# Semigroup varieties satisfying $zxx=zx$ and $zkxyw=zkyxw$

**Samuel J.L. KOPAMU**
University of Goroka,
GOROKA, EHP, Papua New Guinea

samkopamu@gmail.com

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$RB = [xyx = x]$ variety of *rectngular bands*

$Z_\ell = [xy = x]$ variety of *left zero bands*

$Z_r = [xy = y]$ variety of *right zero bands*

$N_1 = [x = y]$ variety of *trivial semigroups*

$\Sigma = \lfloor x^2 = x, xy = yx \rfloor$ variety of *semilattices*

$LN = [x^2 = x, zxy = zyx]$ variety of *left normal bands*

$RN = [x^2 = x, xyz = yxz]$ variety of *right normal bands*

$NB = [x^2 = x, zxyw = zyxw]$ variety of *normal bands*

# METHODS OF CONTRUCTING NEW SEMIGROUPS & SEMIGROUP VARIETIES

# NILPOTENT EXTENSIONS

A semigroup $S$ is said to be an *n-nilpotent extension* of the subsemigroup $S^n$ if $S^{n-1} \neq S^n, n \geq 2$ where $S^n$ is the set of all products of $n$ elements

For any semigroup variety $V$, we denote by

$$V^n = \{S : S^n \in V\}$$

the variety of all *n-nilpotent extensions of* $V$

# N-INFLATIONS

A semigroup $S$ is called a $n$-inflation of the subsemigroup $S^n$ if there existst a retractive endomorphism $f: S \to S^n$ such that $a_1 \ldots a_n = f(a_1) \ldots f(a_n)$ for all $a_1, \ldots, a_n \in S$.

For any semigroup variety $V$, we denote by $V^{<n>} = \left\{ S : S^n = f(S) \in V \right\}$ the variety of all $n$-inflations of members of $V$

# (n,m)-CONSTRUCTIONS

For any semigroup variety $V$, and any ordered pair $(n,m)$ of non-negative integers, the class

$$V^{(n,m)} = \{S : S/\theta(n,m) \in V\}$$

forms a variety, and the map $V \mapsto V^{(n,m)}$ forms an injective endomorphism on the lattice of all semigroup varieties.

# THE (1,0)-CONSTRUCTION CASE

Let $(\Gamma,*)$ be a semilattice and $\{S_\alpha : \alpha \in \Gamma\}$ be a family of pairwise disjoint sets indexed by $\Gamma$ such that for each $(\alpha,\beta) \in \Gamma \times \Gamma$ there exists a map $\phi_{\alpha,\beta} : S_\alpha \to S_{\alpha*\beta}$ such that

$$\phi_{\alpha,\beta}\phi_{\alpha*\beta,\gamma} = \phi_{\alpha,\beta*\gamma} \text{ for all } \alpha,\beta,\gamma$$

On the set $\mathrm{S} = \bigcup\{\mathrm{S}_\alpha : \alpha \in \Gamma\}$ define a binary operation $a \bullet b = a\phi_{\alpha,\beta}$.

Then $(\mathrm{S},\bullet)$ forms a semigroup from $\sum^{(1,0)}$. Conversely, every semigroup in this variety is constructed that way.

# SQUARE EXTENSIONS OF SEMILATTICES

Let $(\Gamma,*)$ be a semilattice and $\{S_\alpha : \alpha \in \Gamma\}$ be a family of pairwise disjoint sets indexed by $\Gamma$ such that for each $(\alpha,\beta) \in \Gamma \times \Gamma$ there map $\phi_{\alpha,\beta} : S_\alpha \to S_{\alpha*\beta}$ which satisfy the following conditions :

(i) $\phi_{\alpha,\beta}\phi_{\alpha*\beta,\gamma} = \phi_{\alpha,\beta*\gamma}$ for all $\alpha,\beta,\gamma \in \Gamma$

(ii) $\alpha \in \mathrm{S}_\alpha$ for all $\alpha \in \Gamma$

(iii) $(\alpha)\phi_{\alpha,\beta} = \alpha * \beta$

(iv) $(a)\phi_{\alpha,\alpha} = \alpha$ for all $a \in S_\alpha, \alpha \in \Gamma$

On the set $\mathrm{S} = \bigcup\{\mathrm{S}_\alpha : \alpha \in \Gamma\}$ define a binary operation $a \bullet b = a\phi_{\alpha,\beta}$, for any $a \in S_\alpha, b \in S_\beta$.

Then $(\mathrm{S},\bullet)$ forms a semigroup which we refer to as [*associative*] *square extension of a semilattice.*

The concept of (associative) square extension of an idempotent groupoid (semigroup) was introduced by A. W. Marczak and J. Plonka

See Reference below :

Novi Sad J. Math.
Vol. 32, No.1, 2002, 159 -166
A NOTE ON SQUARE EXTENSIONS OF BANDS
Igor Dolinka

# IDENTITIES FOR NEWLY CONSTRUCTED SEMIGROUPS & VARIETIES

# IDENTITIES FOR NILPOTENT EXTENSIONS

For any variety $V$ of semigroups, the class

$V^n = \left\{S : S^n \in V\right\}$ forms a variety

such that if

$$V = \left[P(x_1,\ldots,x_k) = Q(x_1\ldots x_k)\right]$$

then

$$V^n = \left[P(X_1,\ldots,X_k) = Q(X_1\ldots X_k)\right]$$

where $X_i = y_{i_1}\ldots y_{i_n}, i = 1,\ldots,k$ where $y_{i_t} = y_{j_k}$

if and only if $i = j$ and $t = k$

# IDENTITIES FOR 2-INFLATIONS

For any variety $V$ of semigroups, the class

If $V = [P(x_1,\ldots,x_k) = Q(x_1 \ldots x_k)]$

and if either $P(x_1,\ldots,x_k)$ or $Q(x_1,\ldots,x_k)$

is a word of length 1 then

$$V^{<2>} = \begin{bmatrix} zP(x_1,\ldots,x_k) = zQ(x_1 \ldots x_k) \\ P(x_1,\ldots,x_k)z = Q(x_1 \ldots x_k)z \end{bmatrix}$$

$z \notin \{x_1,\ldots,x_k\}$.

Keep all identities formed by words of length $> 1$

# IDENTITIES FOR (1,0)-CONSTRUCTION

For any variety $V$ of semigroups, the class

$V^{(1,0)} = \{S : S/\theta(1,0) \in V\}$ forms a variety.

If $V = [P(x_1,\ldots,x_k) = Q(x_1\ldots x_k)]$

then

$$V^{(1,0)} = [zP(x_1,\ldots,x_k) = zQ(x_1\ldots x_k)]$$

where $z \notin \{x_1,\ldots,x_k\}$.

# IDENTITIES FOR SQUARE EXTENSIONS

For any variety $V$ of bands, the class

$$V^{sq} = \left\{ S : S^{sq} \in V \right\}$$

forms a semigroup variety such that if

$$V = \left[ P(x_1, \ldots, x_k) = Q(x_1, \ldots, x_k) \right]$$

then

$$V^{sq} = \left[ \begin{matrix} zP(x_1, \ldots, x_k) = zQ(x_1, \ldots, x_k) \\ P(x_1^2, \ldots, x_k^2) = Q(x_1^2, \ldots, x_k^2) \end{matrix} \right]$$

where $z \notin \{x_1, \ldots, x_k\}$.

# SOME EXAMPLES ….

[Igor Dolinka, 2002].

