Adequate and Ehresmann semigroups

NSAC2013: June 8th, 2013, Novi Sad

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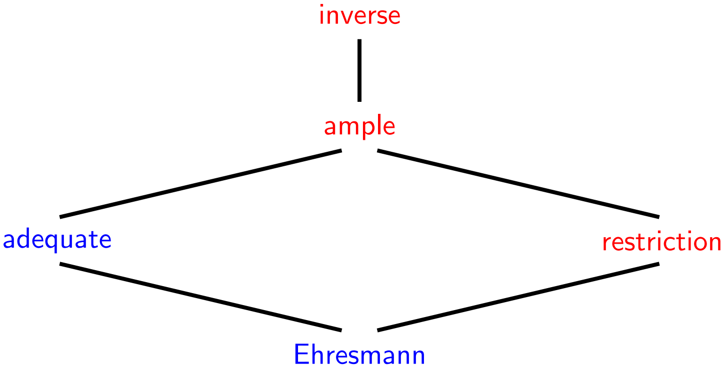
What is this talk about?
Classes of semigroups with semilattices of idempotents



What is this talk about?
Classes of semigroups with semilattices of idempotents
Ehresmann, adequate, restriction, ample semigroups

1. What are these classes? Why are they important? Natural examples.
2. Correspondence with inductive categories.
3. Free objects.
4. ‘Proper’ Ehresmann semigroups.

For any semigroup S, the set of idempotents of S is denoted by $E(S)$.

Inverse semigroups

A semigroup S is **inverse** if and only if S is regular (i.e. for all $a \in S$ there exists $b \in S$ with $a = aba$) and $ef = fe$ for all $e, f \in E(S)$.

Fact An inverse semigroup S is a semigroup such that every $a \in S$ has a unique inverse, i.e. for all $a \in S$ there is a unique $b \in S$ with

$$a = aba \text{ and } b = bab.$$

We write $b = a'$.

Fact If $ef = fe$ for all $e, f \in E(S)$, then $E(S)$ is a **semilattice** i.e. a commutative semigroup of idempotents.

Fact If Y is a semilattice, then Y is partially ordered by

$$e \leq f \Leftrightarrow e = ef \text{ and } e \wedge f = ef.$$

Inverse semigroups as bi-unary semigroups

A **bi-unary** semigroup is a semigroup equipped with two unary operations. We regard bi-unary semigroups as algebras with signature $(2,1,1)$.

Let S be inverse, so for any $a \in S$ we have

$$a = aa'a \text{ and } a' = a'aa'.$$

Then S is a bi-unary semigroup $(S, \cdot, ^+, ^*)$ where

$$a^+ = aa' \text{ and } a^* = a'a.$$

For any $e \in E(S)$ we have $e = eee$ so that $e = e' = e^+ = e^*$.

S satisfies the **identities** Σ :

$$(a^*)^+ = a^*,\ (a^+)^* = a^+,$$

$$a^+a^+ = a^+,\ a^+b^+ = b^+a^+,\ (a^+b^+)^+ = a^+b^+,$$

$$a^+a = a,\ (ab^+)^+ = (ab)^+,\ aa^* = a \text{ and } (a^*b)^* = (ab)^*.$$

Ehresmann semigroups: a variety of bi-unary semigroups

Definition A bi-unary semigroup $(S, \cdot, ^+, ^*)$ is **Ehresmann** if it satisfies the identities Σ.

Let

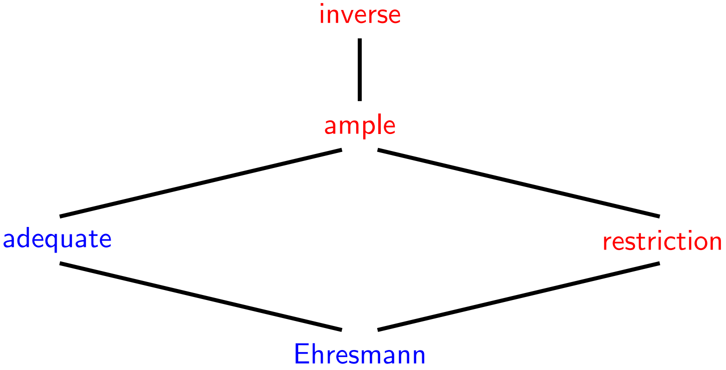
$$E = \{a^* : a \in S\} = \{a^+ : a \in S\}.$$

Then E is a **semilattice**, the semilattice of **projections** of S.

