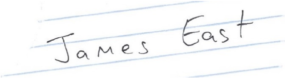
Infinite partition monoids



University of Western Sydney

4th Novi Sad Algebraic Conference

University of Novi Sad

5–9 June 2013

Motivation

Various results on infinite symmetric groups and transformation semigroups by:

- Sierpiński, Banach,
- Galvin, Bergman,
- Higgins, Howie, Maltcev, Mitchell, Péresse, Ruškuc, . . .

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What about other semigroups? Today: partition monoids.

1. Transformation semigroups.

Let:

- X be an infinite set,
- $\mathcal{S}_X = \{\text{permutations of } X\}$
  $= \text{ the symmetric group on } X,$
- $\mathcal{T}_X = \{\text{transformations of } X\}$
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Theorem (Sierpiński, Banach, 1935)

Any countable subset of $\mathcal{T}_X$ is contained in a 2-generated subsemigroup of $\mathcal{T}_X$.

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Theorem (Maltcev, Mitchell, Ruškuc, 2009)

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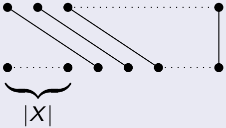
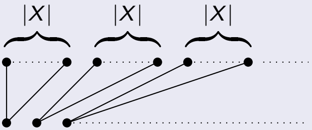
$\alpha =$ 

and $\beta =$ 

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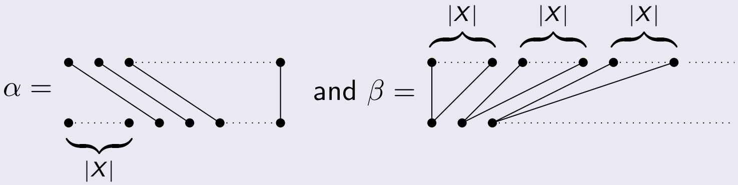

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$\mathcal{T}_X = \langle \mathcal{S}_X, \alpha, \beta \rangle$ where:

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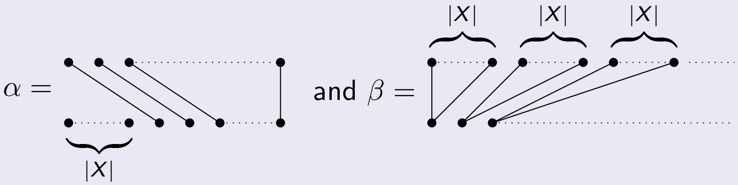

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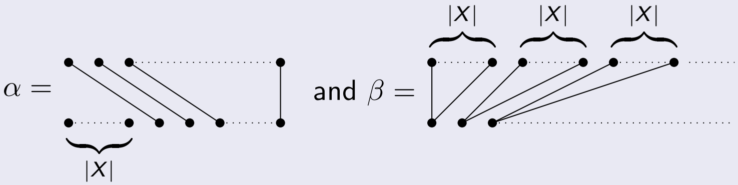

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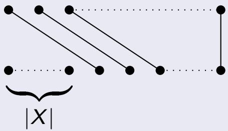
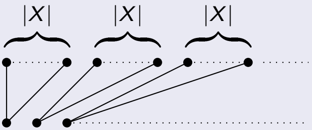
$\alpha =$ 

and $\beta =$ 

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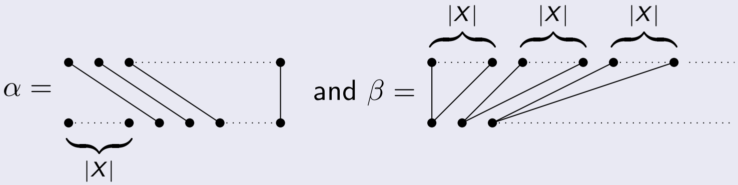

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If $|X|$ is regular, then $\mathcal{T}_X = \langle \mathcal{S}_X, \alpha, \beta \rangle$ iff:

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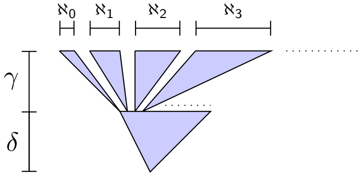

