Monoid varieties with continuum many subvarieties

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A construction

Let W be a set of possibly empty words in an alphabet A not including 0.

The set $W^{\leq} \cup \{0\}$ becomes a monoid denoted by $M(W)$, with

$$u \cdot v := \begin{cases} uv & \text{if } uv \in W^{\leq} \\ 0 & \text{otherwise} \end{cases}$$

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Example: $M(xyx)$ (we'll omit brackets $\{,\}$)

elements: $1, x, y, xy, yz, xyx, 0$.
multiplication: $xy \cdot x = xyx$ but $xy \cdot y = 0$.

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The pseudovariety generated by the $M(W)$ for finite W is the class of finite aperiodic semigroups with central idempotents. Expect equations like

$$x^3 \approx x^4, xt_1xt_2x \approx x^3t_1t_2 \approx t_1t_2x^3.$$

A major source of bad behaviour!

Considered as semigroups

A small sample of bad behaviour

- $M(xyxy)$ has no finite identity basis (O.Sapir, 2000)—"not finitely based" (NFB). Characterised for words in $\{x, y\}^*$.

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- For any word w there are sequences of words $w \leq w_1 \leq w_2 \leq \ldots$ such that $M(w_1), M(w_2), \ldots$ are alternately FB and NFB (O.Sapir and J, 2000).

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- For any word w there are sequences of words $w \leq w_1 \leq w_2 \leq \dots$ such that $M(w_1), M(w_2), \dots$ are alternately FB and NFB (O.Sapir and J, 2000).
- $M(xyxy)$ has an infinite irredundant *semigroup* identity basis (J, 2005).
- The *semigroup* variety of $M(xyx)$ has continuum many semigroup subvarieties (J, 2000).

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...embeds the full powerset lattice $\wp(\mathbb{N})$. So $\mathcal{L}(\mathbb{V}_s(M(xyx)))$

- is of the cardinality of the continuum c,
- order embeds the usual order on $\mathbb{R}$,
- has an antichain of size c,
- contains c many nonfinitely generated subvarieties,
- contains c many NFB subvarieties.

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- A finite aperiodic semigroup with central idempotents generates a hereditarily finitely based semigroup variety if and only if it does not contain $M(xyx)$.

Wenting Zhang and Yanfeng Lou 2008

There is a finitely generated monoid variety with continuum many subvarieties that does not have xyx as an isoterm.

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But the *monoid* subvariety lattice of $\mathbb{V}_s(M(xyx))\ldots$

consists of itself along with just $\mathbb{V}_m(M(xy))$, $\mathbb{V}_m(M(x))$, the variety of semilattice monoids and the trivial variety.

Moving from $\{\cdot\}$ to $\{\cdot, 1\}$.

Stays the same

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Brutally destroyed

- having nonfinitely generated subvarieties
- having NFB subvarieties
- having large numbers of subvarieties
- irredundant axiomatisability

Some positive consequences

In the monoid signature...

J, 2004

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E.W.H.Lee (2009)

If M is an aperiodic monoid with central idempotents then $\mathbb{V}_m(M)$ has a NFB monoid subvariety if and only only if either $xsxyty$ is an isoterm or both $xysxty, xsytxy$ are isoterms.

Wenting Zhang (2013) recently showed the existence of a third limit variety of aperiodic monoids.

Lunch-time chat, NSAC 09

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Crucial pattern

$$x_0 \underline{\ \ ?\ \ } x_1 x_0 \; x_2 x_1 \; x_3 x_2 \; x_4 x_3 \; x_5 x_4 \ldots x_n x_{n-1} \underline{\ \ ?\ \ } x_n$$

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Example

$$x_0\ \underline{xy}\ x_1 x_0\ x_2 x_1\ x_3 x_2\ x_4 x_3\ x_5 x_4 \ldots x_n x_{n-1}\ \underline{xy}\ x_n$$
$$\approx x_0\ \underline{yx}\ x_1 x_0\ x_2 x_1\ x_3 x_2\ x_4 x_3\ x_5 x_4 \ldots x_n x_{n-1}\ \underline{yx}\ x_n$$

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Example

Let G be a finite group failing the law $xyxy \approx yxyx$. Then

1. $\mathbb{V}_m(G)$ and $\mathbb{V}_m(M(xx))$ are Cross varieties.
2. $\mathbb{V}_m(G) \vee \mathbb{V}_m(M(xx))$ contains $M(xyxy)$, so the monoid subvariety lattice embeds $\wp(\mathbb{N})$.

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Theorem

Let M be a finite inherently nonfinitely based monoid. The monoid subvariety lattice of $\mathbb{V}_m(M)$ lattice embeds $\wp(\mathbb{N})$.

Uses Mark Sapir's Zimin word classification of INFB.

$$x_1 x_2 x_1 \ x_3 \ x_1 x_2 x_1 \ x_4 \ x_1 x_2 x_1 \ x_3 \ x_1 x_2 x_1 \ldots$$

Irredundant axiomatisability for a finite monoid

The follow identities are an irredundant system defining a finitely generated monoid variety:

$$xt_1xt_2x \approx x^3t_1t_2 \approx t_1t_2x^3, x^3 \approx x^4$$

with (for each $n > 0$)

$$x_0\ z_0\underline{xy}z_1\ x_1x_0\ x_2x_1\ x_3x_2\ x_4x_3\ x_5x_4 \dots x_nx_{n-1}\ z_0\underline{xy}z_1\ x_n$$
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The variety can be generated by $M(W)$ for a set W containing about 80 words:

$xyyx, xxyy, xtyxy, xytxy, xyxty, xyzyxz, zxyzyx, xyzxzy, \dots$

$\dots, xyabcadbecdexy.$