Compact Notation for Finite Transformations

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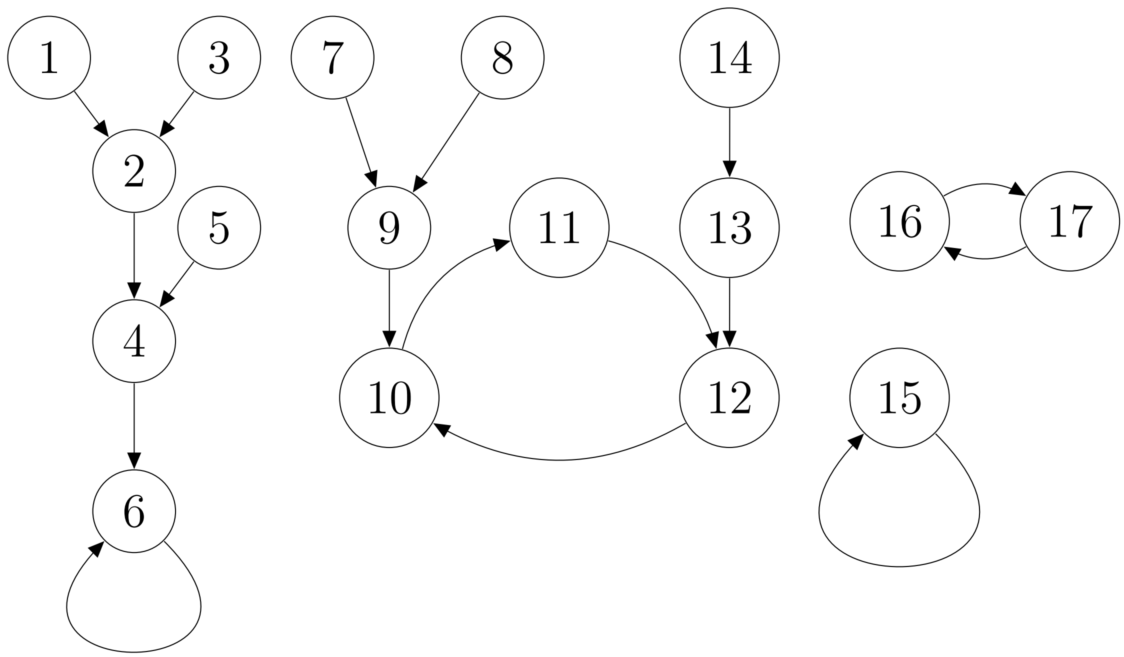

`[[[1,3|2],5|4],6] ([[7,8|9],10],11,[14,13,12]) (16,17)`

Informal Definition

1. Left-to-right comma-separated enumeration indicates a “conveyor belt” of maps. For example, `1,2,3` reads as $1 \mapsto 2$, $2 \mapsto 3$.

2. Parentheses containing a left-to-right comma-separated enumeration of points add the extra map from the last element to the first element of the enumeration, i.e. `(1,...,n)` adds the map $1 \mapsto n$.

3. Square brackets containing a left-to-right enumeration of points leave the image of the last element undefined. Therefore they can be used to denote partial mappings, or for total trajectories terminating in a trivial cycle. If something follows the closing square bracket in a left-to-right order, then the image of the last element is defined by that following element. For example ...`n],k`... defines the map $n \mapsto k$.

4. A vertical bar | ("splat") appearing before the last element in a square bracket turns the preceding sequence into a set and maps all of its elements into the last point. They all 'hit the same wall'. For instance, `[1,2,3|4]` yields the maps $1 \mapsto 4$, $2 \mapsto 4$ and $3 \mapsto 4$. (Note, this is an exception to left-to-right mapping mentioned in (1).)

The Language of Compact Notation Strings

The following context-free grammar defines the language of compact notations. The terminal symbols are `[`,`]`,`(`,`)`,`,`,`|` and the symbols for the n points. The nonterminal symbols are C for components, N for nontrivial trees, T for trees and P for points.

$$
\begin{aligned}
S &\to C^{+} \mid \texttt{()} \\
C &\to \texttt{(}(T\texttt{,})^{+}T\texttt{)} \mid N \\
N &\to \texttt{[}(T\texttt{,})^{+}T \mid P\texttt{]} \mid \texttt{[}(T\texttt{,})^{+}T\texttt{]} \\
T &\to N \mid P \\
P &\to \texttt{1} \mid \texttt{2} \mid \texttt{3} \mid \ldots \mid \texttt{n}
\end{aligned}
$$

Canonical Form

Both `[1,2,[3,4|6]]` and `[[1,2],3,4|6]` denote the same transformation. However, a simple recursive algorithm that starts from the point(s) of each component's cycle can produce a canonical form. All we need to do is to examine the cardinality of the preimage set from outside the cycle, i.e. the number of incoming arrows: 0 leaf, 1 conveyor belt, more than 1 splat.

For instance, here are the conjugacy class representatives of the full transformation semigroup T_4 on four points in canonical form:

`[1,2,3|4]` , `[[1,2],3|4]` , `[1,2|3]` , `[[1,2|3],4]` , `[1,2,3,4]` , `[1,2,3]`, `[1,2][3,4]` , `[1,2]` , `[1,2](3,4)` , `()` , `(1,2)` , `([1,2],3)` , `(1,2,3)` , `([1,2|3],4)` , `([1,2],[3,4])` , `([1,2,3],4)` , `(1,2)(3,4)` , `([1,2],3,4)`, `(1,2,3,4)`

http://arxiv.org/abs/1306.1138, http://sgpdec.sf.net