{X}_4)$$

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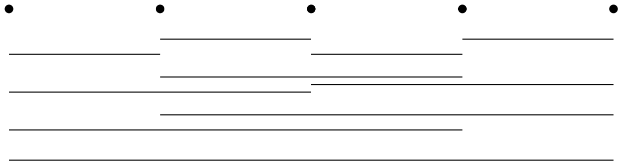
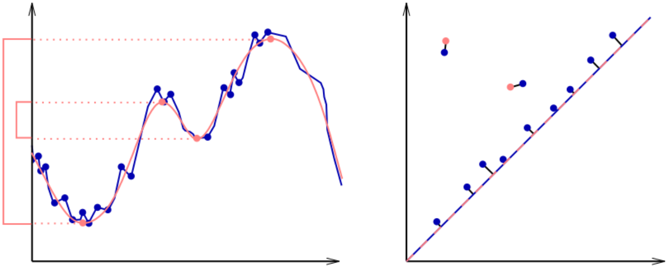


Persistent Homology

Stability of the Persistence Diagram.



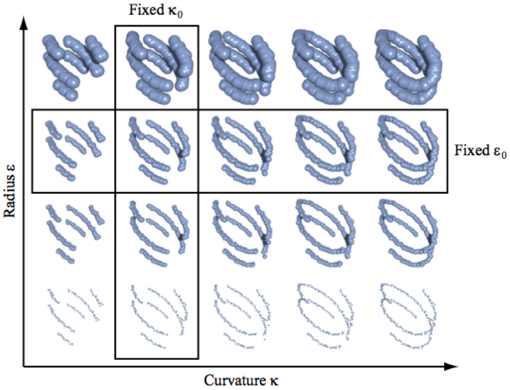
D Cohen-Steiner, H Edelsbrunner, and J Harer, Stability of persistence diagrams. Discrete Comput Geom (2005)

Multidimensional Persistence

What if we have more than one parameter?

Multidimensional Persistence

A bifiltration parametrized along curvature k and radious ϵ



G. Carlsson and A. Zomorodian, The theory of multidimensional persistence. Discrete Comput Geom (2007)

Multidimensional Persistence

$$
\begin{array}{ccccccccc}
\mathbb{X}_{03} & \longrightarrow & \mathbb{X}_{13} & \longrightarrow & \mathbb{X}_{23} & \longrightarrow & \mathbb{X}_{33} & \longrightarrow & \mathbb{X}_{43} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{X}_{02} & \longrightarrow & \mathbb{X}_{12} & \longrightarrow & \mathbb{X}_{22} & \longrightarrow & \mathbb{X}_{32} & \longrightarrow & \mathbb{X}_{42} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{X}_{01} & \longrightarrow & \mathbb{X}_{11} & \longrightarrow & \mathbb{X}_{21} & \longrightarrow & \mathbb{X}_{31} & \longrightarrow & \mathbb{X}_{41} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{X}_{00} & \longrightarrow & \mathbb{X}_{10} & \longrightarrow & \mathbb{X}_{20} & \longrightarrow & \mathbb{X}_{30} & \longrightarrow & \mathbb{X}_{40}
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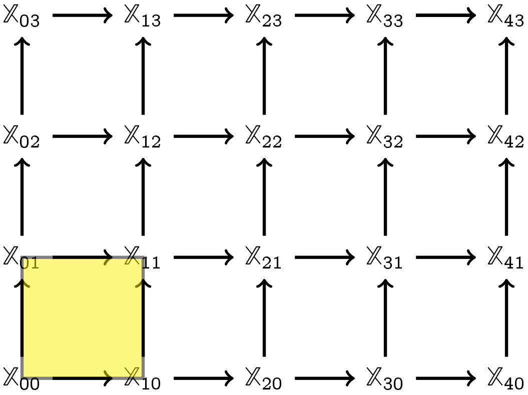

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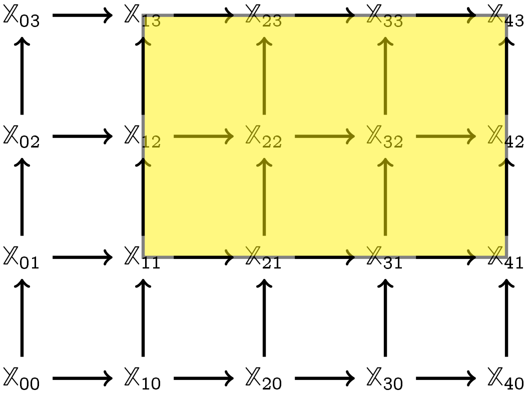

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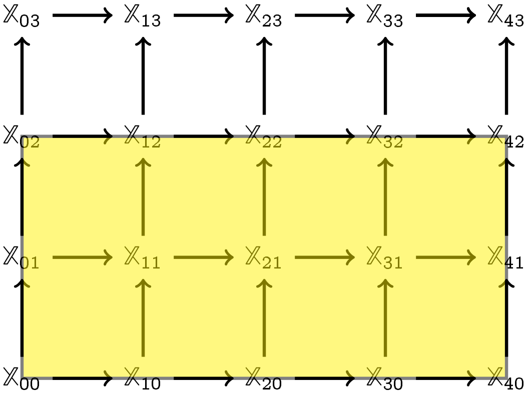

