On the Bergman property for clones

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(joint work with Maja Pech)

Outline

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Definition of cofinality for clones

Observation

If a clone $\mathbb{F}$ is non-finitely generated, then it can be approximated by a chain of proper subclones $\langle \mathbb{F}^{(1)} \rangle \leq \langle \mathbb{F}^{(2)} \rangle \leq \cdots$.

Question

In general, what is the minimal possible length of such a chain?

"Answer"

It is some regular cardinal...

Definition (Cofinality of a clone)

Let $\mathbb{F}$ be a non-finitely generated clone.
By $\mathrm{cf}(\mathbb{F})$ we denote the least cardinal λ such that there exists a chain $(\mathbb{F}_i)_{i<\lambda}$ such that

1. $\forall i < \lambda : \mathbb{F}_i < \mathbb{F}$,
2. $\cup_{i<\lambda} \mathbb{F}_i = \mathbb{F}$.

Observations

- Countable clones are either finitely generated or have cofinality $\aleph_0$.
- Therefore the concept of cofinality becomes interesting only for clones on infinite sets.
- Examples for very large clones are the polymorphism clones of certain homogeneous structures.

Lemma

If $\mathbb{F} \leq O_A$ *has uncountable cofinality, then*

$$\exists n \in \mathbb{N}_+ : \mathbb{F} = \langle \mathbb{F}^{(n)} \rangle_{O_A}.$$

Motivating questions

1. Does the polymorphism clone of the Rado-graph have uncountable cofinality?
2. Does the clone O_A of all functions on an infinite set A have uncountable cofinality?
3. What about other homogeneous structures?

Outline

Relative rank of clones

We adapt Ruškuc' notion of relative rank for semigroups to clones:

Let $\mathbb{F}$ be a clone, and let $M \subseteq \mathbb{F}$.

Definition

A subset $N \subseteq \mathbb{F}$ is called **generating set of $\mathbb{F}$ modulo M** if

$$\langle M \cup N \rangle_{O_A} = \mathbb{F}.$$

The **relative rank of $\mathbb{F}$ modulo M** is the smallest cardinal of a generating set of $\mathbb{F}$ modulo M.

It is denoted by

$$\text{rank}(\mathbb{F} : M)$$

Cofinality and relative rank

Proposition

Let $\mathbb{F} \leq O_A$, $\mathbb{S} \subseteq \mathbb{F}^{(1)}$ *be a transformation semigroup.*
If $\mathrm{cf}(\mathbb{S}) > \aleph_0$ *and if* $\mathrm{rank}(\mathbb{F} : \mathbb{S})$ *is finite, then*

$$\mathrm{cf}(\mathbb{F}) > \aleph_0, \quad \textit{too.}$$

Some concrete cofinality results

Let R denote the Rado-graph.

Observation from Maja's talk

The relative rank of $\mathrm{Pol}(R)$ modulo $\mathrm{End}(R)$ is equal to 1.

Theorem (Dolinka 2012)

$$\mathrm{cf}(\mathrm{End}\,R) > \aleph_0.$$

Corollary

$\mathrm{cf}(\mathrm{Pol}\,R) > \aleph_0$.

Theorem (Malcev, Mitchel, Ruškuc 2009)

For every infinite set A holds $\mathrm{cf}(O_A^{(1)}) > \aleph_0$.

From the proof of Sierpiński's Theorem we have:

The relative rank of O_A modulo $O_A^{(1)}$ is equal to 1.

Corollary

For every infinite set A holds $\mathrm{cf}(O_A) > \aleph_0$.

Outline

Ages

Definition

A class of finitely generated countable structures is called an **age** if it is obtainable as the class of all finitely generated structures that embedd into a given fixed countable structure.

Hereditary property (HP)

$\mathcal{K}$ has the (HP) if $\forall \mathbf{A} \in \mathcal{K}$ if $\mathbf{B} \hookrightarrow \mathbf{A}$, then also $\mathbf{B} \in \mathcal{K}$.

Joint embedding property (JEP)

$\mathcal{K}$ has the (JEP) if

$$\forall \mathbf{A}, \mathbf{B} \in \mathcal{K} \, \exists \mathbf{C} \in \mathcal{K} : \mathbf{A} \hookrightarrow \mathbf{C},\ \mathbf{B} \hookrightarrow \mathbf{C}.$$

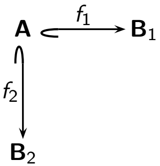
Theorem (Fraïssé)

$\mathcal{K}$ is an age if and only if it contains up to isomorphism only countably many structures, it has the (HP) and the (JEP).

Fraïssé-classes

Amalgamation property (AP)

$\mathcal{K}$ has the (AP) if for all $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ and for all $f_1 : \mathbf{A} \hookrightarrow \mathbf{B}_1$, $f_2 : \mathbf{A} \hookrightarrow \mathbf{B}_2$, there exist $\mathbf{C} \in \mathcal{K}$, $g_1 : \mathbf{B}_1 \hookrightarrow \mathbf{C}$, $g_2 : \mathbf{B}_2 \hookrightarrow \mathbf{C}$, such that the following diagram commutes:



Definition

An age $\mathcal{K}$ is called **Fraïssé-class** if it has the amalgamation property (AP).