We have noted that inverse semigroups are Ehresmann (with $a^+ = aa', a^* = a'a$ and $E = E(S)$).

The converse is not true: any monoid is Ehresmann with $a^+ = 1 = a^*$ for all $a \in M$. Such an Ehresmann semigroup is called **reduced**.

What is this talk about?
Classes of bi-unary semigroups with semilattices of idempotents



What is this talk not really about? Ample and restriction semigroups

If S is inverse, then it satisfies the **ample (type A) identities:**

$$ab^+ = (ab)^+ a \text{ and } b^* a = a(ba)^*.$$

Proof: Let S be inverse, let $a, b \in S$. Then

$$ab^+ = a(a'a)(bb') = a(bb')(a'a) = (abb'a')a = (ab)^+ a.$$

Restriction and ample semigroups (and their one-sided versions) are classes of bi-unary and unary semigroups satisfying the ample identities. These identities allow the essence of techniques for inverse semigroups to be used. Without the identities, we have to think again.

Restriction, adequate, ample semigroups: definitions

1. An Ehresmann semigroup is **restriction** if and only if it satisfies the ample identities:

$$ab^+ = (ab)^+a \text{ and } b^*a = a(ba)^*.$$

2. An Ehresmann semigroup is **adequate** if and only if it satisfies the quasi-identities

$$xz = yz \to xz^+ = yz^+, zx = zy \to z^*x = z^*y$$

and

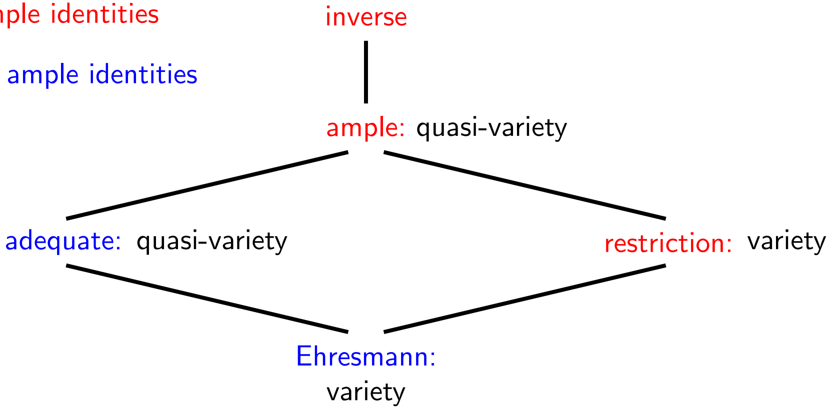
$$x^2 = x \to x = x^+.$$

3. An Ehresmann semigroup is **ample** if it it is both adequate and restriction.

What is this talk about?
Classes of bi-unary semigroups with semilattices of idempotents

ample identities

no ample identities



Restriction, adequate, ample semigroups: observations, examples

1. Inverse semigroups are ample.
2. Any bi-unary subsemigroup of an inverse semigroup is ample.
3. An Ehresmann semigroup S is left restriction iff it embeds into $\mathcal{PT}_S$ where α^+ is the identity map in the domain of α.
4. An Ehresmann semigroup S is left ample iff it embeds into $\mathcal{I}_S$.
5. Monoids are restriction under $a^+ = a^* = 1$, cancellative monoids are ample.
6. Certain semidirect products $Y \rtimes M$ where Y is a semilattice and M is a (cancellative) monoid are restriction (ample).
7. Let Y be a semilattice. Then the **free idempotent generated semigroup** $\mathrm{IG}(Y)$ is adequate but not ample. G, D. Yang, 2013.

Correspondence with inductive categories: Ehresmann semigroups as ordered structures

Let S be an Ehresmann semigroup.