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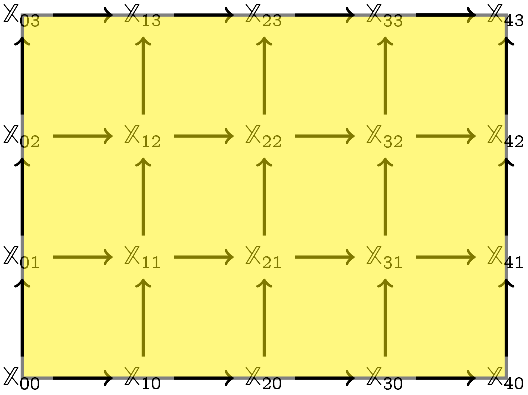
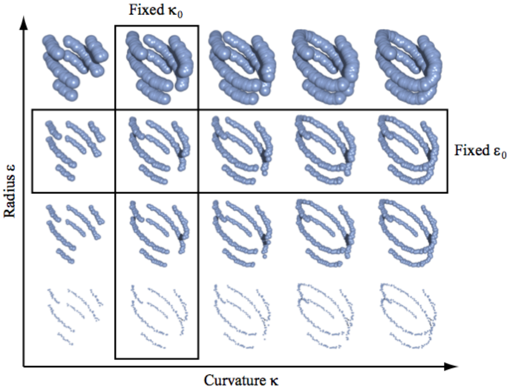


Multidimensional Persistence

The Missing Data Problem



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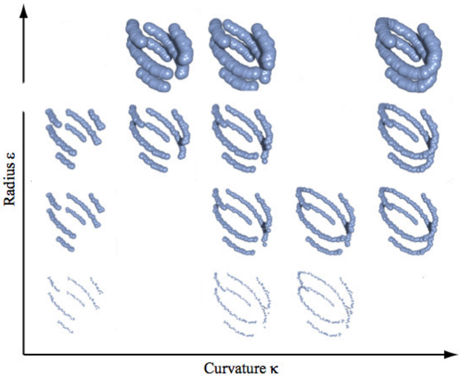

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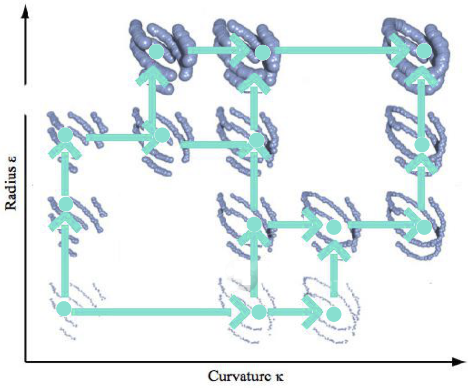

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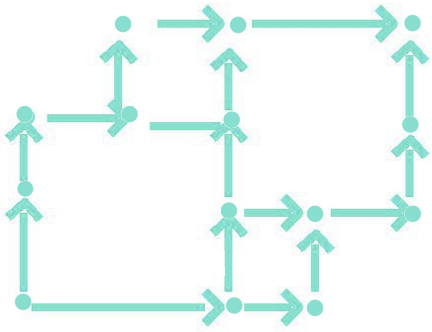

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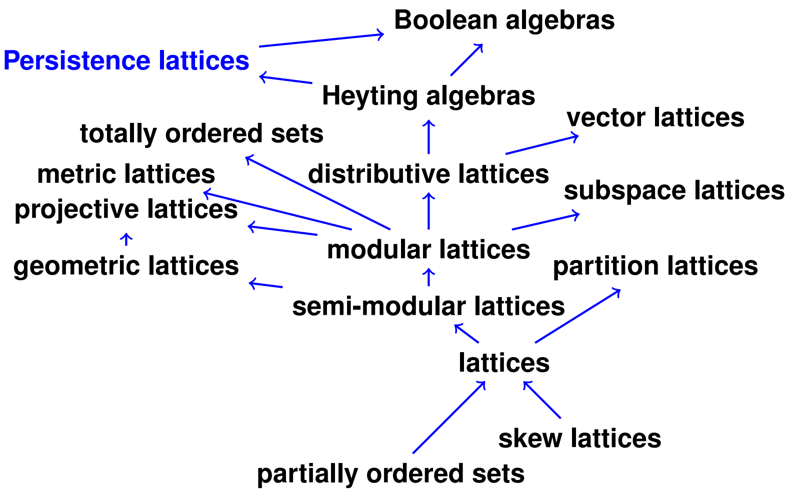
Partially ordered sets

What can the order tell us?

Varieties Of Lattices



Standard Persistence

A Morse-filtration is a partial order on the parameter

$$X_\alpha \subseteq X_\beta \quad \Rightarrow \quad \alpha < \beta$$

$$\begin{array}{c} X_N \\ \uparrow \\ X_{N-1} \\ \uparrow \\ \vdots \\ \uparrow \\ X_2 \\ \uparrow \\ X_1 \end{array}$$

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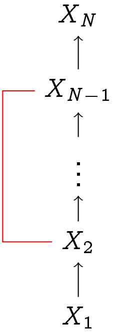

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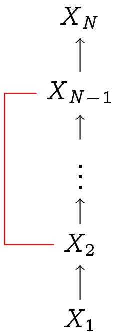
Persistent homology classes

$$\mathrm{H}_*^{i,j}(\mathbb{X}) = \mathrm{im}(\mathrm{H}_*(\mathbb{X}_i) \to \mathrm{H}_*(\mathbb{X}_j))$$



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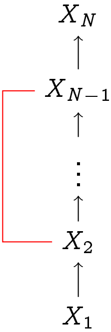

- $\mathrm{H}_*(\mathbb{X}_i) \vee \mathrm{H}_*(\mathbb{X}_j) = \mathrm{H}_*(X_{\max(i,j)})$
- $\mathrm{H}_*(\mathbb{X}_i) \wedge \mathrm{H}_*(\mathbb{X}_j) = \mathrm{H}_*(X_{\min(i,j)})$

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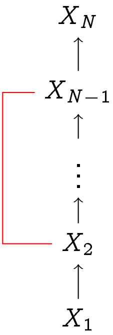

Definition

For any two elements $\mathrm{H}_*(\mathbb{X}_i)$ and $\mathrm{H}_*(\mathbb{X}_j)$, the rank of the persistent homology classes is $\mathrm{im}(\mathrm{H}_*(\mathbb{X}_i \wedge \mathbb{X}_j) \to \mathrm{H}_*(\mathbb{X}_i \vee \mathbb{X}_j))$.