Theorem (Fraïssé 1953)

1. *$\mathcal{K}$ is a Fraïssé-class $\iff$ $\mathcal{K}$ is the age of a countable homogeneous structure,*
2. *any two countable homogeneous structures of the same age are isomorphic.*

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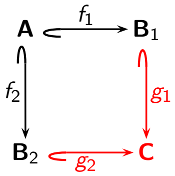

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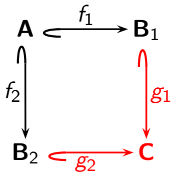

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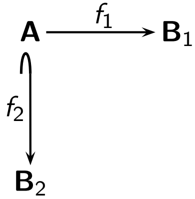
Homo amalgamation property (HAP)

$\mathcal{K}$ has the (HAP) if for all $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$, for all homomorphisms $f_1 : \mathbf{A} \to \mathbf{B}_1$, $f_2 : \mathbf{A} \hookrightarrow \mathbf{B}_2$ there exist $\mathbf{C} \in \mathcal{K}$, $g_1 : \mathbf{B}_1 \hookrightarrow \mathbf{C}$, and $g_2 : \mathbf{B}_2 \to \mathbf{C}$, such that the following diagram commutes:



Theorem (Dolinka 2011)

A countable homogeneous structure $\mathbf{A}$ *is homomorphism homogeneous if and only if* $\mathrm{Age}(\mathbf{A})$ *has the (HAP).*

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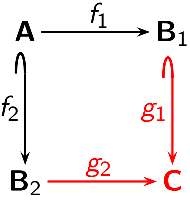

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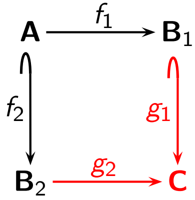

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Strict Fraïssé-classes

If $\mathcal{K}$ is an age, then $\overline{\mathcal{K}} := \{\mathbf{A} \mid \mathbf{A} \text{ countable}, \ \mathrm{Age}(\mathbf{A}) \subseteq \mathcal{K}\}$.

Definition (Dolinka 2011)

A Fraïssé-class $\mathcal{K}$ of relational structures is called **strict Fraïssé-class** if every pair of morphisms in $(\mathcal{K}, \hookrightarrow)$ with the same domain has a pushout in $(\overline{\mathcal{K}}, \to)$.

Observation

Note that these pushouts will always be amalgams. Thus the strict amalgamation property postulates canonical amalgams.

Theorem (Dolinka 2011)

Let $\mathbf{U}$ *be a countable homogeneous structure of age* $\mathcal{K}$*. If*

1. $\mathcal{K}$ *has the strict amalgamation property,*
2. $\mathcal{K}$ *has the (HAP),*
3. *the coproduct of* $\aleph_0$ *copies of* $\mathbf{U}$ *exists and if its age is contained in* $\mathcal{K}$*,*
4. $|\operatorname{End}\mathbf{U}| > \aleph_0$.

Then $\operatorname{cf}(\operatorname{End}\mathbf{U}) > \aleph_0$.

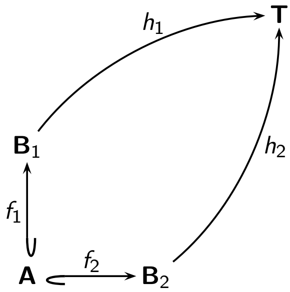
Remark

Dolinka shows more: that End $\mathbf{U}$ *has uncountable* **strong** *cofinality.*

Outline

Kubiś's amalgamated extension property

Let $\mathcal{K}$ be a class of countable, finitely generated structures. We say that $\mathcal{K}$ has the **amalgamated extension property** if



Remark

The strict amalgamation property implies the amalgamated extension property.

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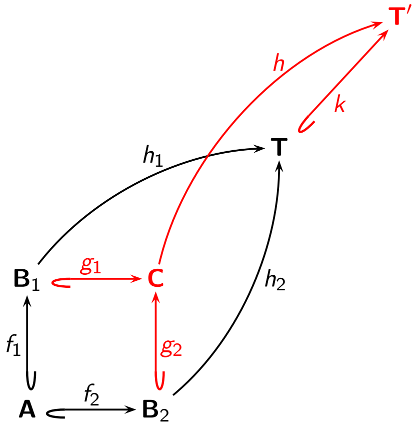

Remark

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Generating polymorphism clones of homogeneous structures

Let us recall a Theorem from Maja's talk:

Theorem

Let $\mathbf{U}$ *be a countable homogeneous structure of age* $\mathcal{K}$ *such that*