***Every square extension A of a rectangular band I is an inflation of I.***

Result. ***A semigroup is a square extension of a semilattice if and only if it satisfies the pair of identities***

$$zx^2 = zx \text{ and } x^2y^2 = y^2x^2$$

Consider this variety of semigroups comprised of all

**SQUARE EXTENSIONS OF SEMILATTICES.**

$$L_2 = \begin{bmatrix} zx^2 = zx \\ x^2y^2 = y^2x^2 \end{bmatrix}$$

$$= \begin{bmatrix} zx^2 = zx \\ x^2y^2 = y^2x^2 \\ zxy = zyx \end{bmatrix}$$

$$\subseteq \begin{bmatrix} zx^2 = zx \\ zxy = zyx \end{bmatrix} = \sum^{(1,0)}$$

Consider the variety of all

# 2-INFLATIONS OF SEMILATTICES

$$\Sigma^{<2>} = \begin{bmatrix} zx^2 = zx \\ x^2z = x^2z \\ xy = yx \end{bmatrix}$$

$$= \begin{bmatrix} zx^2 = zx, \\ x^2z = xz \\ x^2y^2 = y^2x^2 \end{bmatrix}$$

$$\subseteq \begin{bmatrix} zx^2 = zx \\ x^2y^2 = y^2x^2 \end{bmatrix} = L_2$$

The variety $L_2$ is precisely the class of all associative square extensions of semilattices

The variety $\sum^{(1,0)}$ is precisely the class of all (1,0)-constructed semigroups using semilattices

The variety $\sum^{<2>}$ of all 2-inflations of semilattices and the variety of all 2-nilpotent extensions of semilattices coincide

# VARIETIES CONSTRUCTED USING SEMILATTICES

$$\sum^2 = \sum^{<2>} \subseteq L_2 \subseteq \sum^{(1,0)}$$

where

$$\sum^{<2>} = \left[zx^2 = zx, x^2z = xz, xy = yx\right]$$

2-inflations of semilattices

$$= \left[(xy)^2 = xy, (xy)(wz) = (wz)(xy)\right]$$

2-nilpotent extensions of semilattices

# VARIETIES CONSTRUCTED USING NORMAL BANDS

$$NB^2 \neq NB^{<2>}$$

$$NB^{<2>} = \left[ zx^2 = zx, x^2 z = xz, zxyw = zyxw \right]$$

2-inflations of normal bands

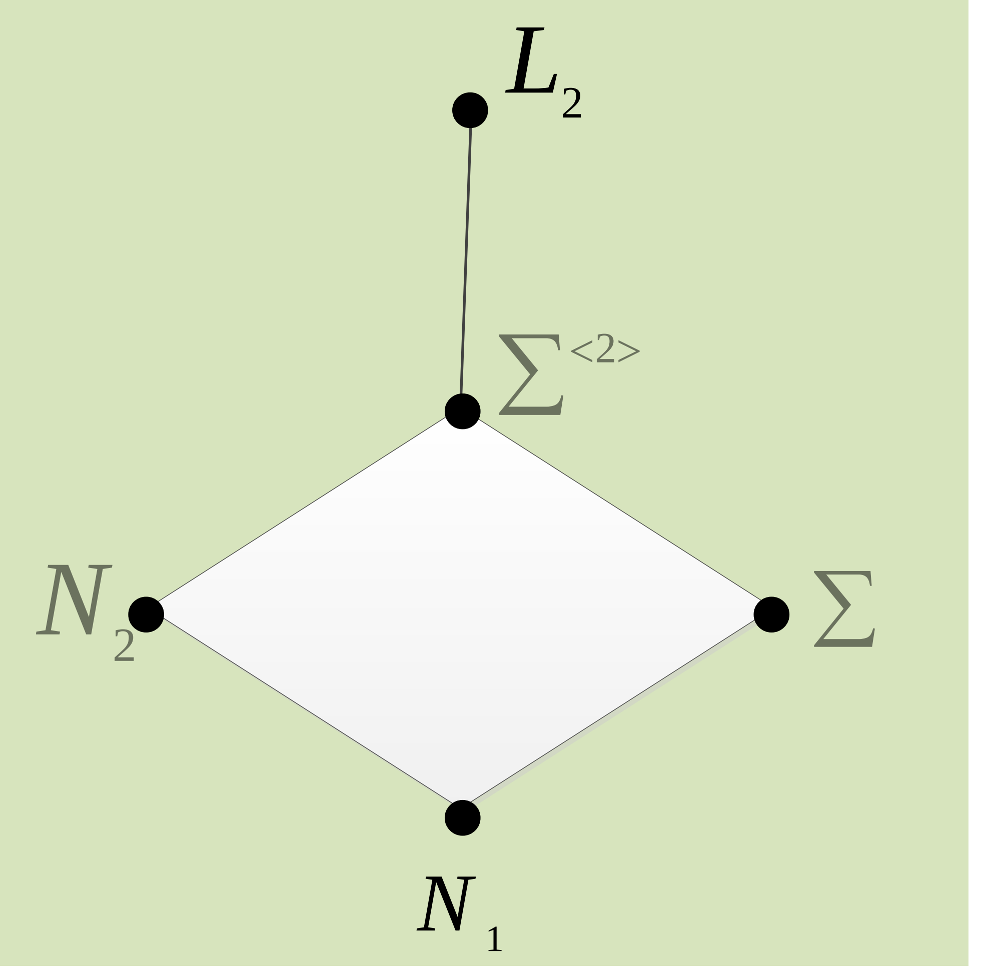
$$NB^2 = \left[ \begin{array}{l} (xy)^2 = xy, \\ (ab)(xy)(wz)(cd) = (ab)(wz)(xy)(cd) \end{array} \right]$$