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Set:

$$\mathbb{X}_{ij} \wedge \mathbb{X}_{k\ell} \Rightarrow \mathbb{X}_{yz}, \text{ with } y = \min(i,k), z = \min(j,\ell)$$

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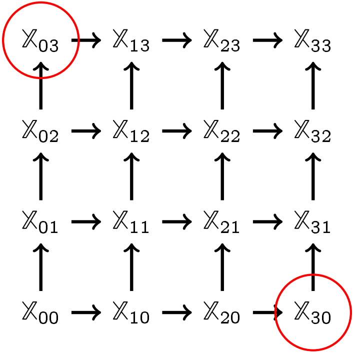

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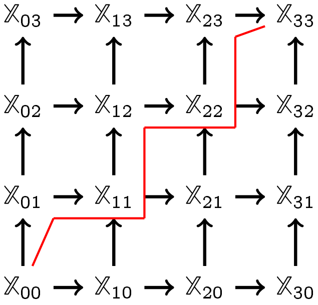

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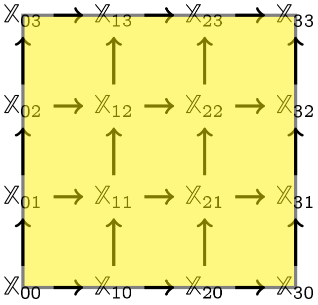

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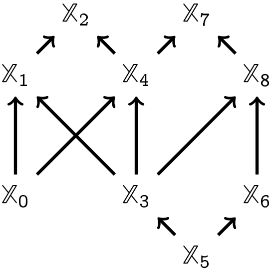

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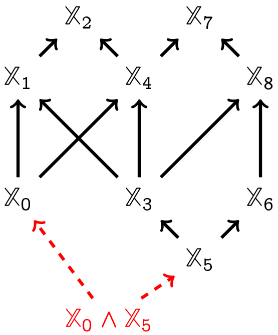

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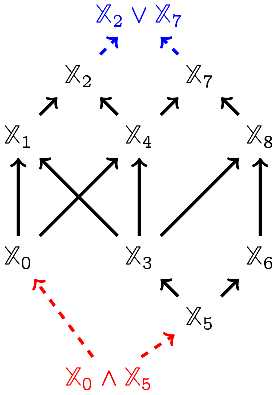

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Diagram is commutative and connected.

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the composition will not commute with identity unless the map is an isomorphism.

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Partial order of vector spaces

For all vector spaces A and B,

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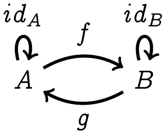

Equalizers and Coequalizers

Equalizers

$$A \overset{f}{\underset{g}{\rightrightarrows}} C$$

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$$E \xrightarrow{\text{e}} A \underset{\text{g}}{\overset{\text{f}}{\rightrightarrows}} C$$

Equalizers and Coequalizers

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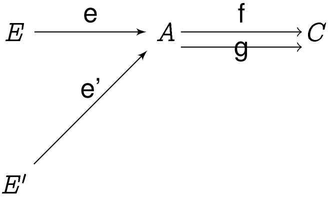

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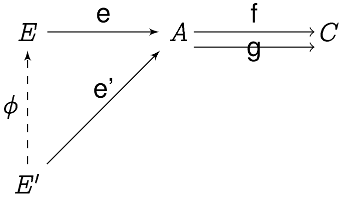

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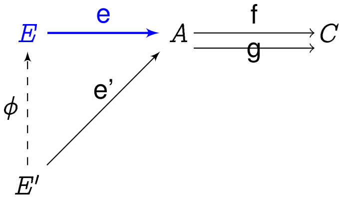
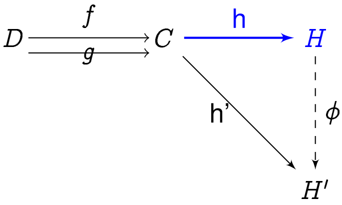


The kernel set is $E = \{\, x \in X \mid f(x) = g(x) \,\} = ker(f - g)$

Equalizers and Coequalizers

Coequalizers



H is the quotient of Y by the equivalence $\langle (f(x), g(x)) | x \in X \rangle$, i.e.,

$$H = C/im(f - g) = coker(f - g)$$

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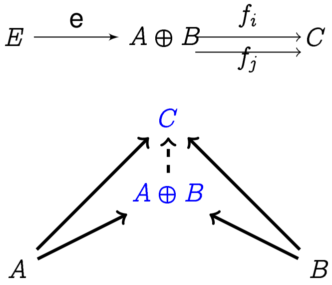

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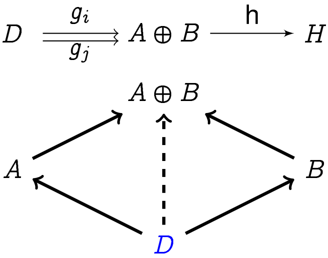
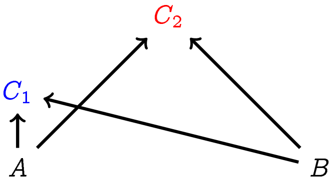


Equalizers and Coequalizers

$$E \xrightarrow{\quad e \quad} A \oplus B \underset{f_j}{\overset{f_i}{\rightrightarrows}} C_1, C_2$$



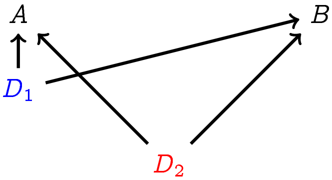
Equalizers and Coequalizers

$$E \xrightarrow{\;\text{e}\;} A \oplus B \underset{f_j}{\overset{f_i}{\rightrightarrows}} C_1 \oplus C_2$$