1. $\mathcal{K}$ *is closed with respect to finite products,*
2. $\mathcal{K}$ *has the (HAP),*
3. $\mathcal{K}$ *has the amalgamated extension property.*

Then $\operatorname{rank}(\operatorname{Pol}\mathbf{U} : \operatorname{End}\mathbf{U}) = 1$

Now we are ready to combine Dolinka's result with the above given Theorem:

Cofinality of polymorphism clones of homogeneous structures

Theorem

Let $\mathbf{U}$ be a countable homogeneous structure of age $\mathcal{K}$. If

1. *$\mathcal{K}$ has the strict amalgamation property,*
2. *$\mathcal{K}$ is closed with respect to finite products,*
3. *$\mathcal{K}$ has the (HAP),*
4. *the coproduct of $\aleph_0$ copies of $\mathbf{U}$ exists and its age is contained in $\mathcal{K}$,*
5. *$|\operatorname{End}\mathbf{U}| > \aleph_0$.*

Then

$$\operatorname{cf}(\operatorname{Pol}\mathbf{U}) > \aleph_0.$$

Examples

The polymorphism clones of the following structures have uncountable cofinality:

- the Rado graph,
- the countable generic poset $\mathbb{P} = (P, \leq)$,
- the countable atomless Boolean algebra,
- the countable universal homogeneous semilattice,
- the countable universal homogeneous distributive lattice,
- the vector-space $\mathbb{F}^\omega$ for any countable field $\mathbb{F}$.

Theorem (Bergman 2006)

Let A be an infinite set. $G = \mathrm{Sym}(A)$ be the group of all permutations of A.
Then every connected Cayley graph of G has finite diameter.

Definition

Any group with this property if said to have the **Bergman property**.

Remark

- *Bergman showed the Bergman-property of* $\mathrm{Sym}(A)$ *to give an alternative proof for the uncountable cofinality of* $\mathrm{Sym}(A)$ *(original proof by Macpherson and Neumann),*
- *Droste and Göbel generalized Bergman's ideas to many other groups,*
- *The Bergman property was defined for semigroups by Maltcev, Mitchel, and Ruškuc.*

The Bergman property for semigroups

Definition (Maltcev, Mitchel, Ruškuc 2009)

A semigroup $\mathbb{S}$ has the **Bergman-property** if for every $U \subseteq \mathbb{S}$ holds

$$U^{+} = S \Rightarrow \exists n \in \mathbb{N}_{+} : \mathbb{S} = \bigcup_{i=1}^{n} U^{i}.$$

Remark

Dolinka (2011) showed the Bergman property for the endomorphism monoids of many homogeneous structures (with the HAP).

The Bergman property for clones

Definition

A clone $\mathbb{F}$ is said to have the **Bergman-property** if for every generating set H of $\mathbb{F}$ and every $k \in \mathbb{N} \setminus \{0\}$ there exists some $n \in \mathbb{N}$ such that every k-ary function from F can be represented by a term of depth at most n from the functions in H.

The main result

Theorem

Let $\mathbf{U}$ *be a countable homogeneous structure of age* $\mathcal{K}$*, such that*

1. $\mathcal{K}$ *has the strict amalgamation property,*
2. $\mathcal{K}$ *is closed with respect to finite products,*
3. $\mathcal{K}$ *has the HAP,*
4. *the coproduct of countably many copies of* $\mathbf{U}$ *in* $(\overline{\mathcal{K}}, \to)$ *exists,*
5. $\operatorname{End}\mathbf{U}$ *is not finitely generated.*

Then $\operatorname{Pol}\mathbf{U}$ *has the Bergman property.*

Strategy of the proof

- We define the notion of strong cofinality for clones,
- we show that a clone has uncountable strong cofinality if and only if it has uncountable cofinality and the Bergman property,
- we show that the clones in question have uncountable strong cofinality.

Definition of strong cofinality for clones

For a set U of functions, by $U^{[k,2]}$ we denote the set of k-ary functions definable from U by terms of depth at most 2.

Definition

For a clone $\mathbb{F} \leq O_A$ and a cardinal λ, a chain $(U_i)_{i<\lambda}$ of proper subsets of $\mathbb{F}$ is called **strong cofinal chain** of length λ for $\mathbb{F}$ if

1. $\bigcup_{i<\lambda} U_i = F$,
2. there exists a $k_0 \in \mathbb{N} \setminus \{0\}$ such that for all $i < \lambda$ and $k \in \mathbb{N} \setminus \{0\}$ with $k \geq k_0$ holds $U_i^{(k)} \subsetneq F^{(k)}$,
3. for all $i < \lambda$ there exists some $j < \lambda$ such that for all $k \in \mathbb{N} \setminus \{0\}$ holds $U_i^{[k,2]} \subseteq U_j$.

The **strong cofinality** of $\mathbb{F}$ is the least cardinal λ such that there exists a strong cofinal chain of length λ for $\mathbb{F}$.

Examples

The polymorphism clones of the following structures have the Bergman property:

- the Rado graph,
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