Fractional Universal Algebra

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The Valued Constraint Satisfaction Problem (VCSP)

- Fix a finite set D.
- A valued constraint (of arity m) over a set of variables V is an expression of the form $R(\mathbf{x})$ where $R : D^m \to \mathbb{Q}_{\geq 0} \cup \{\infty\}$ and $\mathbf{x} \in V^m$.
- The Valued Constraint Satisfaction Problem (VCSP):
  - An instance I of VCSP is a function

$$R_I(x_1, \ldots, x_n) = \sum_{i=1}^{q} R_i(\mathbf{x}_i)$$

    where each $R_i(\mathbf{x}_i)$ is a valued constraint over $V_I = \{x_1, \ldots, x_n\}$.
  - The goal is to find a mapping $\varphi : V_I \to D$ that minimises R_I.

- Valued constraint language = any finite set Γ of functions on D.
- $\mathrm{VCSP}(\Gamma)$ = all VCSP instances in which every R_i is from Γ.
- Want: full classification of problems $\mathrm{VCSP}(\Gamma)$ wrt tractability (assuming **PTIME** $\neq$ **NP**).

Special case 1

- Recall: minimize $R_I(x_1, \ldots, x_n) = \sum_{i=1}^{q} R_i(\mathbf{x}_i)$
- Special case 1: $\mathrm{Im}(R) = \{0, \infty\}$ for each $R \in \Gamma$, $\mathrm{VCSP}(\Gamma) = \mathrm{CSP}(\Gamma)$.
- Computational issue: feasibility, not optimisation
- Galois correspondence between relations and operations
- f is a polymorphism of R if

$$\begin{array}{cccccc} & f & \ldots & f & \\ R & (a_{11}, & \ldots & , a_{1m}) & = 0 \\ \vdots & \vdots & & \vdots & \vdots \\ R & (a_{n1}, & \ldots & , a_{nm}) & = 0 \\ \hline R & (b_1, & \ldots & , b_m) & = 0 \end{array}$$

- Tractability of $\mathrm{CSP}(\Gamma)$ is characterised by polymorphisms of Γ
- Polymorphisms form clones - superposition-closed sets of operations
- Much progress via clones, universal algebras, varieties.
- Dichotomy Conjecture: tractable if Taylor polymorphism, **NP**-c o/w

Special case 2

- Recall: minimize $R_I(x_1, \ldots, x_n) = \sum_{i=1}^{q} R_i(\mathbf{x}_i)$
- Special case 2: $\mathrm{Im}(R) \subseteq \mathbb{Q}_{\geq 0}$ for each $R \in \Gamma$.
- Finite-valued VCSPs
- Computational issue: optimisation, not feasibility
- Galois correspondence between rational-valued functions and fractional operations [Cohen, Cooper, Jeavons'06]
  - Proof uses Farkas' lemma from the theory of linear programming
- Tractability characterised by fractional polymorphisms.

Fractional polymorphisms

- An n-ary fractional operation on D is a probability distribution μ on the set $\{f \mid f : D^n \to D\}$ of all n-ary operations on D.
- For a function $R : D^n \to \mathbb{Q}_{\geq 0}$, a fractional operation μ is said to be a fractional polymorphism of R if, for all $\mathbf{a}_1, \ldots, \mathbf{a_n} \in D^m$,

$$\mathbb{E}_{f\sim\mu}(R(f(\mathbf{a}_1, \ldots, \mathbf{a}_n))) \leq \frac{1}{n} \cdot (R(\mathbf{a}_1) + \ldots R(\mathbf{a}_n)),$$

or, in expanded form,

$$\sum_{f:D^n \to D} \Pr_{\mu}[f] \cdot R(f(\mathbf{a}_1, \ldots, \mathbf{a}_n)) \leq \frac{1}{n}(R(\mathbf{a}_1) + \ldots + R(\mathbf{a}_n)).$$