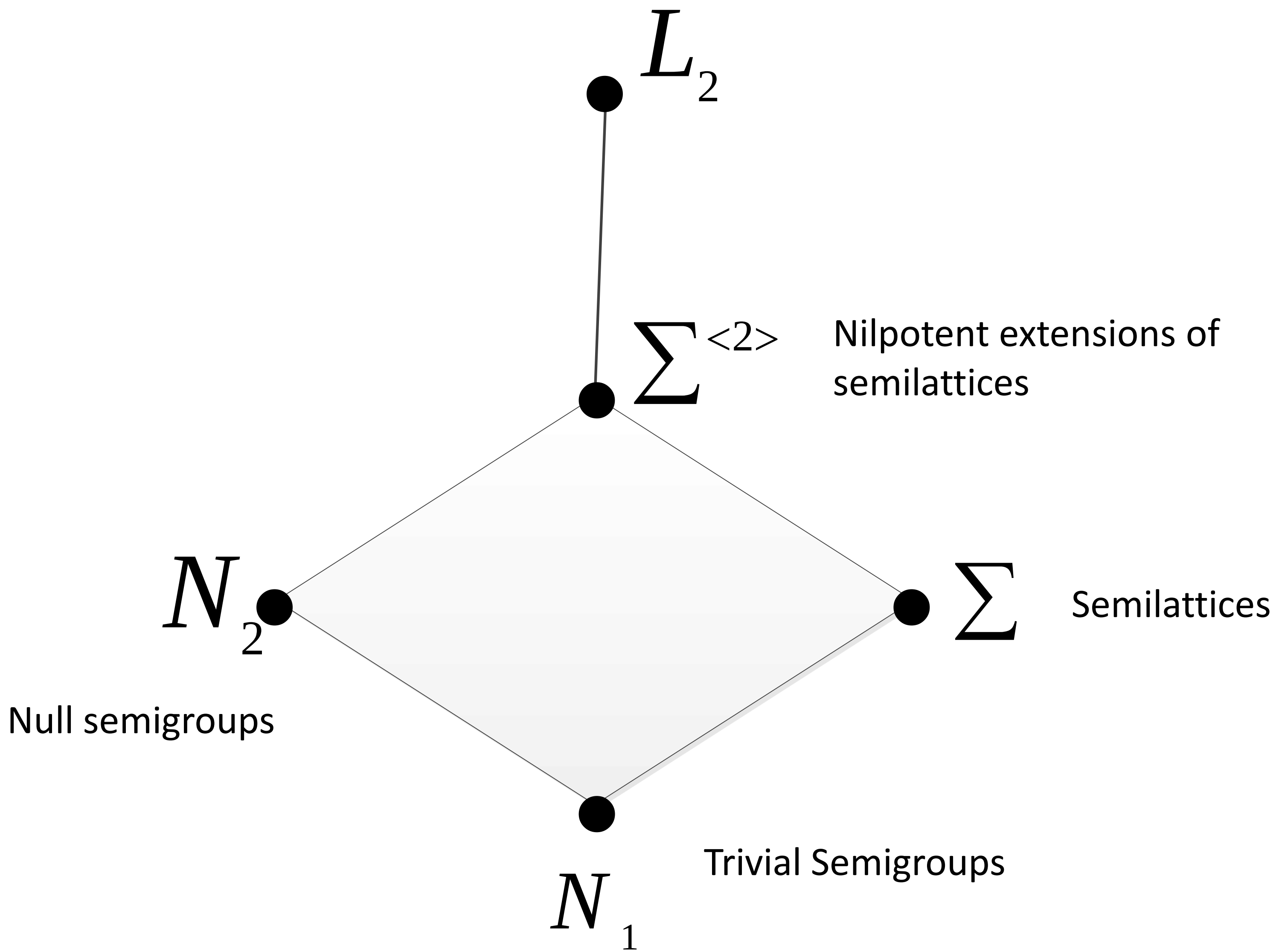
2-nilpotent extensions of normal bands

# SUBVARIETY LATTICES

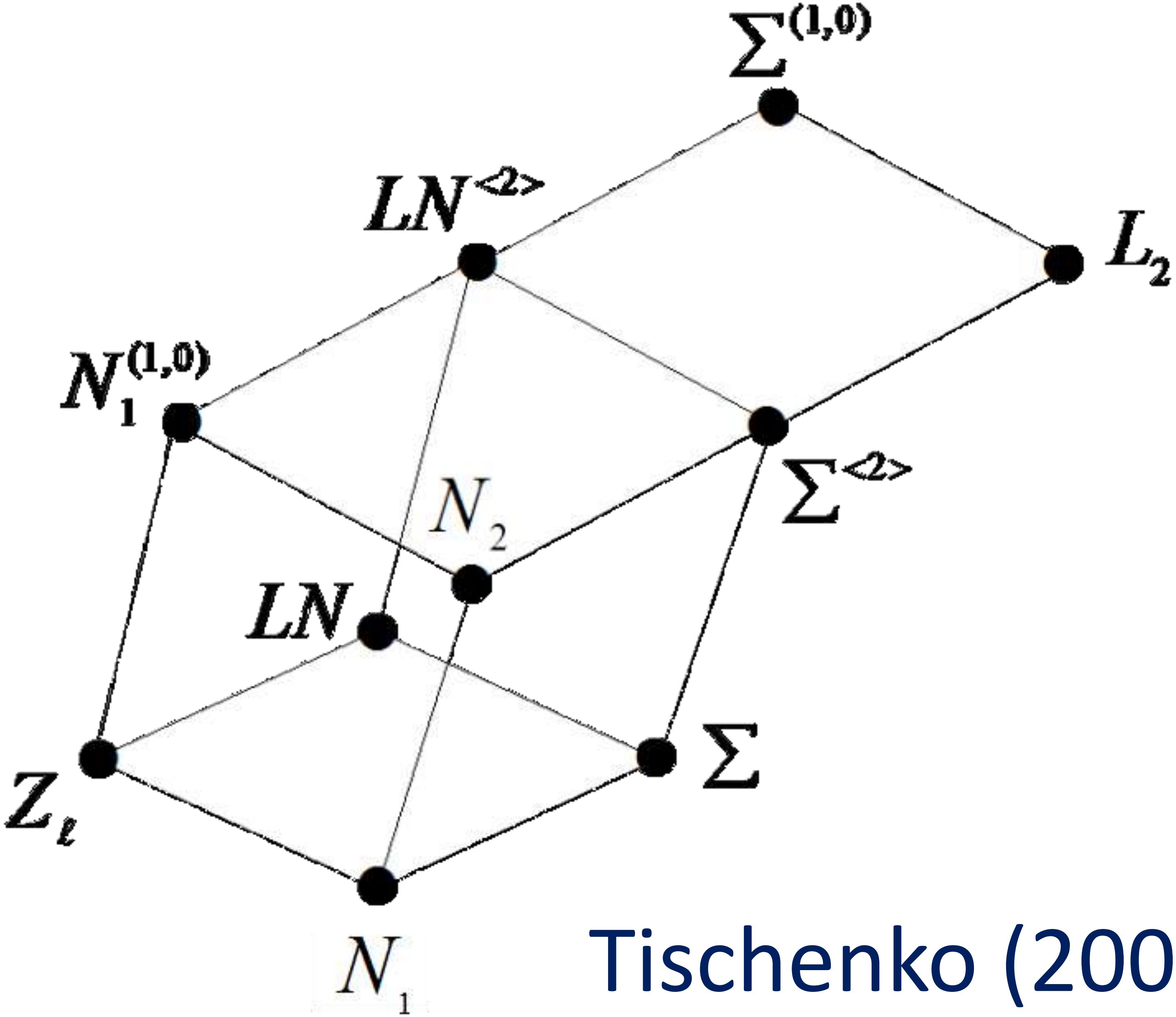
**The lattice of all subvarieties of the variety of all**