Define $\leq_r$ and $\leq_\ell$ on S by

$$a \leq_r b \Leftrightarrow a = a^+ b, \; (a \leq_\ell b \Leftrightarrow a = ba^*).$$

Then $\leq_r, \leq_\ell$ are partial orders.

Fact If S satisfies the ample identity, then $\leq_r = \leq_\ell$ and is compatible with multiplication.

Without the ample identity, they are not, in general, equal, nor compatible on both sides with multiplication.

Define $\cdot$ on S by

$$\exists a \cdot b \Leftrightarrow a^* = b^+ \text{ and then } a \cdot b = ab.$$

Then $\leq_r$ and $\leq_\ell$ are compatible with $\cdot$ where defined.

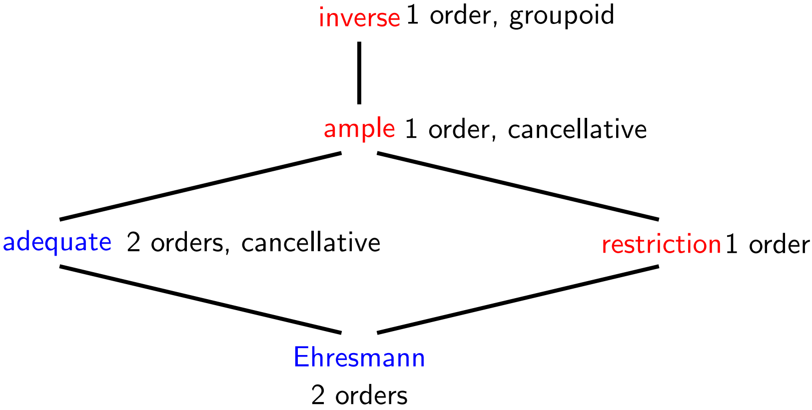
Correspondence with inductive categories

An **Ehresmann category** is a small category ordered by two partial orders, possessing restrictions and co-restrictions.

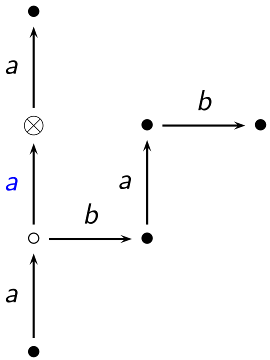
Theorem Lawson, 1986 The category of Ehresmann semigroups and morphisms is isomorphic to the category of Ehresmann categories and strongly ordered functors.

There are corollaries all the way 'up'. The existence of the ample identities means we need just one partial order; for adequate and ample semigroups the Ehresmann categories are cancellative. In the inverse case they are inductive groupoids and we recover the Ehresmann-Schein-Nambooripad Theorem.

Correspondence with inductive categories: Going up to Ehresmann-Schein-Nambooripad



Free objects:
Free inverse semigroup FIS(X): Munn 1972



tree T: word in $FIS(X)$ is

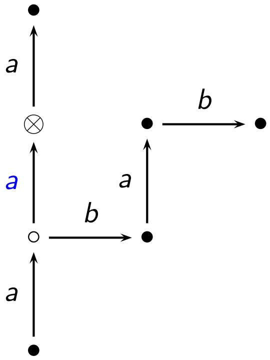
$$(a'a)(babb'a'b')a(aa')$$
$$= a^*(bab)^+aa^+$$

trunk is a

For operation TS : within the Cayley graph of free group on X, glue end of T onto start of S.

Free objects:
Free ample/restriction semigroup FAmS(X)
Fountain, Gomes, G 2009

Trees in FIS(X) with trunk in X^* e.g.

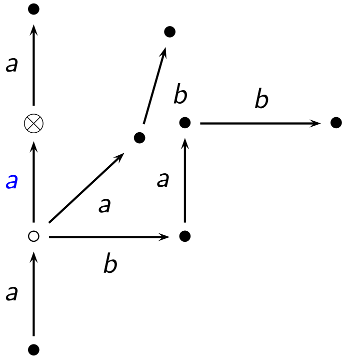


for $^+$ take $\otimes$ to $\circ$

for * take $\circ$ to $\otimes$

Free objects:
Free adequate/Ehresmann semigroup FAdS(X)
Kambites 2011

X-trees: birooted X-labelled trees with trunk in X^* e.g.