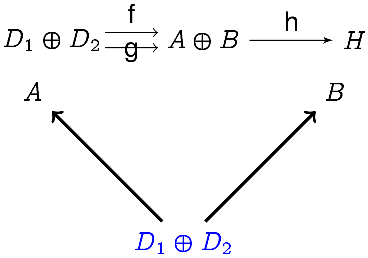
$$A \longrightarrow {\color{blue} C_1 \oplus C_2} \longleftarrow B$$

Equalizers and Coequalizers

$$D_1, D_2 \overset{f}{\underset{g}{\rightrightarrows}} A \oplus B \xrightarrow{h} H$$



Equalizers and Coequalizers



Formal Definition

Meet Operation

The *join* of two elements A and B is the *equalizer* of $A \wedge B \to A \oplus B \rightrightarrows C_k$ given by:

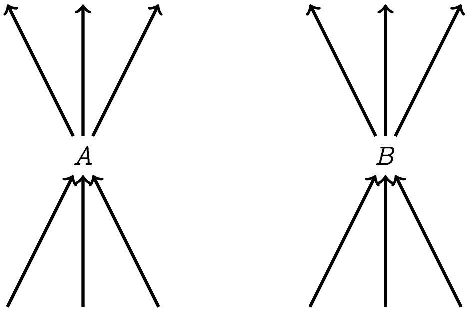
$$A \wedge B = \{x \in A \oplus B | f_i(x) = f_j(x), \text{for all } i, j \in I\}$$

Join Operation

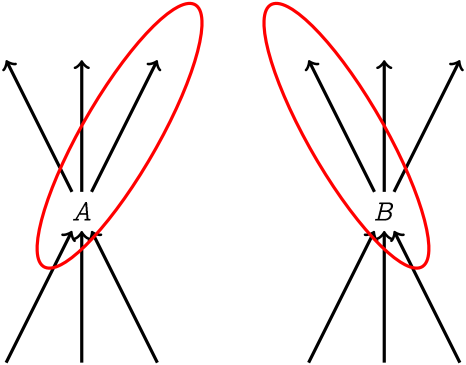
The *meet* of two elements A and B is the *coequalizer* of $D_k \rightrightarrows A \oplus B \to A \vee B$ given by:

$$A \vee B = A \oplus B / \langle g_i(x) \sim g_j(x) \mid x \in D_k, \text{for all } i, j \in I \rangle$$

Intuition



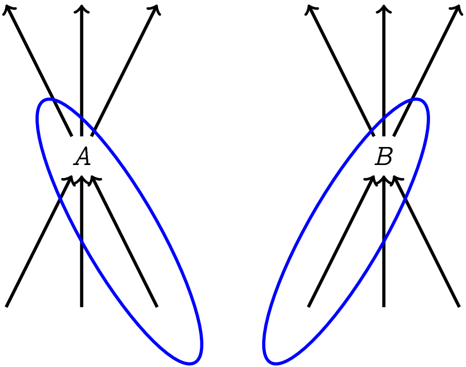
Intuition



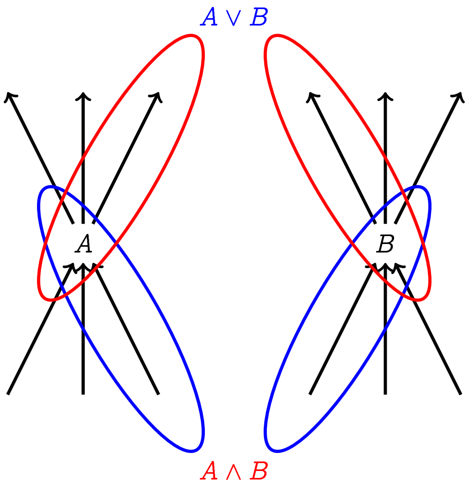
$A \wedge B$

Intuition

$A \vee B$



Intuition



Completeness

Theorem (JPC & PŠ 2013)

The persistence lattice is a complete lattice with

$$\bigwedge A_k = \{ x \in \oplus_k A_k : f_{A_i}(x) = f_{A_j}(x) \},$$

$$\bigvee_k A_k = (\oplus_k A_k)/\langle \bigcup \theta_{A_i A_j} \rangle.$$

where $\theta_{A_i A_j} = \langle (f_{A_i}(x), f_{A_j}(x)) \rangle$

Algebraic Properties

What lattice do we get?

Algebraic Properties

Theorem (JPC & PŠ 2013)

Let A and B be vector spaces. Then,

$$0 \to A \wedge B \to A \oplus B \to A \vee B \to 0$$

is a short exact sequence.

Algebraic Properties

Theorem (JPC & PŠ 2013)

Let A and B be vector spaces. Then,

$$0 \to A \wedge B \to A \oplus B \to A \vee B \to 0$$

is a short exact sequence.

Sketch of the Proof.

The equalizer map $f : A \wedge B \to A \oplus B$ is injective.
The coequalizer map $g : A \oplus B \to A \vee B$ is surjective.
Moreover $\operatorname{im} f = \ker g$ so that

$$A \vee B \cong A \oplus B / f(A \wedge B).$$

□

Algebraic Properties

Theorem (JPC & PŠ 2013)

The persistence lattice of a given persistence diagram is distributive.

Algebraic Properties

Theorem (JPC & PŠ 2013)

The persistence lattice of a given persistence diagram is distributive.

Proof.