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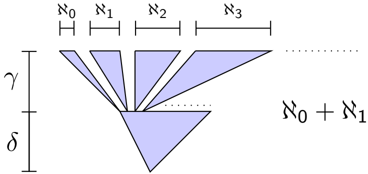

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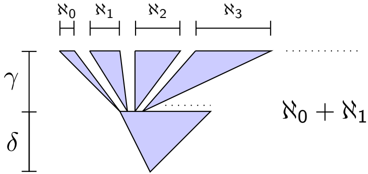

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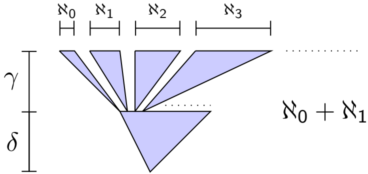

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- $|X|$ is singular if $X = \bigcup_{i \in I} X_i$ with $|I| < |X|$ and $|X_i| < |X|$,
- $|X|$ is regular otherwise.

Theorem (East, Mitchell, Péresse, 2013)

If $|X|$ is singular, then $\mathcal{T}_X = \langle \mathcal{S}_X, \alpha, \beta \rangle$ iff:

- α is is surjective and $|X \setminus X\alpha| = |X|$, and
- β is surjective and $\{x \in X : |x\alpha^{-1}| \geq \mu\}$ has size $|X|$ for all cardinals $\mu < |X|$.

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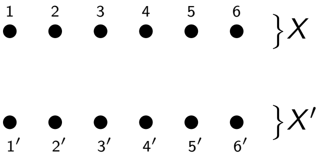

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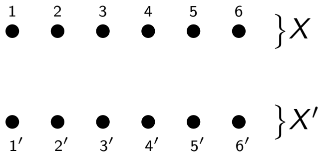

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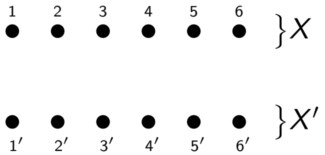

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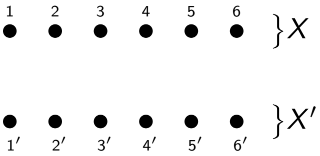

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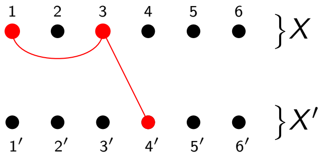

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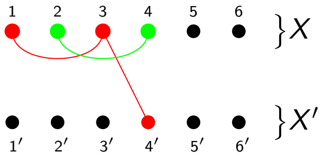

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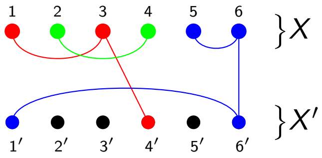

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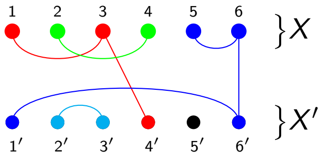

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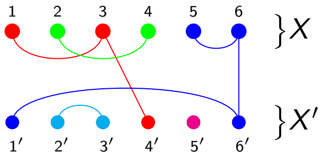

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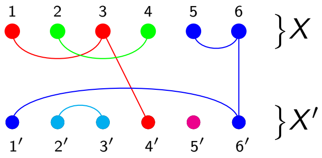

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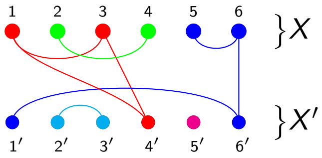

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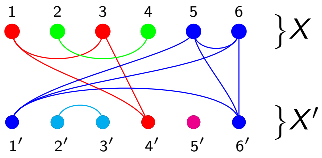

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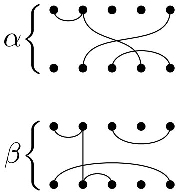

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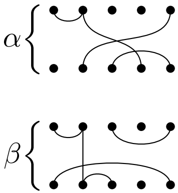

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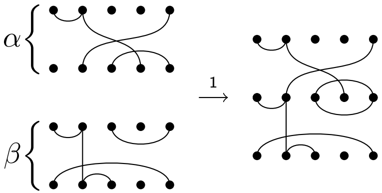

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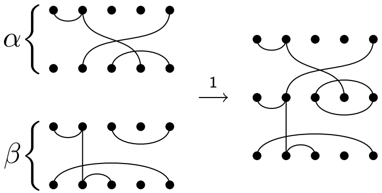

