On Automorphisms, Derivations and Elementary Operators

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This talk is based on a paper

- D. Eremita, I. Gogić, D. Ilišević, *Generalized skew derivations implemented by elementary operators* (2013), to appear in Algebras and Representation Theory

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Examples of semiprime rings:

- Any prime ring is obviously a semiprime ring.
- Any reduced ring is a semiprime ring.
- Any J-semisimple ring is semiprime. In particular, semisimple rings and von Neumann regular rings are all semiprime.
- Any direct product of semiprime rings is semiprime.
- Any matrix ring over a semiprime ring is a semiprime ring.

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An interesting class of additive maps $d : R \to R$ including both automorphisms and generalized derivations is the class of **generalized skew derivations**, that is, those satisfying

$$d(xy) = \delta(x)y + \sigma(x)d(y) \qquad (x, y \in R),$$

for some map $\delta : R \to R$ and automorphism $\sigma \in \mathrm{Aut}(R)$.

Since by assumption R is semiprime, it is easy to see that the map δ is automatically additive and it is uniquely determined by d. Moreover, δ is a σ-**derivation** (skew-derivation), i.e. δ satisfies

$$\delta(xy) = \delta(x)y + \sigma(x)\delta(y) \qquad (x, y \in R).$$

We decompose d as

$$d = \delta + \rho,$$

where $\rho := d - \delta$, and note that

$$\rho(x) = \sigma(x)d(1) \qquad (x \in R).$$

On the other hand, an attractive and fairly large class of additive maps $\phi : R \to R$ is the class of **generalized elementary operators**, that is, those which can be expressed as a finite sum

$$\phi(x) = \sum_i a_i x b_i \qquad (x \in R),$$

where the **coefficients** a_i, b_i are elements of the Utumi right quotient ring Q_{mr}. If all a_i, b_i lie in R, then we say that ϕ is an **elementary operator**.

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Motivated by the fact that (generalized) elementary operators comprise both (X-)inner automorphisms $x \mapsto pxp^{-1}$ and (X-)inner generalized derivations $x \mapsto px - xq$, we consider the following question:

Problem

Describe the form of generalized skew derivations which are implemented by (generalized) elementary operators.

The notion of a right quotient ring was introduced by Yuzo Utumi in 1956. An overring Q of a ring R is said to be a **right quotient ring** of R if given $p, q \in Q$, with $p \neq 0$, there exists $a \in R$ satisfying $pa \neq 0$ and $qa \in R$.

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Utumi proved that for every semiprime ring (or more generally, for any ring without total left zero divisors) there exists a maximal right quotient ring, called the **Utumi right quotient ring** of R and denoted by Q_{mr}.

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A right ideal I of R is said to be **dense** if for every $x, y \in R$, with $x \neq 0$ there exists $a \in R$ such that $xa \neq 0$ and $ya \in I$. Note that this is equivalent to saying that R is a right quotient ring of I.

The basic and in fact the characteristic four properties of Q_{mr} are:

(i) R is a subring of Q_{mr}.

(ii) For any $q \in Q_{mr}$ there exists a dense right ideal I of R such that $qI \subseteq R$.

(iii) If $0 \neq q \in Q_{mr}$ and I is a dense right ideal of R, then $qI \neq 0$.

(iv) For any dense right ideal I of R and a right R-module homomorphism $f : I_R \to R_R$ there exists $q \in Q_{mr}$ such that f is a left multiplication by q.

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The center of Q_{mr} is called the **extended centroid** of R and it is denoted by C. It is well known that C is a von Neumann regular self-injective ring. Moreover, C is a field if and only if R is a prime ring.

For any subset $S \subseteq Q_{mr}$ there exists a unique idempotent $\varepsilon(S)$ in C such that

$$\mathrm{rann}_{Q_{mr}}(Q_{mr} S Q_{mr}) = (1 - \varepsilon(S))Q_{mr} \quad \text{and} \quad \varepsilon(S)x = x \text{ for all } x \in S,$$

where $\mathrm{rann}_{Q_{mr}}(X)$ denotes the right annihilator in Q_{mr} of a subset $X \subseteq Q_{mr}$. The idempotent $\varepsilon(S)$ is called the **central support** of S. Whenever $S = \{x\}$ for some $x \in Q_{mr}$ we write $\varepsilon(x)$ for $\varepsilon(S)$.

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An automorphism $\sigma \in \mathrm{Aut}(R)$ (resp. σ-derivation $\delta : R \to R$) is said to be X-**inner** if there exists an element $p \in Q_{mr}^{\times}$ (resp. $q \in Q_{mr}$) such that $\sigma(x) = pxp^{-1}$ (resp. $\delta(x) = qx - \sigma(x)q$) for all $x \in R$. In this case we say that an element p (resp. q) **implements** σ (resp. δ).