Let A, B and X be vector spaces such that $X \vee A = X \vee B$ and $X \wedge A = X \wedge B$ in order to show that $A \cong B$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A \wedge X & \longrightarrow & A \oplus X & \longrightarrow & A \vee X & \longrightarrow & 0 \\
\updownarrow \cong & & \updownarrow \cong & & & & \updownarrow \cong & & \updownarrow \cong \\
0 & \longrightarrow & B \wedge X & \longrightarrow & B \oplus X & \longrightarrow & B \vee X & \longrightarrow & 0
\end{array}$$

□

Algebraic Properties

Definition

A bounded distributive lattice L is a Heyting algebra if, for all $A, B \in L$, $A \Rightarrow B$ is the biggest X such that $A \wedge X \leq B$, i.e.,

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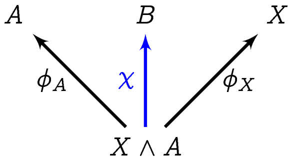

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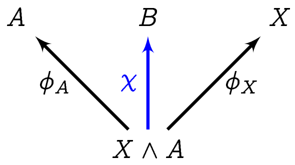

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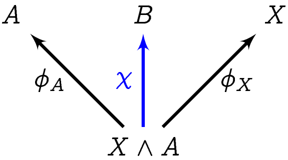


Example

- The open sets of any top space X under $\cap$, $\cup$, $\emptyset$, X and $U \Rightarrow V = int((X - U) \cup V)$
- Complete distributive lattices with $x \Rightarrow y = \bigvee\{z : x \wedge z \leq y\}$

Algebraic Properties

Theorem (JPC & PŠ 2013)

The persistence lattice of a given persistence diagram is distributive, complete and bounded. It is completely distributive thus constituting a ***complete Heyting algebra****.*

Algebraic Properties

Arrow Operation for standard persistence

$A \Rightarrow B$ is the biggest X such that $A \wedge X \to B$

$$\begin{array}{c} \vdots \\ \uparrow \\ X_i \\ \uparrow \\ B \\ \uparrow \\ A \\ \uparrow \\ X_j \\ \uparrow \\ \vdots \end{array}$$

$$A \Rightarrow B = \begin{cases} B, & \text{if } B \leq A \\ 1, & \text{if } A \leq B \end{cases}$$

Algebraic Properties

Arrow operation for multidimensional persistence

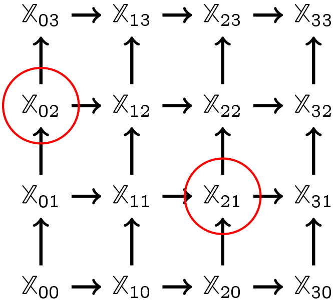
$A \Rightarrow B$ is the biggest X such that $A \wedge X \to B$

$$\begin{array}{ccccccc}
\mathbb{X}_{03} & \to & \mathbb{X}_{13} & \to & \mathbb{X}_{23} & \to & \mathbb{X}_{33} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{X}_{02} & \to & \mathbb{X}_{12} & \to & \mathbb{X}_{22} & \to & \mathbb{X}_{32} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{X}_{01} & \to & \mathbb{X}_{11} & \to & \mathbb{X}_{21} & \to & \mathbb{X}_{31} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{X}_{00} & \to & \mathbb{X}_{10} & \to & \mathbb{X}_{20} & \to & \mathbb{X}_{30}
\end{array}$$

Algebraic Properties

Arrow operation for multidimensional persistence

$A \Rightarrow B$ is the biggest X such that $A \wedge X \to B$



Algebraic Properties

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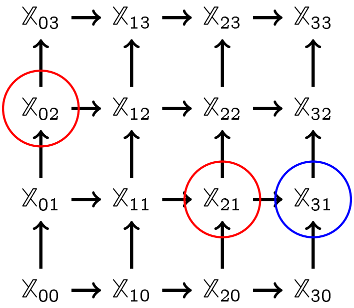


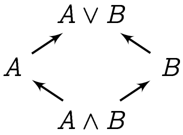
Algebraic Properties

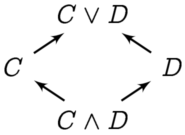
Stability

A B C D

Algebraic Properties

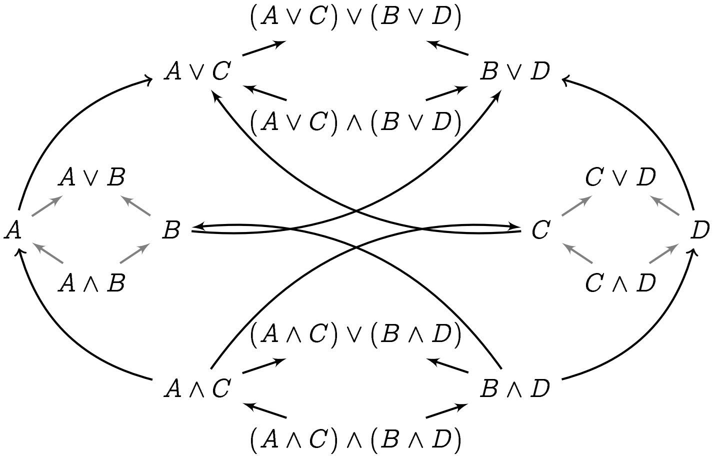
Stability





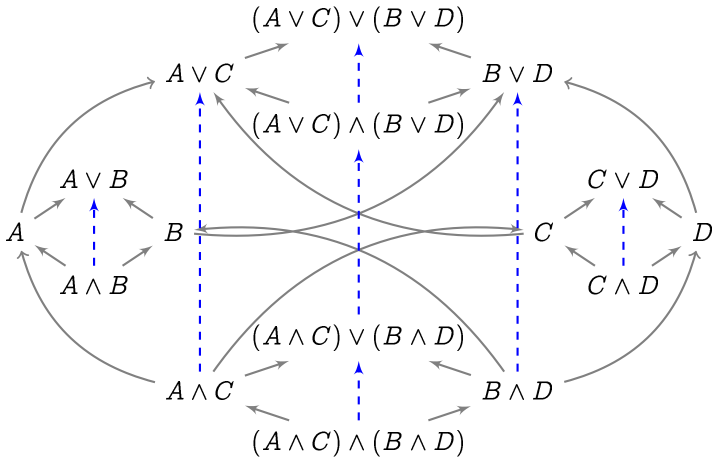
Algebraic Properties

Stability



Algebraic Properties

Stability



(Other) Open Problems

- Other views on stability
- General decompositions and diagrams
- New algorithms and analysis
- Impact of the Heyting algebra structure
- Study of the dual space