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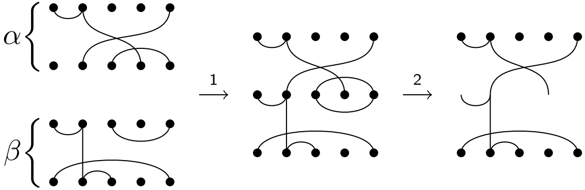

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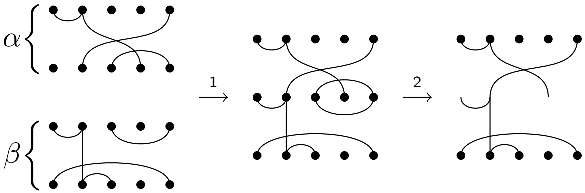

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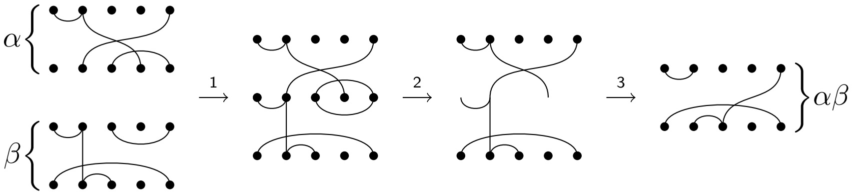

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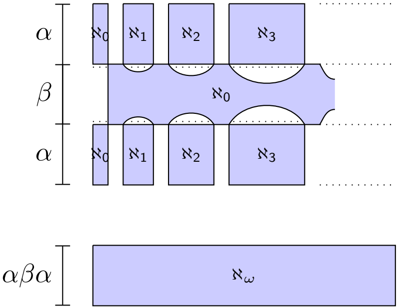

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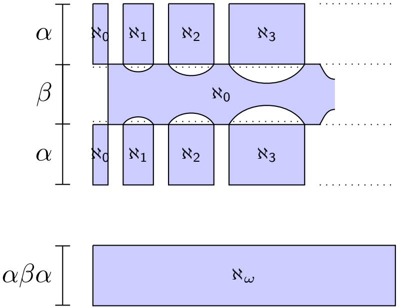
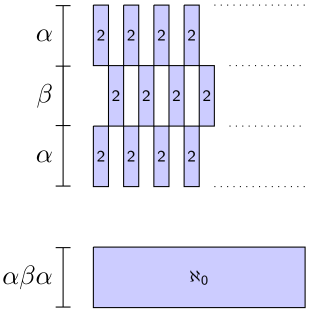


Blocks of singular cardinality can be made from smaller blocks.

Interesting things happen when X is infinite:



2. Partition monoids.

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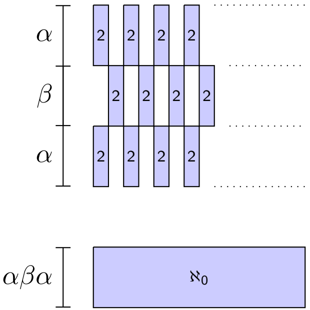
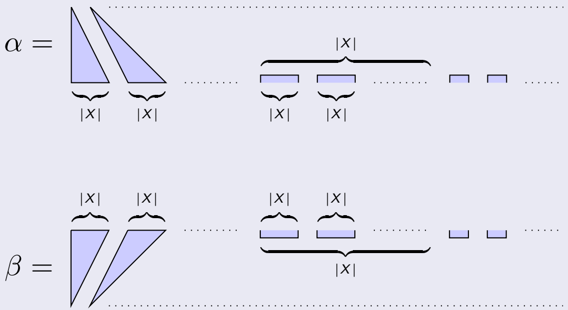


(Countably) infinite blocks can be made from finite blocks.

2. Partition monoids.

Theorem

$\mathcal{P}_X = \langle \mathcal{S}_X, \alpha, \beta \rangle$ where



Proof: Let $\gamma \in \mathcal{P}_X$.

