On Automorphisms, Derivations and Elementary Operators

Ilja Gogić

Department of Mathematics, University of Zagreb
and
Department of Mathematics and Informatics, University of Novi Sad

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This talk is based on a paper

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An associative ring $R$ is said to be **semiprime** if the zero-ideal is the intersection of prime ideals.

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**Examples of semiprime rings:**
- Any prime ring is obviously a semiprime ring.
- Any reduced ring is a semiprime ring.
- Any $J$-semisimple ring is semiprime. In particular, semisimple rings and von Neumann regular rings are all semiprime.
- Any direct product of semiprime rings is semiprime.
- Any matrix ring over a semiprime ring is a semiprime ring.
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An interesting class of additive maps $d: R \to R$ including both automorphisms and generalized derivations is the class of **generalized skew derivations**, that is, those satisfying

$$d(xy) = \delta(x)y + \sigma(x)d(y) \quad (x, y \in R),$$

for some map $\delta: R \to R$ and automorphism $\sigma \in \text{Aut}(R)$. 
Since by assumption $R$ is semiprime, it is easy to see that the map $\delta$ is automatically additive and it is uniquely determined by $d$. Moreover, $\delta$ is a $\sigma$-derivation (skew-derivation), i.e. $\delta$ satisfies

$$\delta(xy) = \delta(x)y + \sigma(x)\delta(y) \quad (x, y \in R).$$

We decompose $d$ as

$$d = \delta + \rho,$$

where $\rho := d - \delta$, and note that

$$\rho(x) = \sigma(x)d(1) \quad (x \in R).$$
On the other hand, an attractive and fairly large class of additive maps $\phi : R \to R$ is the class of **generalized elementary operators**, that is, those which can be expressed as a finite sum

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\phi(x) = \sum_i a_i x b_i \quad (x \in R),
$$

where the **coefficients** $a_i, b_i$ are elements of the Utumi right quotient ring $Q_{mr}$. If all $a_i, b_i$ lie in $R$, then we say that $\phi$ is an **elementary operator**.
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Motivated by the fact that (generalized) elementary operators comprise both \((X-)\)inner automorphisms \( x \mapsto pxp^{-1} \) and \((X-)\)inner generalized derivations \( x \mapsto px - xq \), we consider the following question:

**Problem**

Describe the form of generalized skew derivations which are implemented by (generalized) elementary operators.
The notion of a right quotient ring was introduced by Yuzo Utumi in 1956. An overring $Q$ of a ring $R$ is said to be a right quotient ring of $R$ if given $p, q \in Q$, with $p \neq 0$, there exists $a \in R$ satisfying $pa \neq 0$ and $qa \in R$. Utumi proved that for every semiprime ring (or more generally, for any ring without total left zero divisors) there exists a maximal right quotient ring, called the Utumi right quotient ring of $R$ and denoted by $Q_{mr}$. A right ideal $I$ of $R$ is said to be dense if for every $x, y \in R$, with $x \neq 0$ there exists $a \in R$ such that $xa \neq 0$ and $ya \in I$. Note that this is equivalent to saying that $R$ is a right quotient ring of $I$. 
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A right ideal \( I \) of \( R \) is said to be **dense** if for every \( x, y \in R \), with \( x \neq 0 \) there exists \( a \in R \) such that \( xa \neq 0 \) and \( ya \in I \). Note that this is equivalent to saying that \( R \) is a right quotient ring of \( I \).
The basic and in fact the characteristic four properties of $Q_{mr}$ are:

(i) $R$ is a subring of $Q_{mr}$.

(ii) For any $q \in Q_{mr}$ there exists a dense right ideal $I$ of $R$ such that $qI \subseteq R$.

(iii) If $0 \neq q \in Q_{mr}$ and $I$ is a dense right ideal of $R$, then $qI \neq 0$.

(iv) For any dense right ideal $I$ of $R$ and a right $R$-module homomorphism $f : I_R \rightarrow R_R$ there exists $q \in Q_{mr}$ such that $f$ is a left multiplication by $q$. 

The center of $Q_{mr}$ is called the extended centroid of $R$ and it is denoted by $C$. It is well known that $C$ is a von Neumann regular self-injective ring. Moreover, $C$ is a field if and only if $R$ is a prime ring.
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For any subset $S \subseteq Q_{mr}$ there exists a unique idempotent $\varepsilon(S)$ in $C$ such that

$$\text{rann}_{Q_{mr}}(Q_{mr} SQ_{mr}) = (1 - \varepsilon(S))Q_{mr}$$

and

$$\varepsilon(S)x = x \quad \text{for all } x \in S,$$

where $\text{rann}_{Q_{mr}}(X)$ denotes the right annihilator in $Q_{mr}$ of a subset $X \subseteq Q_{mr}$. The idempotent $\varepsilon(S)$ is called the **central support** of $S$. Whenever $S = \{x\}$ for some $x \in Q_{mr}$ we write $\varepsilon(x)$ for $\varepsilon(S)$. 
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An automorphism $\sigma \in \text{Aut}(R)$ (resp. $\sigma$-derivation $\delta : R \to R$) is said to be $X$-inner if there exists an element $p \in Q_{mr}^\times$ (resp. $q \in Q_{mr}$) such that $\sigma(x) = pxp^{-1}$ (resp. $\delta(x) = qx - \sigma(x)q$) for all $x \in R$. In this case we say that an element $p$ (resp. $q$) implements $\sigma$ (resp. $\delta$).
Theorem (D. Eremita, I. G. and D. Ilišević, 2013)