HVALA

HVALA

THANK YOU

HVALA

THANK YOU

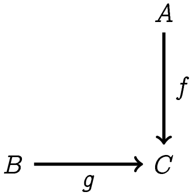
OBRIGADO

THE B-SIDES

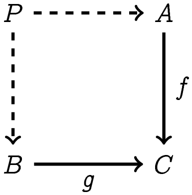
Implementation

Implementing pullbacks and pushouts

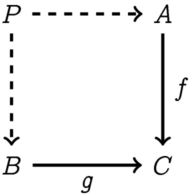
Pullback



Pullback



Pullback



Compute $\ker(A \oplus B \xrightarrow{(f,g)} C)$

Algorithm

We start out with two maps f, g represented by matrices F, G. To compute the pullback of f and g, we construct the matrix corresponding to $(f, -g)$:

$$\begin{array}{|c|c|} \hline F & -G \\ \hline \end{array}$$

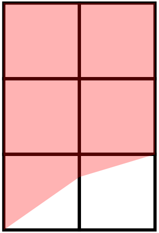
Algorithm

We start out with two maps f, g represented by matrices F, G. To compute the pullback of f and g, we construct the matrix corresponding to $(f, -g)$: Compute kernel

I	
	I
F	$-G$

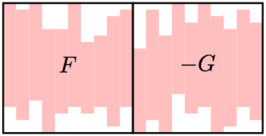
Algorithm

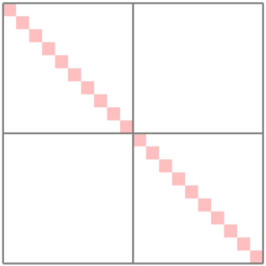
We start out with two maps f, g represented by matrices F, G. To compute the pullback of f and g, we construct the matrix corresponding to $(f, -g)$:
Compute kernel



Algorithm

Computing the pullback

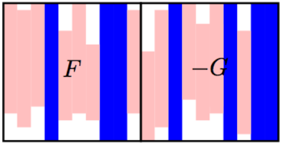


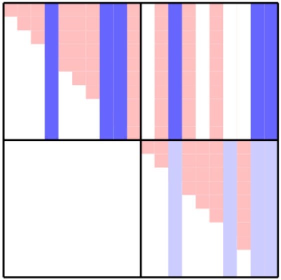


P. Škraba and M. Vejdemo-Johansson, Persistence modules: algebra and algorithms. Mathematics of Computation (submitted, 2013)

Algorithm

Computing the pullback





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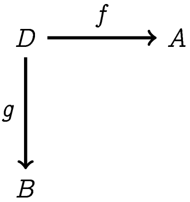
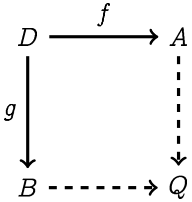
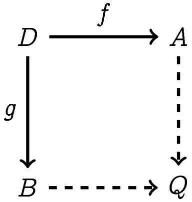
Pushout



Pushout



Pushout



Compute $\mathrm{coker}(D \xrightarrow{(f,g)} A \oplus B)$

Esakia Duality

Using a duality for Heyting algebras

Algebraic Properties

HA := Heyting algebras and homomorphisms $\cong$ **Esa** := Esakia Spaces and homeomorphisms

Algebraic Properties

Esakia Spaces

An Esakia Space $(X, \leq \tau)$ is a set X equipped with a partial order $\leq$ and a topology τ such that:

- (X, τ) is compact;
- $x \not\leq y$ implies $\exists$ U of X st. $x \in U$ and $y \notin U$;
- for each clopen C of (X, τ), the ideal $\downarrow C$ is clopen.

Esakia spaces are *Hausdorff* and *zero-dimensional*, constituting Stone spaces.

Algebraic Properties

Esakia Duality for Standard Persistence

join-irreducibles: all nonzero elements

basic opens: $N_a = \{\, I \text{ prime ideal} \mid a \in I \,\}$

$\tau = \langle N_a, X - N_a \mid a \in X \rangle$

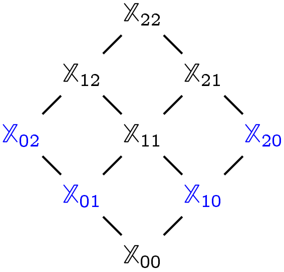
Algebraic Properties

Esakia Duality for Multidimensional Persistence

join-irreducibles: $\mathbb{X}_{0i}$ and $\mathbb{X}_{j0}$ with $i \neq j$

basic opens: $N_a = \{\, I \text{ prime ideal} \mid a \in I \,\}$

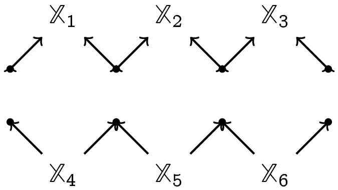
$\tau = \langle N_a, X - N_a \mid a \in X \rangle$