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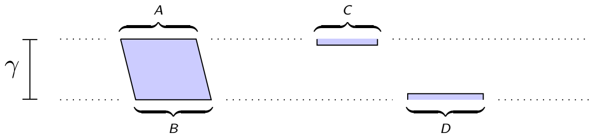
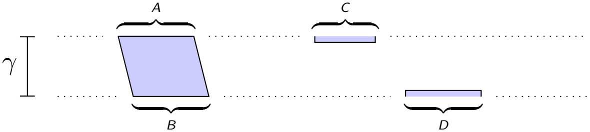


Proof: Let $\gamma \in \mathcal{P}_X$. We'll show that $\gamma = \alpha\pi\beta$ for some $\pi \in \mathcal{S}_X$.



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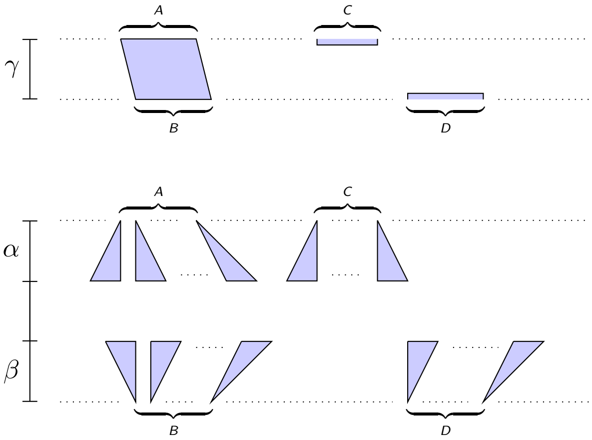

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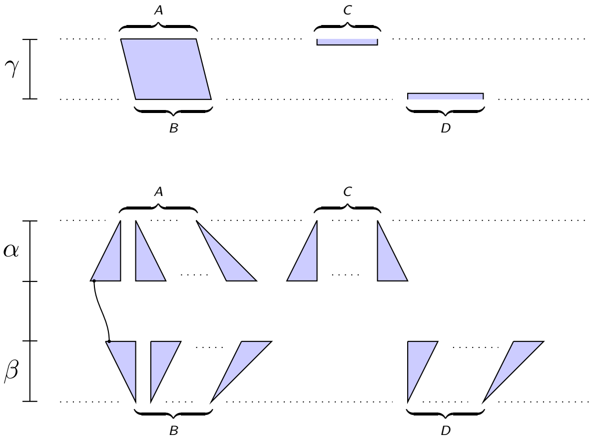

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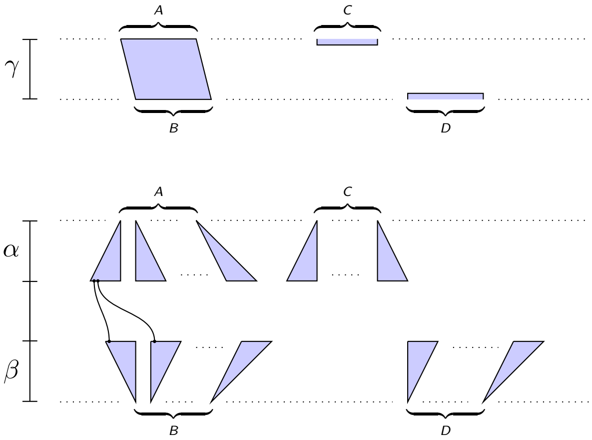

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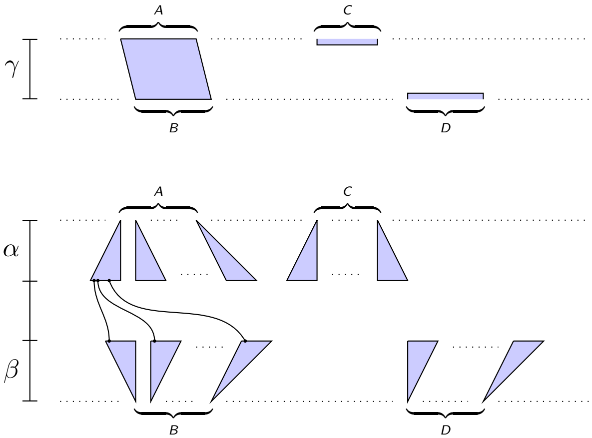