- For a function $R : D^n \to \mathbb{Q}_{\geq 0}$, being submodular on a lattice $(D, \vee, \wedge)$ means having the binary fractional polymorphism with $\Pr[\vee] = \Pr[\wedge] = 1/2$

$$\frac{1}{2} \cdot R(\mathbf{a}_1 \vee \mathbf{a}_2) + \frac{1}{2} \cdot R(\mathbf{a}_1 \wedge \mathbf{a}_2) \leq \frac{1}{2} \cdot (R(\mathbf{a}_1) + R(\mathbf{a}_2)).$$

Dichotomy for finite valued VCSPs

Theorem (Thapper, Živný '12; Kolmogorov '12; Thapper, Živný '13)

$\mathrm{VCSP}(\Gamma)$ *is tractable iff* Γ *has a binary commutative fractional polymorphism.*

- A binary fractional polymorphism μ is commutative if each operation f in $\mathrm{supp}(\mu) = \{f \mid \mathsf{Pr}_\mu[f] > 0\}$ is commutative.
- One algorithm based on linear programming works for all tractable cases.
- Proofs use a combination of techniques from LP and clone theory.
- Curiously, the tractability condition is equivalent to requiring either
  - one binary fractional polymorphism μ with commutative f in $\mathrm{supp}(\mu)$, or
  - symmetric fractional polymorphisms of all arities

Tight dichotomy for $|D| = 3$

Theorem (Huber,AK,Powell '12)

Let $|D| = 3$. *If we can name the elements of* D *as* $-1, 0, 1$ *so that*

- Γ *is submodular wrt* $-1 < 0 < 1$ *or*
- Γ *is* α*-bisubmodular for some rational* $\alpha \in (0, 1]$

then $\mathrm{VCSP}(\Gamma)$ *is tractable. Otherwise,* $\mathrm{VCSP}(\Gamma)$ *is* **NP***-hard.*

- submodularity wrt $-1 < 0 < 1$ = having binary fractional polymorphism with $\Pr[\vee] = \Pr[\wedge] = 1/2$ (where $\vee$ = max and $\wedge$ = min wrt $-1 < 0 < 1$)
- α-bisubmodularity wrt $-1 > 0 < 1$ = having binary fractional polymorphism with $\Pr[\vee_0] = \alpha/2$, $\Pr[\vee_1] = (1-\alpha)/2$, $\Pr[\wedge_0] = 1/2$, where
  - $1 \vee_0 -1 = -1 \vee_0 1 = 0$ and $x \vee_0 y = \max(x, y)$ otherwise
  - $1 \vee_1 -1 = -1 \vee_1 1 = 1$ and $x \vee_1 y = \max(x, y)$ otherwise
  - $1 \wedge_0 -1 = -1 \wedge_0 1 = 0$ and $x \wedge_0 y = \min(x, y)$ otherwise

  here max and min are wrt the order $-1 > 0 < 1$.

VCSP: The general case

- Recall: minimize $R_I(x_1, \ldots, x_n) = \sum_{i=1}^{q} R_i(\mathbf{x}_i)$
- Computational issues: both feasibility and optimisation
- Tractability of $\mathrm{VCSP}(\Gamma)$ is determined by weighted polymorphisms of Γ
- An n-ary weighted polymorphism of Γ is a probability distribution on the set of n-ary polymorphisms of Γ

$$\begin{array}{ccccc} & f & \ldots & f & \\ R & (a_{11}, & \ldots & , a_{1m}) & < \infty \\ \vdots & \vdots & & \vdots & \vdots \\ R & (a_{n1}, & \ldots & , a_{nm}) & < \infty \\ \hline R & (b_1, & \ldots & , b_m) & < \infty \end{array}$$

satisfying the same inequality as for fractional polymorphisms.

- Galois correspondence between general-valued functions and weighted operations [Cohen,Cooper,Creed,Jeavons,Živný '13]
- Dichotomies known for special cases:
  - $D = \{0, 1\}$ [Cohen,Cooper,Jeavons,AK '06]
  - Γ contains all 0-1 valued unaries [Kolmogorov,Živný '12]

Fractional universal algebra

- Fractional algebras: algebras with fractional operations
- New challenge: build a theory of fractional algebras
- Cohen et al. started a VCSP-oriented theory of weighted clones: they provide a Galois correspondence and characterisations of Galois-closed sets
- Not much is know beyond that: this direction is wide open!