**SQUARE EXTENSIONS OF SEMILATTICES**



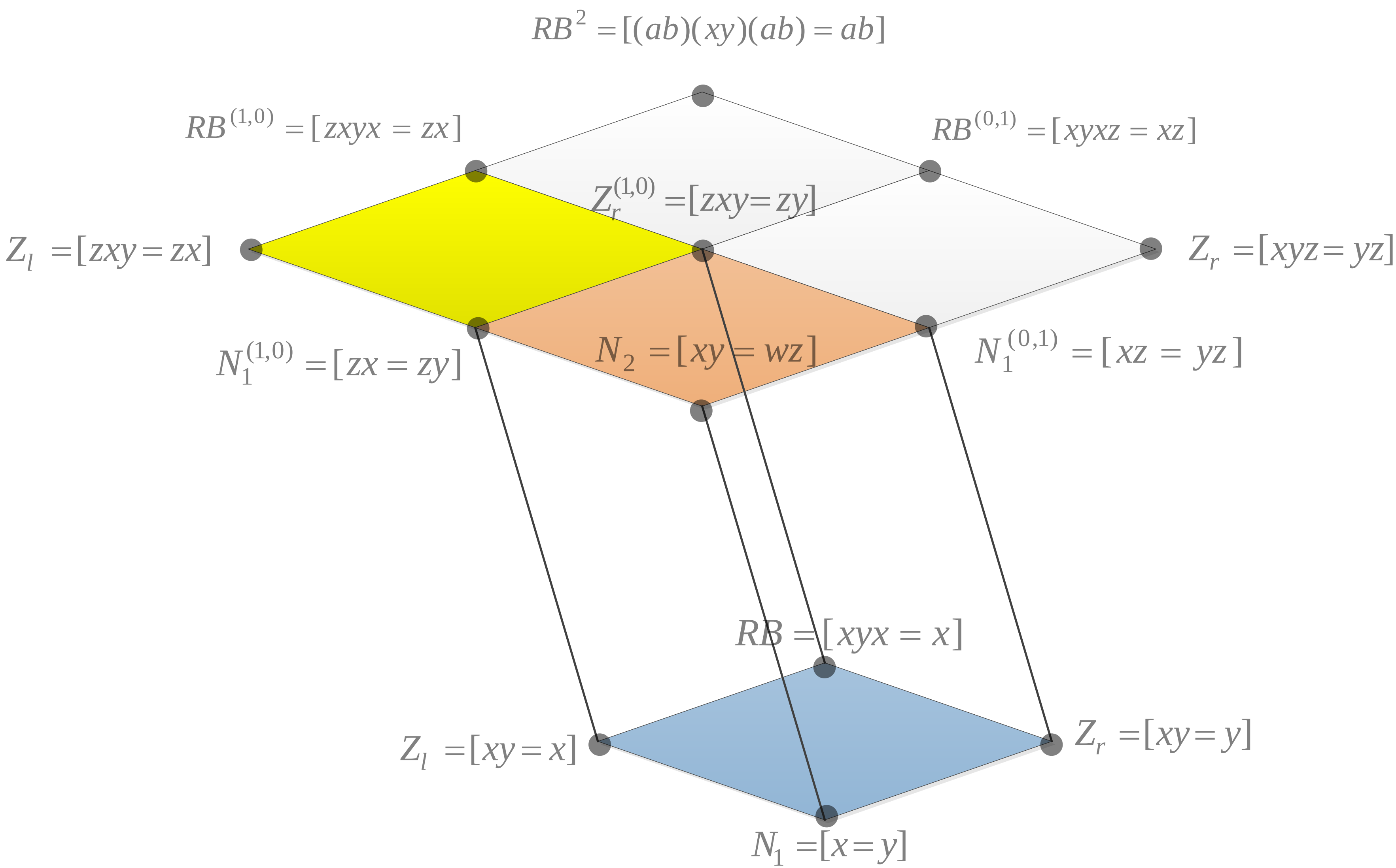


$L_2$
$\Sigma^{<2>}$
Nilpotent extensions of semilattices
$N_2$
$\Sigma$
Semilattices
Null semigroups
$N_1$
Trivial Semigroups

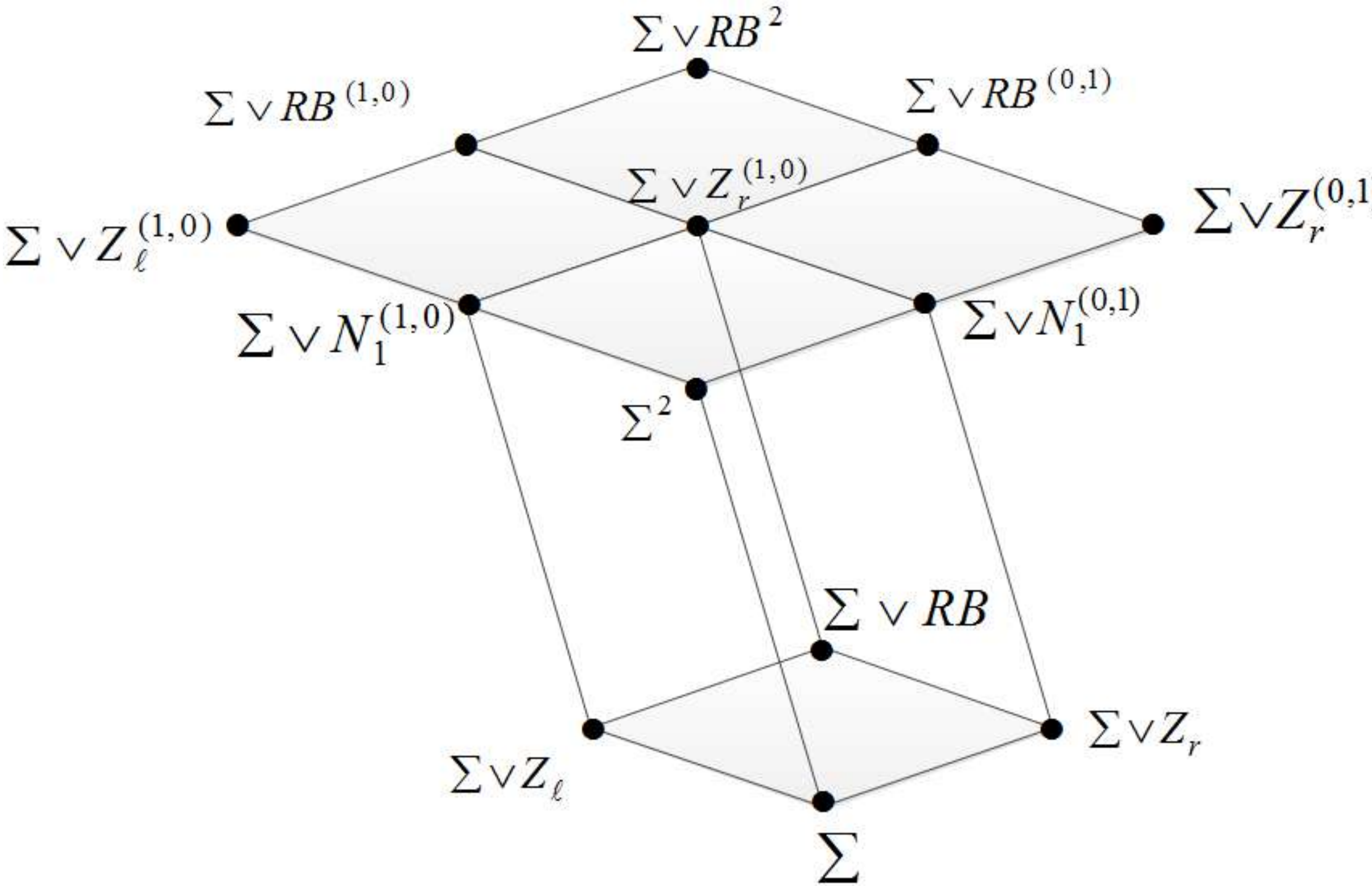



Tischenko (2007)