Theorem (D. Eremita, I. G. and D. Ilišević, 2013)

Suppose that $\sigma : R \to R$ is a ring epimorphism and let $a \in Q_{mr}$. If the map $\rho_a : R \to Q_{mr}$, given by $\rho_a : x \mapsto \sigma(x)a$ is implemented by a generalized elementary operator, then there exists an invertible element $p \in Q_{mr}^{\times}$ such that

$$\varepsilon(a)\sigma(x) = \varepsilon(a)pxp^{-1} \qquad (x \in R).$$

In particular, $\rho_a(x) = pxp^{-1}a$ $(x \in R)$.

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Corollary

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Problem

If an automorphism $\sigma \in \mathrm{Aut}(R)$ is implemented by an elementary operator (with all coefficients lying in R), is σ in fact an inner automorphism of R?

The answer is negative (in general):

Example

Let R be the C^*-algebra consisting of all elements $x \in C([1,\infty], \mathrm{M}_2(\mathbb{C}))$ (i.e. all continuous functions from the interval $[1,\infty] \subseteq \overline{\mathbb{R}}$ to the C^*-algebra $\mathrm{M}_2(\mathbb{C})$ of 2×2 complex matrices) such that

$$x(n) = \begin{bmatrix} \lambda_n(x) & 0 \\ 0 & \lambda_{n+1}(x) \end{bmatrix} \qquad (n \in \mathbb{N})$$

for some convergent sequence $(\lambda_n(x))$ of complex numbers. Then R admits an outer $*$-automorphism σ which is implemented by an elementary operator.

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Problem

Characterize the class of all unital semiprime rings R with the property that all automorphisms of R which are implemented by elementary operators are necessarily inner.

Theorem (D. Eremita, I. G. and D. Ilišević, 2013)

If a σ-derivation $\delta : R \to R$ is implemented by a generalized elementary operator, then δ is X-inner, and for each element $q \in Q_{mr}$ which implements δ there exists an invertible element $p \in Q_{mr}^{\times}$ such that

$$\varepsilon(q)\sigma(x) = \varepsilon(q)pxp^{-1} \qquad (x \in R).$$

In particular, $\delta(x) = qx - pxp^{-1}q$ $(x \in R)$.

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Corollary

Let δ be a non-zero σ-derivation of a prime ring R. If δ is implemented by a generalized elementary operator, then both σ and δ are X-inner.

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Corollary

Let δ be a non-zero σ-derivation of a prime ring R. If δ is implemented by a generalized elementary operator, then both σ and δ are X-inner.

However, this Corollary is not true for general semiprime rings.

Example

Let $R := \mathrm{M}_n(\mathbb{C}) \oplus \mathrm{M}_n(\mathbb{C}) \oplus \mathrm{M}_n(\mathbb{C})$. Since each right ideal of $\mathrm{M}_n(\mathbb{C})$ is of the form $p\mathrm{M}_n(\mathbb{C})$ for some projection $p \in \mathrm{M}_n(\mathbb{C})$, we have $Q_{mr}(\mathrm{M}_n(\mathbb{C})) = \mathrm{M}_n(\mathbb{C})$, and hence $Q_{mr}(R) = R$. For $1 \leq i \leq 3$, let ε_i be the central idempotent of R with one non-zero entry 1 at i-th coordinate, and let p be a non-central invertible matrix in $\mathrm{M}_n(\mathbb{C})$. We define maps $\sigma, \delta : R \to R$ by

$$\sigma(x) = \sigma(x_1, x_2, x_3) := (px_1p^{-1}, x_3, x_2) \quad \text{and} \quad \delta(x) := \varepsilon_1 x - \sigma(x)\varepsilon_1.$$

Obviously σ is an automorphism of R and δ is a non-zero inner σ-derivation which is implemented by an elementary operator

$$\phi : x \mapsto \varepsilon_1 x 1_R - (p.\varepsilon_1)x(p^{-1}.\varepsilon_1)$$

However, σ is not an (X-)inner automorphism of R since σ is not the identity on the center of R (for example, $\sigma(\varepsilon_2) = \varepsilon_3$).

Finally, if d is a generalized σ-derivation, then using a decomposition $d = \delta + \rho$, one obtains:

Corollary

If a generalized σ-derivation d of R is implemented by a generalized elementary operator, then δ is an X-inner σ-derivation, and for each element $q \in Q_{mr}$ which implements δ, there exists an invertible element $p \in Q_{mr}^{\times}$ such that

$$\varepsilon(r)\sigma(x) = \varepsilon(r)pxp^{-1} \qquad (x \in R),$$

where $r := d(1) - q$. In particular, $d(x) = qx - pxp^{-1}r$ $(x \in R)$.

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Remark

All these results are also true when σ is only an epimorphism.