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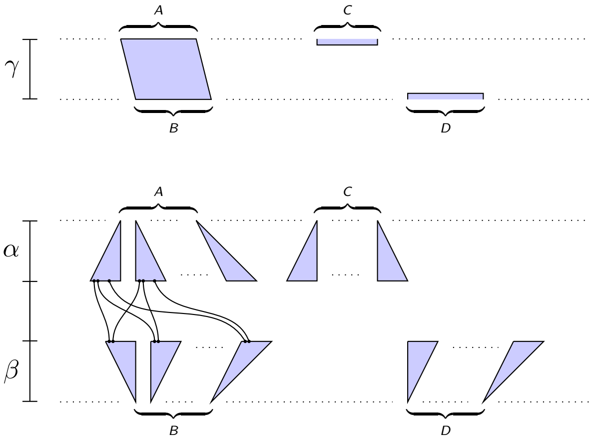

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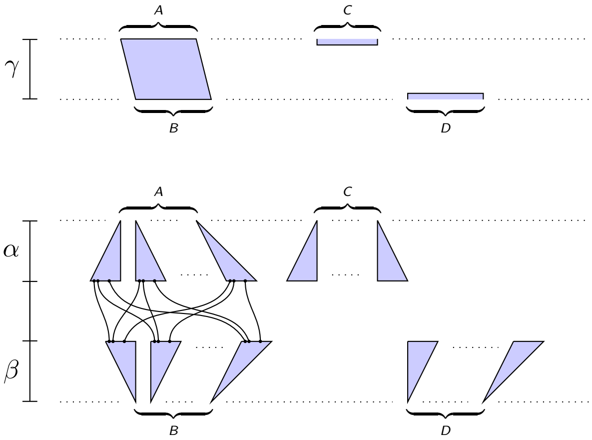

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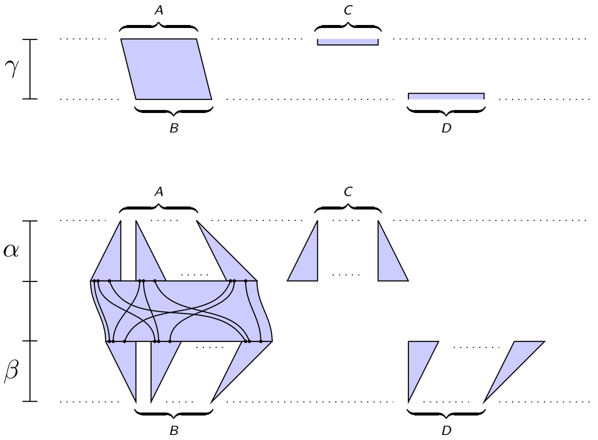

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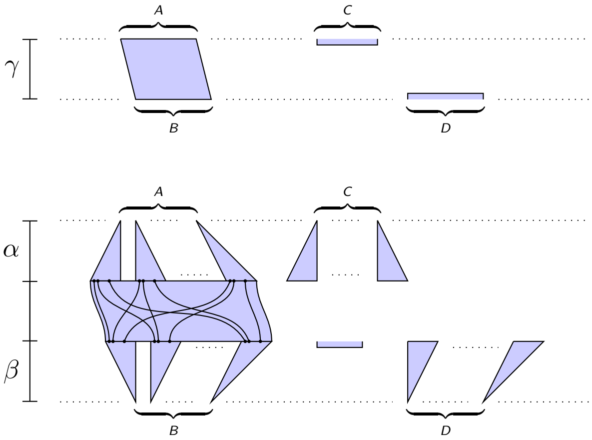

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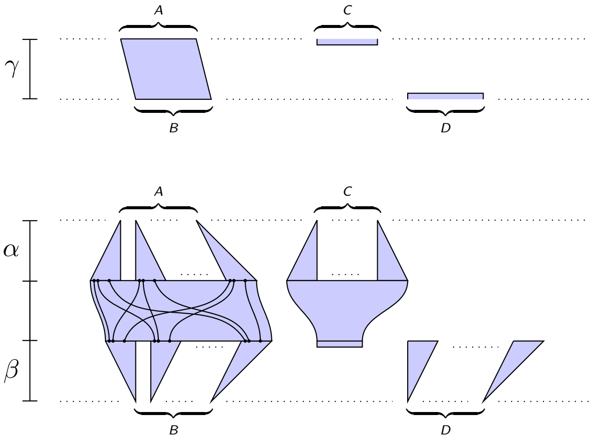