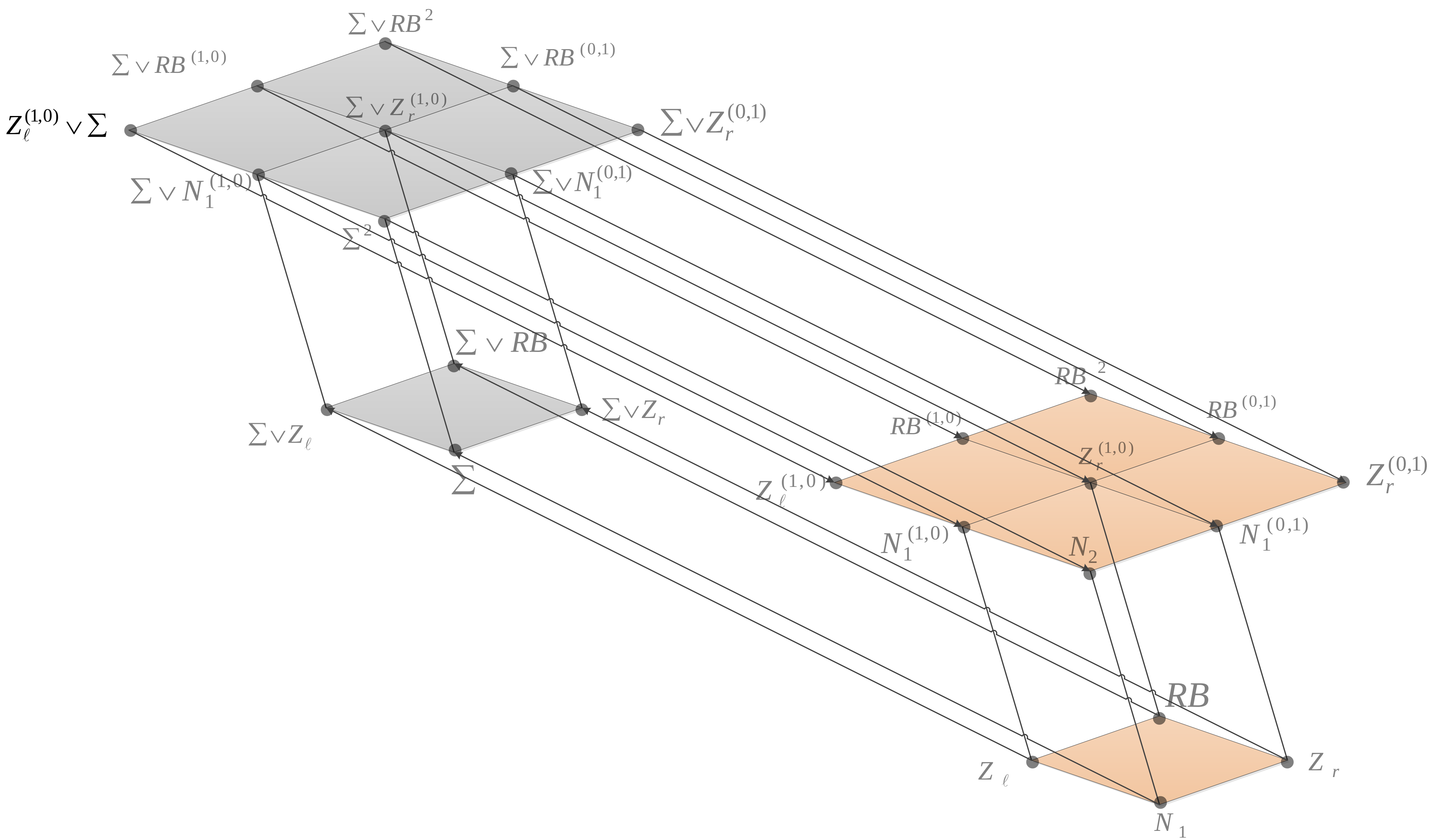
# MELNIK: All varieties of 2-nilpotent extensions of rectangular bands



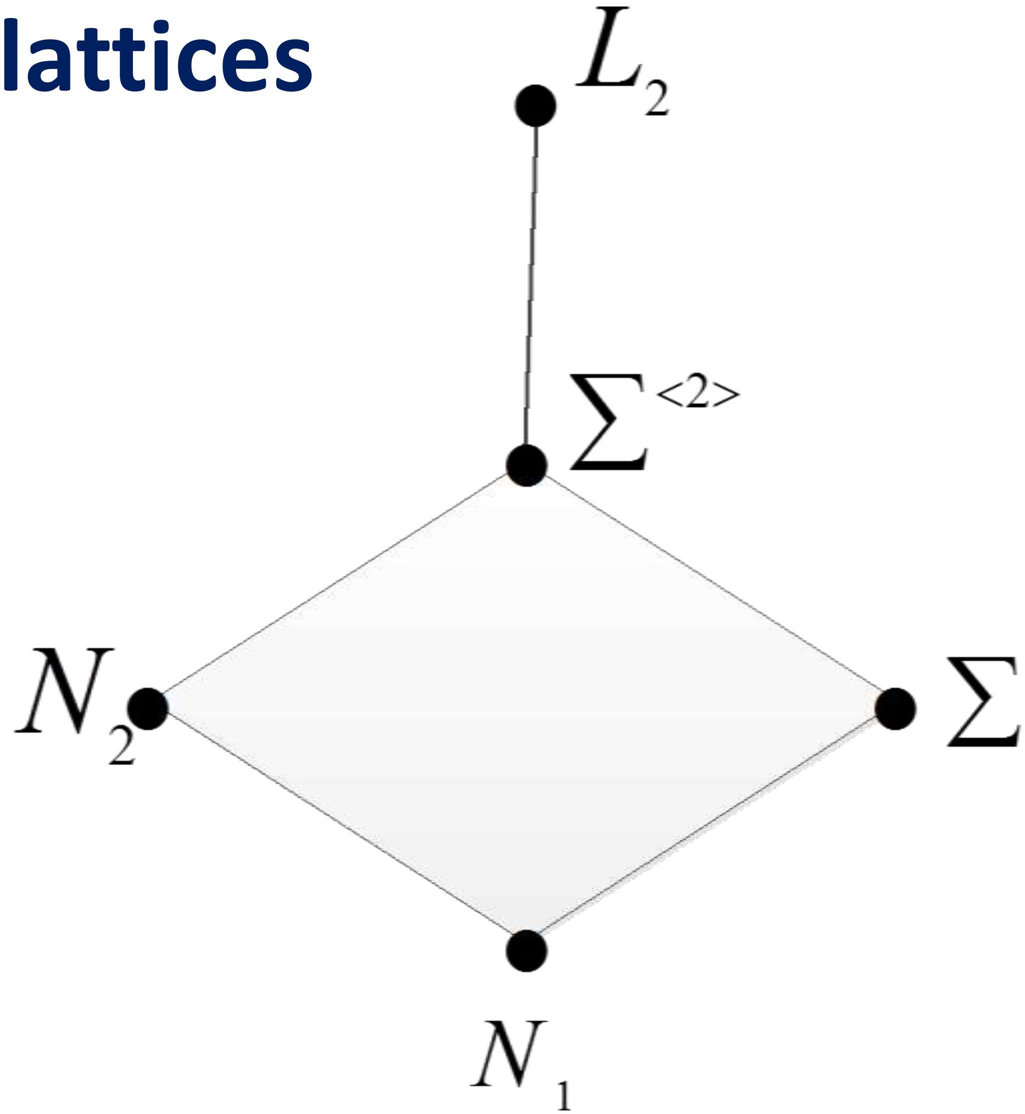
# PETRICH: Joins of 2-nilpotent extensions of rectangular band varieties with the semilattice variety

