Small expansions of $(\omega, <)$ and $(\omega + \omega^*, <)$

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Definitions, Notations, Problems and Results

- For $\mathcal{M} = (M, \ldots)$, $A \subseteq M$: $\varphi(A) = \{a \in A \mid \mathcal{M} \models \varphi(a)\}$.
- $A \subseteq M$ is *minimal set* iff for every $\varphi \in For_M$ exactly one of $\varphi(A)$ and $\neg\varphi(A)$ is infinite.
  We also say that $\mathrm{CB}(A) = 1 = \deg(A)$.
- If there are pairwise disjoint formulas $\varphi_1, \varphi_2, \ldots, \varphi_n$ such that $\varphi_i(A)$ is minimal set, then we say that $\mathrm{CB}(A) = 1$ and $\deg(A) = n$.
- Structure is minimal if its underlying set is minimal.
  CB rank and degree of structure is rank and degree of underlying set. $\mathrm{CB}(\varphi) = \mathrm{CB}(\varphi(M))$, $\deg(\varphi) = \deg(\varphi(M))$.

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- Consider $(\omega, <, P_k)$, where $P(x)$ says "k divides x". $\mathrm{CB}(x = x) = 1$, $\deg(x = x) = k$.
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  If f is function increasing faster then any linear, we can make such example: $Q(x) \overset{\mathrm{def}}{\iff} (\exists y)(x = f(y))$. Then $(\omega, <, Q)$ is of CB rank =2.

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Assume L is a discrete ordered set. No proper expansion of $(\omega + L, <)$ is minimal.

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Theorem 2:

Every expansion of $(\omega, <)$ with CB rank 1 and degree $k > 1$ is essentially unary: It is definitionally equivalent to $(\omega, <, P_k)$ where $P_k(x)$ is $x \equiv 0 (\mathrm{mod} k)$.

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Conjecture

If CB rank of expansion of $(\omega, <)$ is 2, then it is essentially unary.

Technical details

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$$p_E = \{n < x \mid n \in \omega\} \cup \{\varphi^*(x) \mid \varphi(x) \in For_{L_E}, |\omega \setminus \varphi(\omega)| < \aleph_0\}.$$

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If $a \models p_0$ and $D < a$, then D has a maximum definable by a formula using the same parameters as does formula defining D.

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If $\mathrm{cl}(E) \neq \emptyset$ is nonempty, then $p_E(\mathbb{U}) = \{x \mid \omega < x < \mathrm{cl}(E)\}$.

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- ω-type over ω-sequence is unique.

Theorem 1

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Now assume CB(ω) =CB($x = x$) $= 1$ and deg(ω) =deg($x = x$) $= k > 1$.

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Theorem 2

Assume $\mathcal{M}$ is expansion of $(\omega + L, <)$, such that CB(ω) =CB($x = x$) $= 1$ and deg(ω) =deg($x = x$) $= k > 1$. Then it is definitionally equivalent to $(\omega + L, <, P_k)$.