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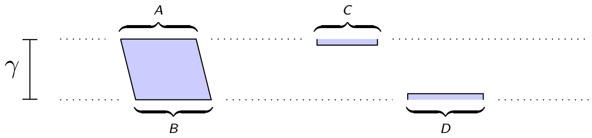

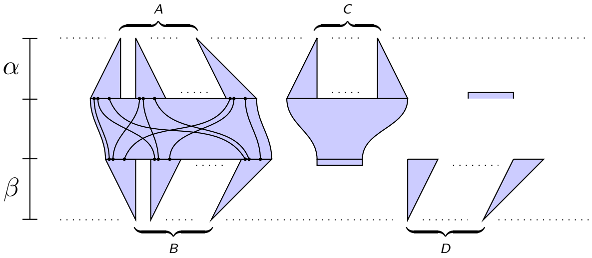

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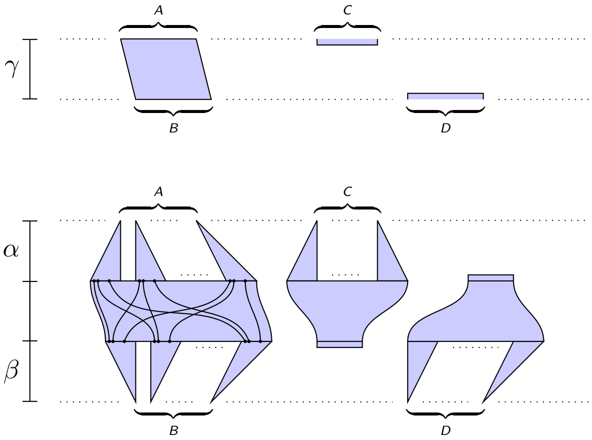

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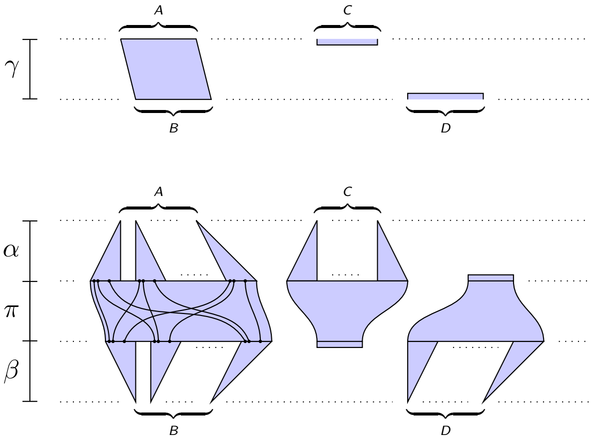
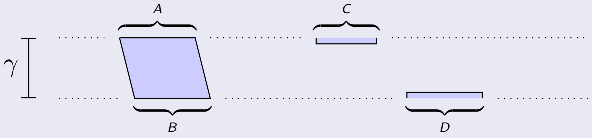


2. Partition monoids.

Some notation

Let $\gamma \in \mathcal{P}_X$ and let $\mu \leq |X|$ be a cardinal.



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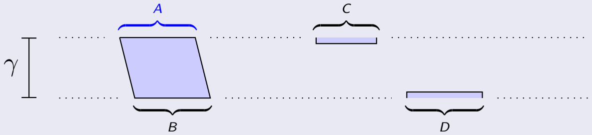

Define:

- $c_u(\gamma, \mu)$ = number of connected upper blocks of size $\geq \mu$,

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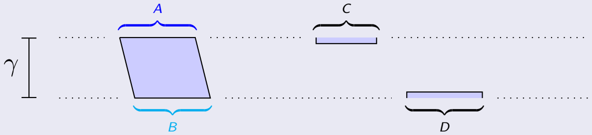

Define:

- $c_u(\gamma, \mu) =$ number of connected upper blocks of size $\geq \mu$,
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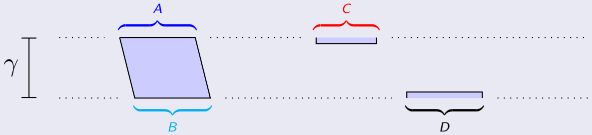