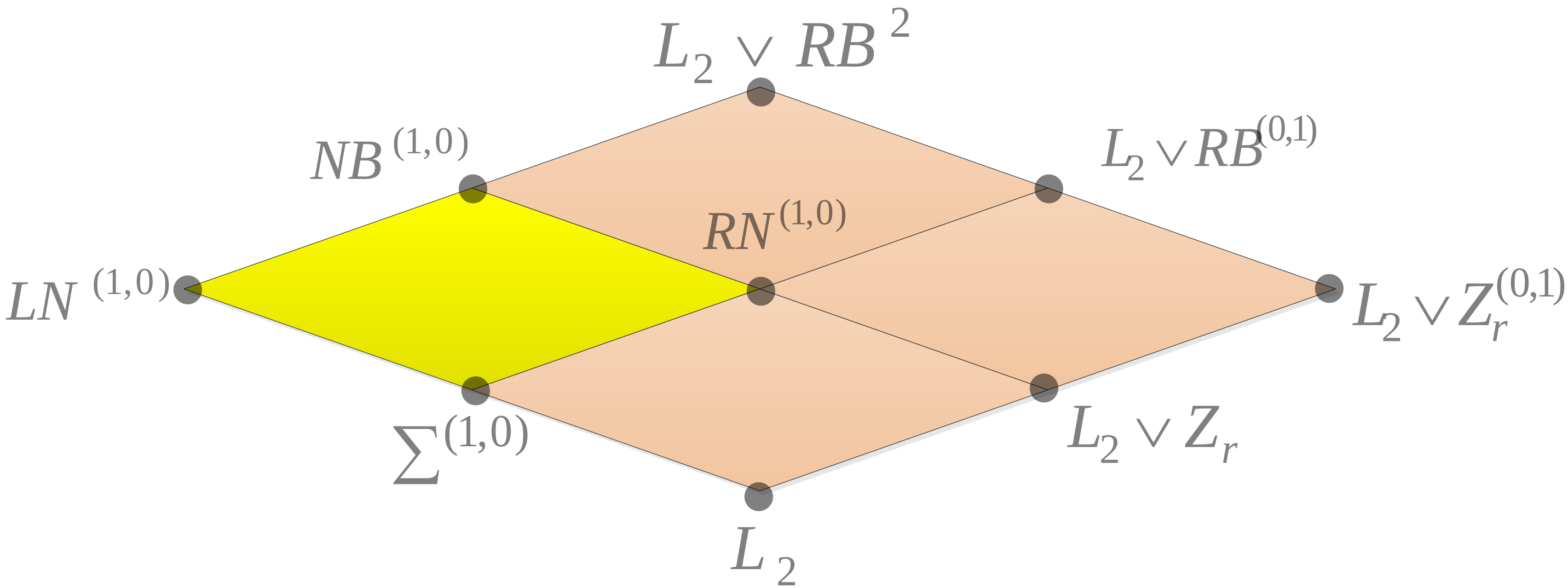


# All varieties of 2-nilpotent extensions of normal bands



# All varieties of square extensions of semilattices





$L_2 \vee RB^2$
$NB^{(1,0)}$
$L_2 \vee RB^{(0,1)}$
$RN^{(1,0)}$
$LN^{(1,0)}$
$L_2 \vee Z_r^{(0,1)}$
$\sum^{(1,0)}$
$L_2 \vee Z_r$
$L_2$

Lemma 1 $\quad Z_\ell \vee L_2 = \sum^{(1,0)}$

Proof. For any semigroup $S$ in $\sum^{(1,0)}$, the Greens relation $L$ forms a congruence on $S$ and the relation

$$\gamma = \{(a,b) : a^2 = b^2, a,b \in S\}$$

forms a congruence on $S$ such that $L \cap \gamma = 1_S$.

Since $S/L \in L_2$ and $S/\gamma \in LN$, we concl de that

$$\sum^{(1,0)} \subseteq L_2 \vee LN = L_2 \vee (\sum \vee Z_\ell) = L_2 \vee Z_\ell$$

The equality holds since the reverse inclusion also holds trivially.

Lemma 2 $(Z_\ell^{(1,0)} \vee \sum) \vee L_2 = LN^{(1,0)}$

Proof.

$$
\begin{aligned}
(Z_\ell^{(1,0)} \vee \textstyle\sum) \vee L_2 &= Z_\ell^{(1,0)} \vee (\textstyle\sum \vee L_2) \\
&= Z_\ell^{(1,0)} \vee L_2 \\
&= (Z_\ell^{(1,0)} \vee Z_\ell) \vee L_2 \\
&= Z_\ell^{(1,0)} \vee (Z_\ell \vee L_2) \\
&= Z_\ell^{(1,0)} \vee \textstyle\sum^{(1,0)} \\
&= (Z_\ell \vee \textstyle\sum)^{(1,0)} \\
&= LN^{(1,0)}
\end{aligned}
$$

Lemma 3 $(Z_r^{(1,0)} \vee \sum) \vee L_2 = RN^{(1,0)}$

Proof.

$$
\begin{aligned}
(Z_r^{(1,0)} \vee \sum) \vee L_2 &= Z_r^{(1,0)} \vee (\sum \vee L_2) \\
&= Z_r^{(1,0)} \vee L_2 \\
&= (Z_r^{(1,0)} \vee Z_\ell) \vee L_2 \\
&= Z_r^{(1,0)} \vee (Z_\ell \vee L_2) \\
&= Z_r^{(1,0)} \vee \sum\nolimits^{(1,0)} \\
&= (Z_r \vee \sum)^{(1,0)} \\
&= RN^{(1,0)}
\end{aligned}
$$

Lemma 4 $(RB^{(1,0)} \vee \sum) \vee L_2 = NB^{(1,0)}$

Proof.