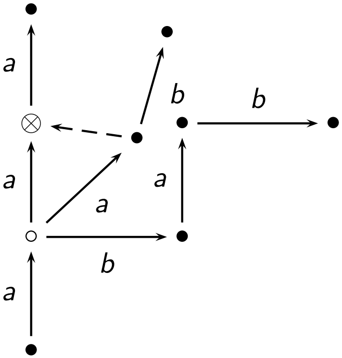


Tree T: word $a^*(bab)^+(ab)^+aa^+$

TS: glue end of T to start of S
and take trunk preserving retracts

for $^+$ take $\otimes$ to $\circ$
for * take $\circ$ to $\otimes$

Free objects:
Free adequate/Ehresmann semigroup FAdS(X)
Kambites 2011



if we have ample condition, then

$a^*(bab)^+(ab)^+aa^+$
$= a^*(bab)^+ab^+a^+$

‘Proper’ Ehresmann semigroups: Proper inverse semigroups

Let S be an inverse semigroup.

1. $\sigma = \langle E(S) \times E(S) \rangle$ is the least group congruence on S.
2. S is **proper** if
$$(a^+ = b^+ \text{ and } a\,\sigma\,b) \text{ implies } a = b;$$
this definition is left/right dual.
3. If S is proper, $S \to E(S) \times S/\sigma$ given by
$$s \mapsto (s^+, s\sigma)$$
is clearly a SET embedding.
4. S is proper if and only if it embeds into a semidirect product $Y \rtimes G$ where Y is a semilattice and G is a group O’Carroll 1981.
5. S has a **proper cover** McAlister 1974. That is, there exists a proper inverse semigroup $\widehat{S}$ and an idempotent separating morphism $\widehat{S} \twoheadrightarrow S$.
6. The free inverse semigroup is proper.

‘Proper’ Ehresmann semigroups: Proper ample and restriction semigroups

1. Let S be Ehresmann; put $\sigma = \langle E \times E \rangle$.
2. S/σ is **reduced**, if S is ample then S/σ is **cancellative**.
3. A restriction/ample S is **proper** iff the following condition and its dual hold:
$$(a^+ = b^+ \text{ and } a\,\sigma\,b) \text{ implies } a = b.$$
4. Results involving semidirect products for restriction/ample semigroups analogous to those in the inverse case hold where **group** is replaced by **monoid/cancellative monoid** Lawson 1986, Cornock, G 2012.
5. The free restriction semigroup is proper.

‘Proper’ Ehresmann semigroups: What makes such results involving semidirect products work?

Let S be an Ehresmann monoid.

1. Suppose that $S = \langle X \rangle_{(2,1,1,0)}$. Put $T = \langle X \rangle_{(2,0)}$ *so that* T *is the* **monoid** *generated by* X.
2. $S = \langle T \cup E \rangle_{(2)}$ so that any $s \in S$ can be written as

   $$s = t_0 e_1 t_1 \dots e_n t_n,$$

   for some $t_0, \dots, t_n \in T$ and $e_1, \dots, e_n \in E$.
3. If the ample identities hold then $S = ET$.
4. The above is what is behind results connecting restriction/ample/inverse monoids to semidirect products $Y \rtimes T$ of a semilattice Y and a monoid T.

'Proper' Ehresmann semigroups: Losing our identity

- Without the ample identities, we must think again.
- Adequate and Ehresmann semigroups do not behave like inverse, ample or restriction semigroups.

'Proper' Ehresmann monoids:
Structure of Ehresmann monoids: $\mathcal{E}$

- The old notion of 'proper' is no good - it leads inexorably to a semidirect product construction, which is no longer appropriate.
- Want condition **P** for Ehresmann monoids such that:
  (i) all monoids satisfying **P** have their structure described by monoids acting on semilattices;
  (ii) if $S \in \mathcal{E}$ then there exists $\widehat{S} \in \mathcal{E}$ satisfying **P** and a projection-separating morphism
  $$\widehat{S} \twoheadrightarrow S,$$
  i.e. $\widehat{S}$ is a **cover** of S;
  (iii) the free objects in $\mathcal{E}$ satisfy **P**.