Suppose that $\sigma : R \to R$ is a ring epimorphism and let $a \in Q_{mr}$. If the map $\rho_a : R \to Q_{mr}$, given by $\rho_a : x \mapsto \sigma(x)a$ is implemented by a generalized elementary operator, then there exists an invertible element $p \in Q^\times_{mr}$ such that

$$\varepsilon(a)\sigma(x) = \varepsilon(a)pxp^{-1} \quad (x \in R).$$

In particular, $\rho_a(x) = pxp^{-1}a \ (x \in R)$. 
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**Corollary**

*If an epimorphism $\sigma : R \rightarrow R$ is implemented by a generalized elementary operator, then $\sigma$ is an $X$-inner automorphism of $R$.***
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**Corollary**

If an epimorphism $\sigma : R \to R$ is implemented by a generalized elementary operator, then $\sigma$ is an $X$-inner automorphism of $R$.

**Problem**

If an automorphism $\sigma \in \text{Aut}(R)$ is implemented by an elementary operator (with all coefficients lying in $R$), is $\sigma$ in fact an inner automorphism of $R$?
The answer is negative (in general):

**Example**

Let $R$ be the $C^*$-algebra consisting of all elements $x \in C([1, \infty], M_2(\mathbb{C}))$ (i.e. all continuous functions from the interval $[1, \infty] \subseteq \mathbb{R}$ to the $C^*$-algebra $M_2(\mathbb{C})$ of $2 \times 2$ complex matrices) such that

$$x(n) = \begin{bmatrix} \lambda_n(x) & 0 \\ 0 & \lambda_{n+1}(x) \end{bmatrix} \quad (n \in \mathbb{N})$$

for some convergent sequence $(\lambda_n(x))$ of complex numbers. Then $R$ admits an outer $\ast$-automorphism $\sigma$ which is implemented by an elementary operator.
The answer is negative (in general):

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for some convergent sequence $(\lambda_n(x))$ of complex numbers. Then $R$ admits an outer $*$-automorphism $\sigma$ which is implemented by an elementary operator.

**Problem**

Characterize the class of all unital semiprime rings $R$ with the property that all automorphisms of $R$ which are implemented by elementary operators are necessarily inner.
Theorem (D. Eremita, I. G. and D. Ilišević, 2013)

If a \( \sigma \)-derivation \( \delta : R \to R \) is implemented by a generalized elementary operator, then \( \delta \) is \( X \)-inner, and for each element \( q \in Q_{mr} \) which implements \( \delta \) there exists an invertible element \( p \in Q_{mr}^{\times} \) such that

\[
\varepsilon(q)\sigma(x) = \varepsilon(q)pxp^{-1} \quad (x \in R).
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In particular, \( \delta(x) = qx - pxp^{-1}q \) (\( x \in R \)).
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In particular, \( \delta(x) = qx - pxp^{-1}q \) \((x \in R)\). 

Corollary

Let \( \delta \) be a non-zero \( \sigma \)-derivation of a prime ring \( R \). If \( \delta \) is implemented by a generalized elementary operator, then both \( \sigma \) and \( \delta \) are \( X \)-inner.
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Corollary

Let \( \delta \) be a non-zero \( \sigma \)-derivation of a prime ring \( R \). If \( \delta \) is implemented by a generalized elementary operator, then both \( \sigma \) and \( \delta \) are \( X \)-inner.

However, this Corollary is not true for general semiprime rings.
Example

Let $R := M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$. Since each right ideal of $M_n(\mathbb{C})$ is of the form $pM_n(\mathbb{C})$ for some projection $p \in M_n(\mathbb{C})$, we have $Q_{mr}(M_n(\mathbb{C})) = M_n(\mathbb{C})$, and hence $Q_{mr}(R) = R$. For $1 \leq i \leq 3$, let $\varepsilon_i$ be the central idempotent of $R$ with one non-zero entry 1 at $i$-th coordinate, and let $p$ be a non-central invertible matrix in $M_n(\mathbb{C})$. We define maps $\sigma, \delta : R \to R$ by

$$\sigma(x) = \sigma(x_1, x_2, x_3) := (px_1 p^{-1}, x_3, x_2) \quad \text{and} \quad \delta(x) := \varepsilon_1 x - \sigma(x)\varepsilon_1.$$ 

Obviously $\sigma$ is an automorphism of $R$ and $\delta$ is a non-zero inner $\sigma$-derivation which is implemented by an elementary operator

$$\phi : x \mapsto \varepsilon_1 x 1_R - (p \cdot \varepsilon_1) x (p^{-1} \cdot \varepsilon_1).$$

However, $\sigma$ is not an ($X$-)inner automorphism of $R$ since $\sigma$ is not the identity on the center of $R$ (for example, $\sigma(\varepsilon_2) = \varepsilon_3$).
Finally, if $d$ is a generalized $\sigma$-derivation, then using a decomposition $d = \delta + \rho$, one obtains:

**Corollary**

If a generalized $\sigma$-derivation $d$ of $R$ is implemented by a generalized elementary operator, then $\delta$ is an $X$-inner $\sigma$-derivation, and for each element $q \in Q_{mr}$ which implements $\delta$, there exists an invertible element $p \in Q_{mr}^\times$ such that

$$\varepsilon(r)\sigma(x) = \varepsilon(r)p xp^{-1} \quad (x \in R),$$

where $r := d(1) - q$. In particular, $d(x) = qx - p xp^{-1}r \quad (x \in R)$. 
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**Remark**

All these results are also true when $\sigma$ is only an epimorphism.