Other applications

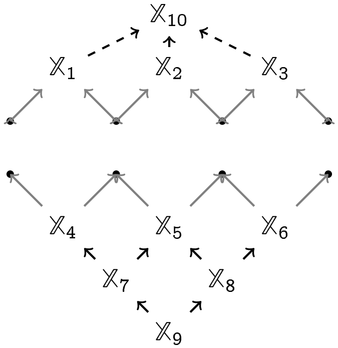
Other applications in the framework

The Largest Injective



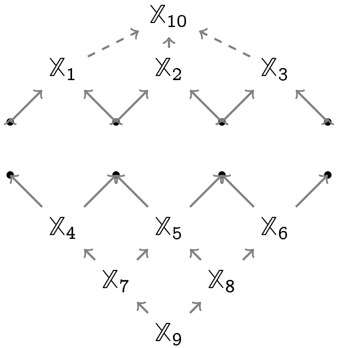
$$\mathrm{im}\left(\mathrm{H}_*(\mathbb{X}_i) \wedge \mathrm{H}_*(\mathbb{X}_j) \to \mathrm{H}_*(\mathbb{X}_i) \vee \mathrm{H}_*(\mathbb{X}_j)\right) \qquad \forall i, j$$

The Largest Injective



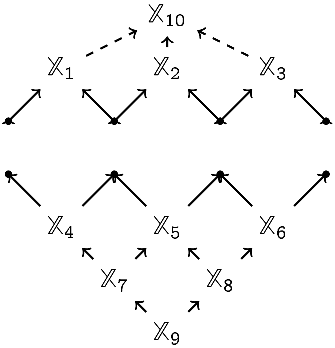
$$\mathrm{im}\left(\mathrm{H}_*(\mathbb{X}_i) \wedge \mathrm{H}_*(\mathbb{X}_j) \to \mathrm{H}_*(\mathbb{X}_i) \vee \mathrm{H}_*(\mathbb{X}_j)\right) \qquad \forall i, j$$

The Largest Injective



$$\mathrm{im}\left(\bigwedge_i \mathrm{H}_*(\mathbb{X}_j) \to \bigvee_i \mathrm{H}_*(\mathbb{X}_i)\right)$$

The Largest Injective



$$\operatorname{im}\left(\bigwedge_{i\in\mathtt{sources}} \mathrm{H}_*(\mathbb{X}_j) \to \bigvee_{j\in\mathtt{sinks}} \mathrm{H}_*(\mathbb{X}_j)\right)$$

Algorithmic Applications

Bifiltrations: associativity.

$$
\begin{array}{ccccccc}
\mathbb{X}_{03} & \longrightarrow & \mathbb{X}_{13} & \longrightarrow & \mathbb{X}_{23} & \longrightarrow & \mathbb{X}_{33} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
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\end{array}
$$

Algorithmic Applications

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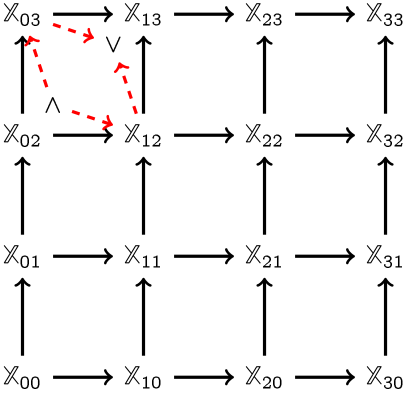

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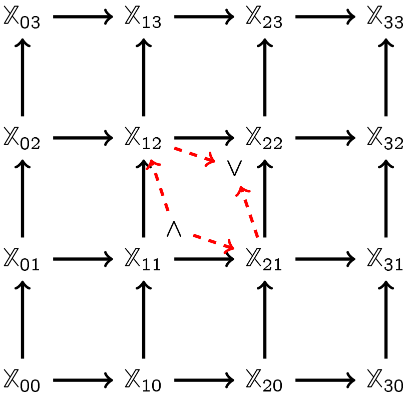

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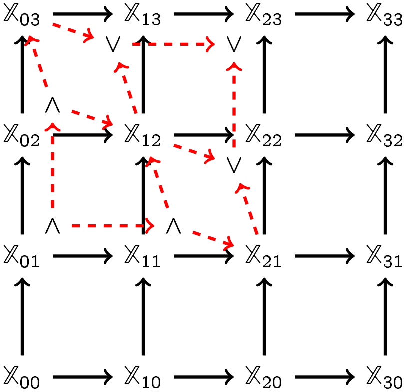
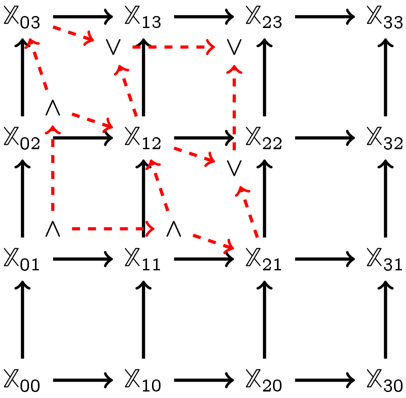


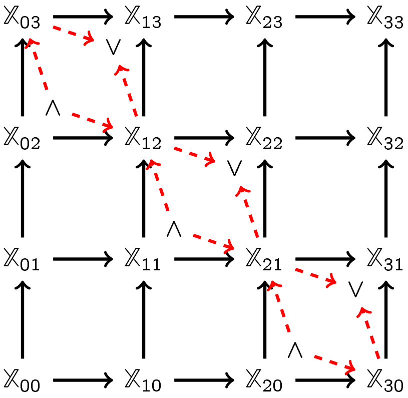
Algorithmic Applications

Bifiltrations: associativity.



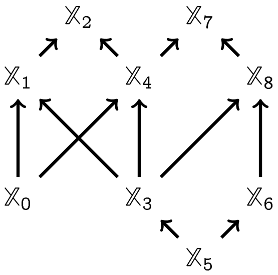
Algorithmic Applications

Bifiltrations: sections.