$$(RB^{(1,0)} \vee \sum) \vee L_2 = RB^{(1,0)} \vee (\sum \vee L_2)$$

$$= RB^{(1,0)} \vee L_2$$

$$= (RB^{(1,0)} \vee Z_\ell) \vee L_2$$

$$= Z_r^{(1,0)} \vee (Z_\ell \vee L_2)$$

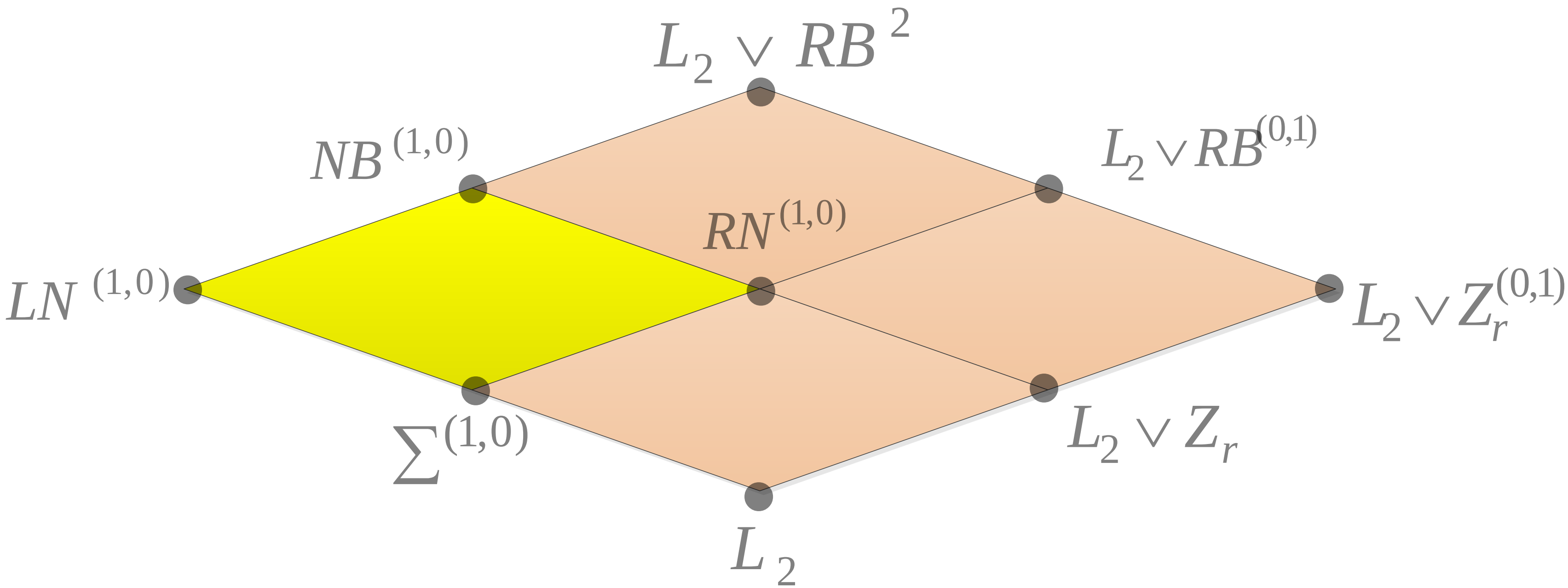
$$= RB^{(1,0)} \vee \sum^{(1,0)}$$

$$= (RB \vee \sum)^{(1,0)}$$

$$= NB^{(1,0)}$$

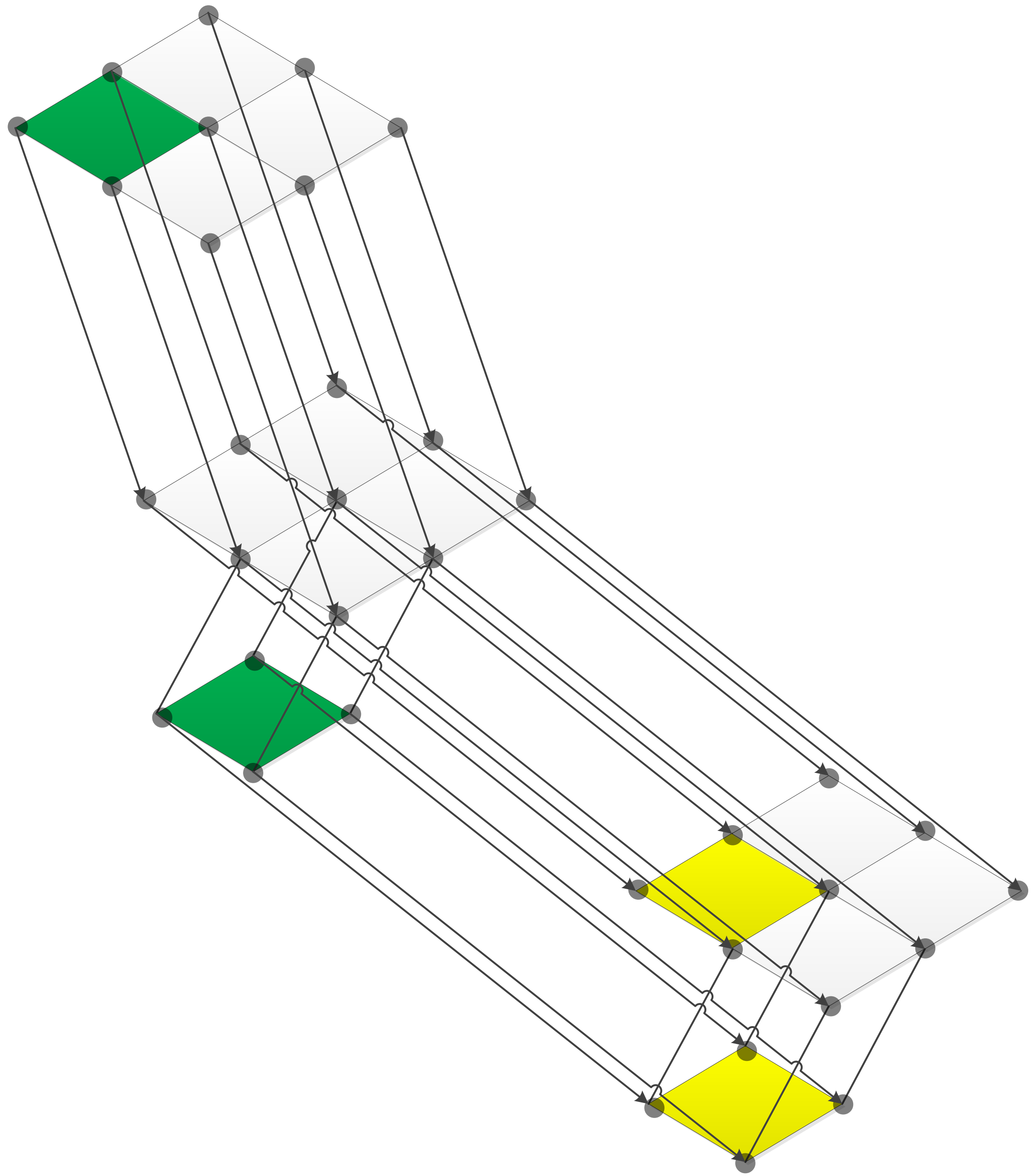
Lemma 5 $Z_r \vee L_2 = [zx^2 = zx, xyz = yxz]$

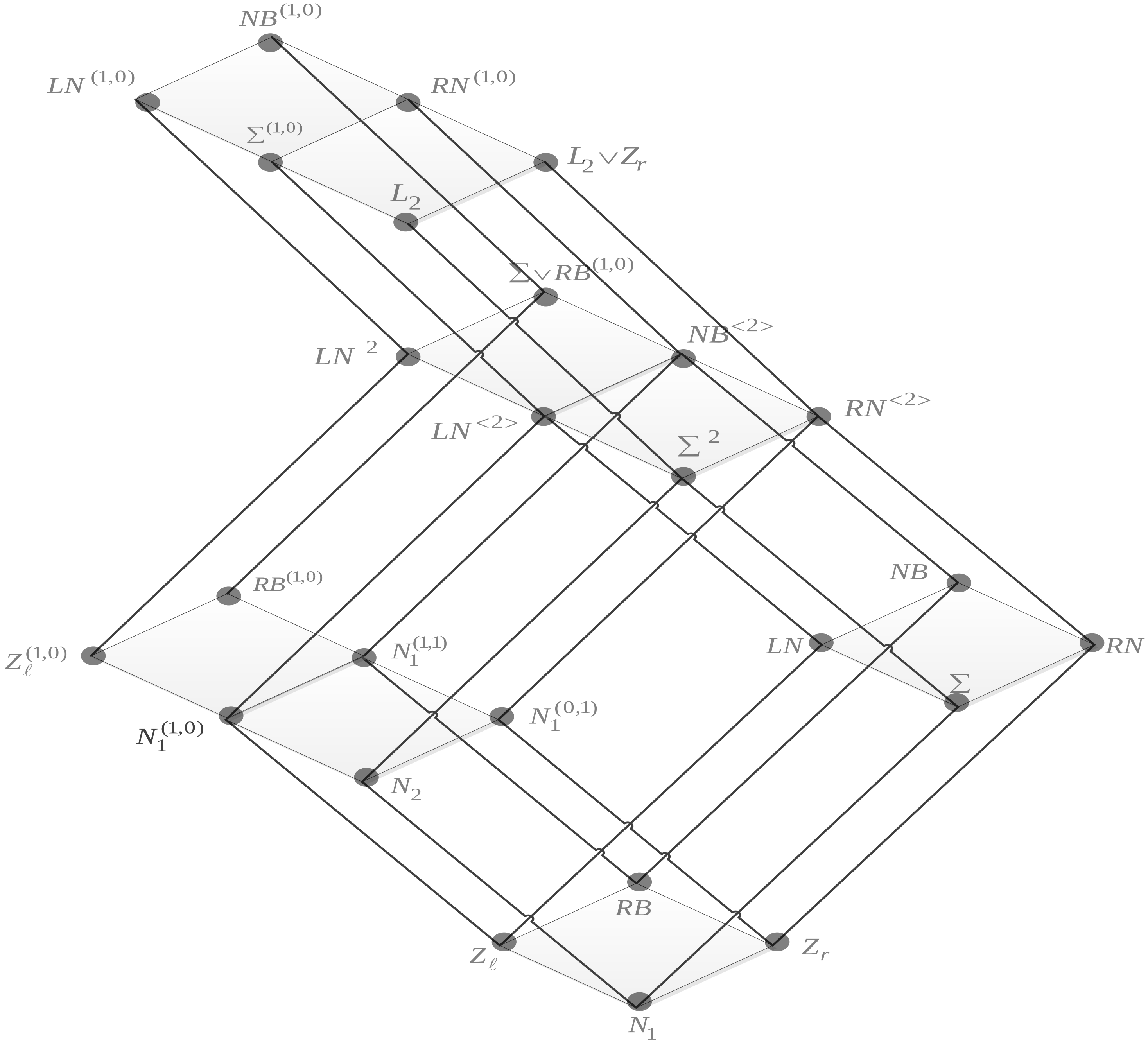
Proof.