‘Proper’ Ehresmann monoids: Generators and T-normal form Branco, Gomes, G

Suppose that $M \in \mathcal{E}$ and $M = \langle E \cup T \rangle_{(2)}$ where T is a submonoid of M. Then any $x \in M$ can be written as

$$x = t_0 e_1 t_1 \ldots e_n t_n,$$

where $n \geq 0, e_1, ..., e_n \in E, t_1, ..., t_{n-1} \in T \setminus \{1\}, t_0, t_n \in T$ and for $1 \leq i \leq n$

$$e_i < (t_i e_{i+1} \ldots t_n)^+ \text{ and } e_i < (t_0 e_1 t_1 \ldots t_{i-1})^*.$$

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$$x = (t_0 e_1)(t_1 e_2) \ldots (e_n t_n),$$

where $n \geq 0, e_1, ..., e_n \in E, t_1, ..., t_{n-1} \in T \setminus \{1\}, t_0, t_n \in T$ and for $1 \leq i \leq n$

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‘Proper’ Ehresmann monoids: Generators and T-normal form Branco, Gomes, G

Let $M \in \mathcal{E}$ with $M = \langle E \cup T \rangle_{(2)}$ where T is a submonoid of M.

M is said to be T-**proper** if it satisfies the following condition and its dual: for all $s, t \in T, e, f \in E$

$$(se)^+ = (te)^+ \text{ and } se \,\sigma\, te, \text{ then } se = te.$$

Note If M is restriction, then M is M-proper if and only if it is proper.

Fact The free adequate monoid $\mathrm{FAdM}(X)$ on X is X^*-proper.

T-proper Ehresmann monoids: Ehresmann monoids and actions on E Branco, Gomes, G

Let $M \in \mathcal{E}$ with $M = \langle E \cup T \rangle_{(2)}$ where T is a submonoid of M.

T acts on E on the left/right via

$$t \cdot e = (te)^+, \; e \circ t = (et)^*$$

via order preserving maps.

If S is restriction, the action of T on E is by morphisms.

The actions are linked by the **compatibility** identities: for $e, f \in E, t \in T$

$$e(t \cdot f) = e(t \cdot ((e \circ t)f)) \text{ and } (e \circ t)f = ((e(t \cdot f)) \circ t)f.$$

T-proper Ehresmann monoids: The construction Branco, Gomes, G

Let T be a monoid acting by order-preserving maps on the left and right of a semilattice E satisfying the compatibility conditions.

Theorem There is an Ehresmann monoid $\mathcal{P}(T, E)$ such that

1. the semilattice of projections of $\mathcal{P}(T, E)$ is E;
2. T is a submonoid of $\mathcal{P}(T, E)$;
3. $\mathcal{P}(T, E) = \langle T \cup E \rangle_{(2)}$;
4. $\mathcal{P}(T, E)$ is T-proper;
5. $\mathcal{P}(T, E)/\sigma \cong T$;
6. if $T = X^*$, then $\mathcal{P}(T, E)$ is adequate;
7. the free adequate monoid on X is of the form $\mathcal{P}(X^*, E)$.

A note on the construction Extend the action of T on E to the action of the free product $T * E$ on E. Let $w^+ = w \cdot 1$ and $w^* = 1 \circ w$. Put

$$\sim = \langle (u^+ u, u), (u, uu^*) \rangle.$$

Then

$$\mathcal{P}(T, E) = (T * E)/\sim.$$

T-proper covers for Ehresmann monoids
Branco, Gomes, G and Jones

Theorem Let $M \in \mathcal{E}$ and let T be a submonoid of M such that $M = \langle T \cup E \rangle_{(2)}$. Then M has a T-proper cover; in particular, $\mathcal{P}(T, E)$ is a cover for M.

Questions:

1. Is $\mathcal{P}(T, E)$ always adequate for T cancellative?
2. An O'Carroll type embedding theorem? A McAlister P-theorem?