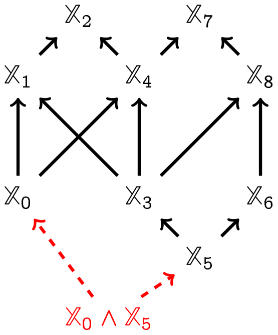
Algorithmic Applications

General Diagrams: common features.



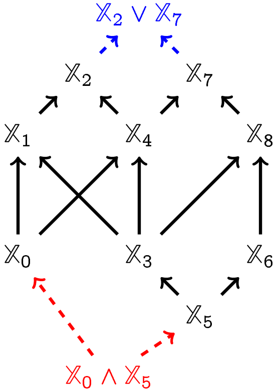
Algorithmic Applications

General Diagrams: common features.



Algorithmic Applications

General Diagrams: common features.



Algorithmic Applications

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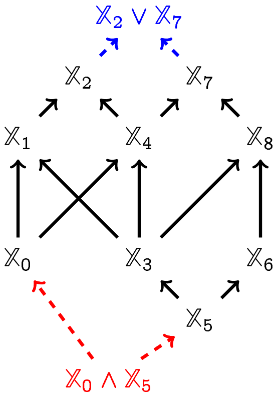
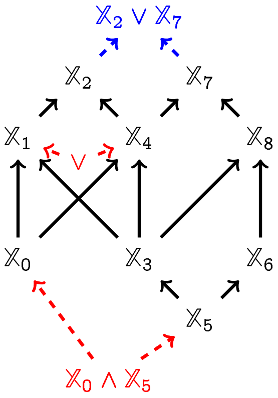


Algorithmic Applications

General Diagrams: associativity.



Algorithmic Applications

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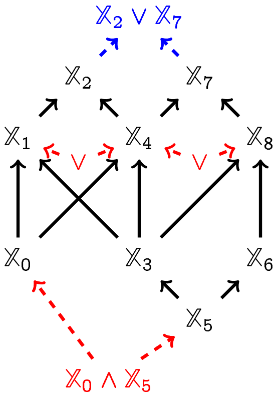

Algorithmic Applications

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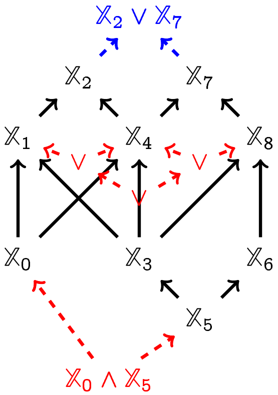
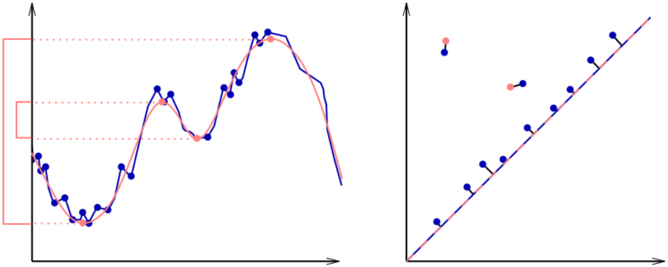


Stability

Stability Teorems

Stability

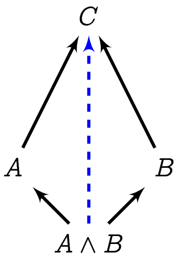
Stability of the Persistence Diagram.



D Cohen-Steiner, H Edelsbrunner, and J Harer, Stability of persistence diagrams. Discrete Comput Geom (2005)

Stability

Stability



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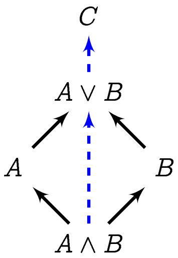

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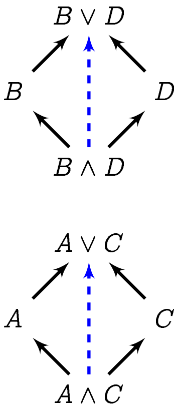

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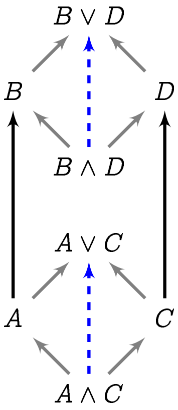

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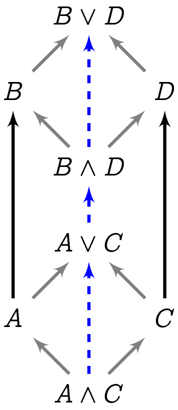


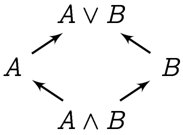
Stability

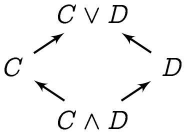
Stability

$A \quad B \quad C \quad D$

Stability

Stability





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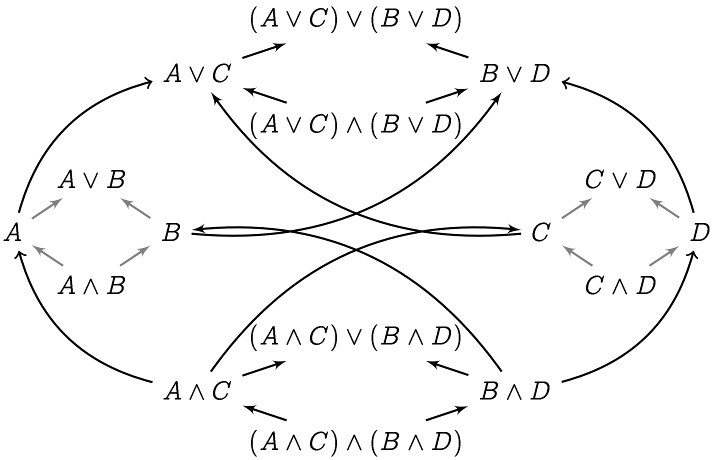

Stability

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