$L_2 \vee RB^2$
$NB^{(1,0)}$
$L_2 \vee RB^{(0,1)}$
$RN^{(1,0)}$
$LN^{(1,0)}$
$L_2 \vee Z_r^{(0,1)}$
$\Sigma^{(1,0)}$
$L_2 \vee Z_r$
$L_2$

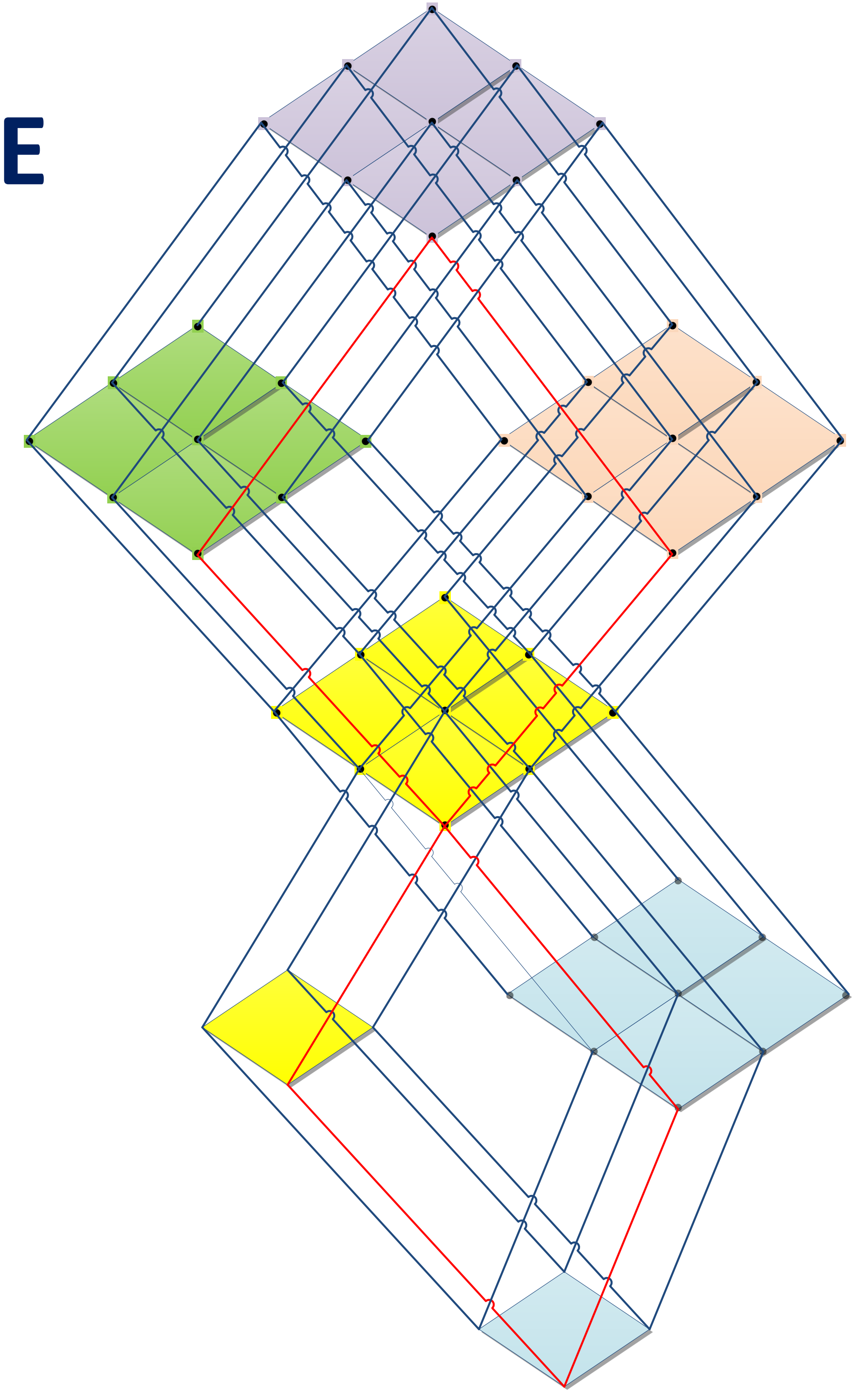
# (1,0) construction of normal bands





$NB^{(1,0)}$
$LN^{(1,0)}$
$RN^{(1,0)}$
$\Sigma^{(1,0)}$
$L_2 \vee Z_r$
$L_2$
$\Sigma \vee RB^{(1,0)}$
$LN^2$
$NB^{<2>}$
$RN^{<2>}$
$LN^{<2>}$
$\Sigma^2$
$RB^{(1,0)}$
$NB$
$Z_\ell^{(1,0)}$
$N_1^{(1,1)}$
$LN$
$RN$
$\Sigma$
$N_1^{(0,1)}$
$N_1^{(1,0)}$
$N_2$
$RB$
$Z_\ell$
$Z_r$
$N_1$

# CONJECTURE



For each integer $n \geq 2$ define the following families of semigroup varieties :

$$L_n = \begin{bmatrix} z(x_1 \ldots x_{n-1})^2 = z(x_1 \ldots x_{n-1}) \\ x^n y^n = y^n x^n \end{bmatrix}$$

$$\Sigma^n = \begin{bmatrix} (x_1 \ldots x_n)^2 = (x_1 \ldots x_n) \\ (x_1 \ldots x_n)(y_1 \ldots y_n) = (y_1 \ldots y_n)(x_1 \ldots x_n) \end{bmatrix}$$

$$(\Sigma^{n-1})^{(1,0)} = \begin{bmatrix} z(x_1 \ldots x_{n-1})^2 = z(x_1 \ldots x_{n-1}) \\ z(x_1 \ldots x_{n-1})(y_1 \ldots y_{n-1}) = z(y_1 \ldots y_{n-1})(x_1 \ldots x_{n-1}) \end{bmatrix}$$

so that

$$\Sigma^n \subseteq L_n \subseteq (\Sigma^{n-1})^{(1,0)}$$