Model theory, stability, applications

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June 6, 2013

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- There was a greater interest in mathematical rigor, and a concern whether reasoning involving certain infinite quantities was sound.
- In addition to logicians such as Cantor, Frege, Russell, major mathematicians of the time such as Hilbert and Poincaré participated in these developments.
- Out of all of this came the beginnings of mathematical accounts of higher level or "metamathematical" notions such as set, truth, proof, and algorithm (or effective procedure).

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- Classical foundational issues are still present in modern mathematical logic, especially set theory.
- But various relations between logic and other areas have developed: set theory has close connections to analysis, proof theory to computer science, category theory and recently homotopy theory.
- We will discuss in more detail the case of model theory. Early developments include Malcev's applications to group theory, Tarski's analysis of definability in the field of real numbers, and Robinson's rigorous account of infinitesimals (nonstandard analysis).

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- "First order" refers to the quantifiers ranging over elements or individuals rather than sets.

Model theory II

- A *model* of T is simply a first order structure M consisting of an underlying set or universe M together with a distinguished collection of relations (subsets of M^n), functions $M^n \to M$ and "constants" corresponding to the symbols of L, in which the sentences of T are true. It is natural to allow several universes (many-sorted framework).

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- On the other hand, the axioms for topological spaces, and topological spaces themselves have on the face of it a "second order" character. (A set X is given the structure of a topological space by specifying a collection of *subsets* of X satisfying various properties..).

Definable sets I

- Another key notion is that of a definable set.
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- Our familiar number systems already provide quite different behaviour or features of definable sets.

Definable sets II

- In the structure $(\mathbb{N}, +, \times, 0, 1)$, subsets of $\mathbb{N}$ definable by formulas $\phi(x)$ which begin with a sequence of quantifiers $\exists y_1 \forall y_2 \exists y_3 ... \forall y_n$ get more complicated as n increases.

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- The collection of definable subsets of $\mathbb{N}$ is called the arithmetical hierarchy, and already with one existential quantifier we can define "noncomputable" sets. In fact the study of definability in $(\mathbb{N}, +, \times, 0, 1)$ is precisely recursion theory.

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- However in the structure $(\mathbb{R}, +, \cdot)$, the hierarchy collapses, one only needs one block of existential quantifiers to define definable sets. Moreover the definable sets have a geometric feature: they are the so-called semialgebraic sets, namely finite unions of subsets of $\mathbb{R}^n$ of form $\{\bar{x} : f(\bar{x}) = 0 \wedge \bigwedge_{i=1,..k} g_i(\bar{x}) > 0\}$ where f and the g_i are polynomials with coefficients from $\mathbb{R}$.

Definable sets III

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- The model-theoretic problem of describing definable sets in $(\mathbb{C}, +, \times)$ up to definable bijection is essentially the same as the central problem in algebraic geometry, namely classification of algebraic varieties up to birational isomorphism.
- But the model theory of the structure $(\mathbb{C}, +, \times)$ or its first order theory ACF_0, has little bearing on the problem, and it is rather definability in richer (but still tame) structures such as fields equipped with a derivation, valuation, automorphism... which has consequences and applications.

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- Attached to a first order theory there are at least two categories, $Mod(T)$ the category of models of T, and $Def(T)$ the category of definable sets, where the latter can be identified with $Def(M)$, the category of definable sets in a "big" model M of T.

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- The classification of first order theories concerns finding meaningful dividing lines. The "logically perfect" first order theories are the stable theories, to be discussed below.

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- For example the *spectrum problem*: given a (complete) theory, we have the function $I(-T)$ from (infinite) cardinals to cardinals, where $I(\kappa, T)$ is the number of models of T of cardinality κ, up to isomorphism. What are the possible such functions, as T varies?

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- Shelah solved the problem for countable theories, in the process identifying the class of *stable* first order theories, and developing stability theory, the detailed analysis of the categories $Mod(T)$ and $Def(T)$ for an arbitrary stable theory T.

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- The (or a) definition of stability is not particularly enlightening but is a good example of a "model-theoretic" property: T is stable if there is no model M of T definable relation $R(x, y)$ and $a_i, b_i \in M$ for $i = 1, 2, ..$ such that $R(a_i, b_j)$ if $i < j$.

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- ACF_0 is the canonical example of a stable theory. Another (complete) example is the theory of infinite vector spaces over a fixed division ring.
- More recently it was discovered (Sela) that the first order theory of the free group $(F_2, \cdot)$ is stable, yielding new connections between model theory and geometric group theory.

Finite rank stable theories I

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- The building blocks of all definable sets in a finite rank stable theory (in a sense that I will say something about if there is time) are what I will call the *minimal* definable sets.

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- A very influential conjecture of Boris Zilber was that that any minimal set (in a finite rank stable theory) is of three possible (mutually exclusive) types:
- (a) "field like": up to $\sim$, X has definably the structure of an algebraically closed field

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- (b) "vector space like": up to $\sim$, X has a definable commutative group structure such that moreover any definable subset of $X \times ..X$ is up to finite Boolean combination and translation, a definable subgroup.

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- A counterexample was found by Hrushovski in the late 80's, and the methods for constructing such examples have become a subarea of model theory.
- However the conjecture has been proved for some very rich finite rank stable theories (originally via so-called Zariski geometries, but other proofs were found later), and in the last part of the talk (if there is time) I will discuss a couple of examples and applications with algebraic-geometric and number-theoretic features.

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- Algebraic geometry (ACF_0) lives on the field of constants $\mathcal{C}$.
- The Zilber conjecture is valid in this context (and gave rise to new results in diophantine geometry over function fields): minimal sets of type (a) are algebraic curves in $\mathcal{C}$, minimal sets of type (b) are related to simple "nonconstant" abelian varieties, and there is an interest in identifying minimal sets of type (c).

- A recent application (with J, Nagloo) is to transcendence (algebraic independence) questions regarding an intensively studied class of ordinary differential equations, in the complex domain, namely the Painlevé equations.

Theorem 0.1

Consider the Painlevé II family of second order ODE's: $y'' = 2y^3 + ty + \alpha$ *where* $\alpha \in \mathbb{C}$. *Then the solution set* Y_α *of the relevant equation (as a definable set in* $\mathcal{U}$*) is strongly minimal iff* $\alpha \notin \mathbb{Z} + 1/2$, *and moreover for all such* α, Y_α *is of type (c) (trivial). Moreover any "generic" equation in each of the Painlevé families I-VI, is strongly minimal and "strongly trivial" implying that if* $y_1, .., y_n$ *are distinct solutions, then* $y_1, y_1', ..., y_n, y_n'$ *are algebraically independent over* $\mathbb{C}(t)$.

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- Some time ago, with Scanlon, we showed that strongly minimal compact complex manifolds of type (b) are (nonalgebraic simple) complex tori.
- Minimal sets of kind (c) are quite rare, and it is conjectured that in the Kähler context they are closely related to hyperkäeler manifolds, another intensively studied class of compact complex manifolds.