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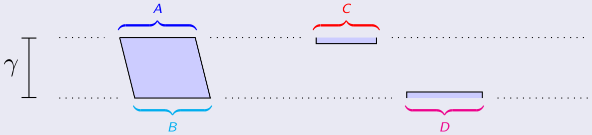

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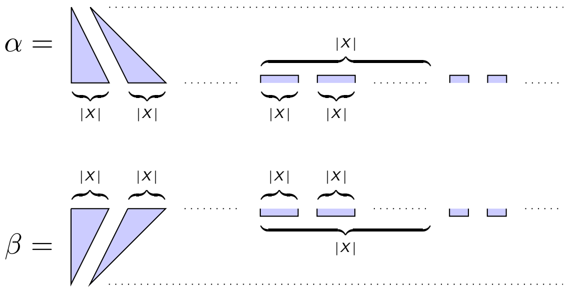
From theorem:



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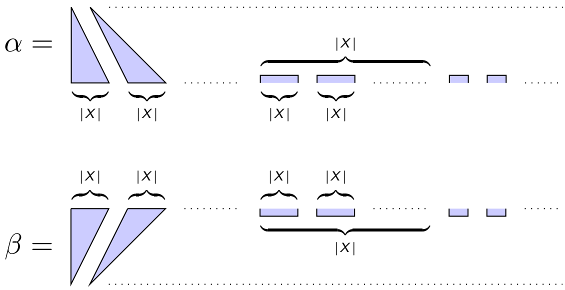
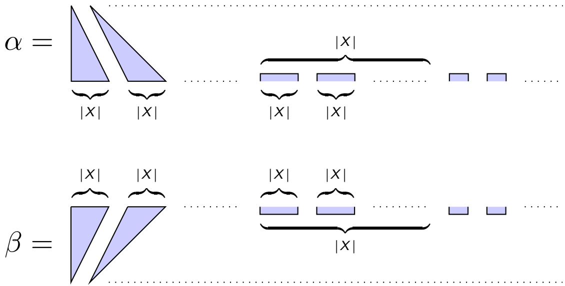


Here:

- $c_u(\alpha, 2) = 0$

2. Partition monoids.

From theorem:



Here:

- $c_u(\alpha, 2) = 0 = d_u(\alpha, 1)$,

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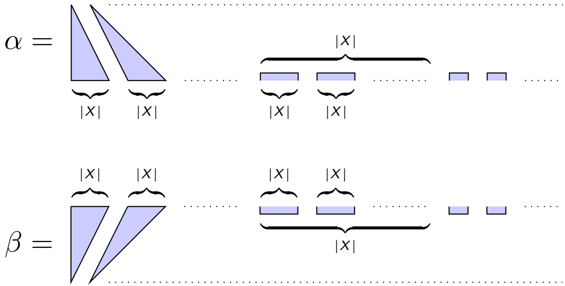

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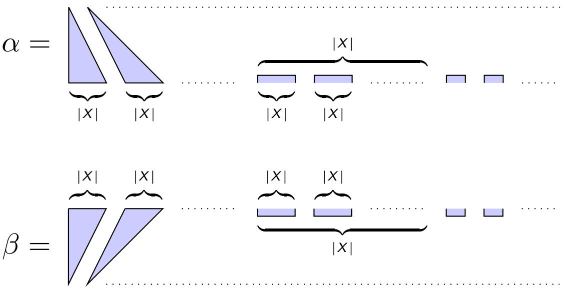

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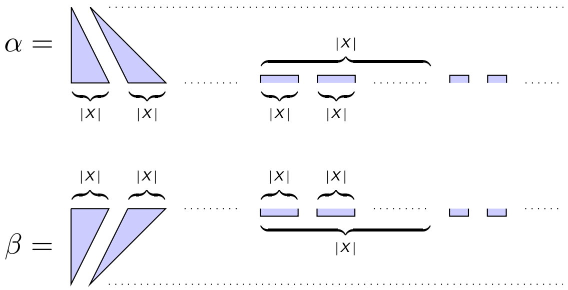

Here:

- $c_u(\alpha, 2) = 0 = d_u(\alpha, 1)$,
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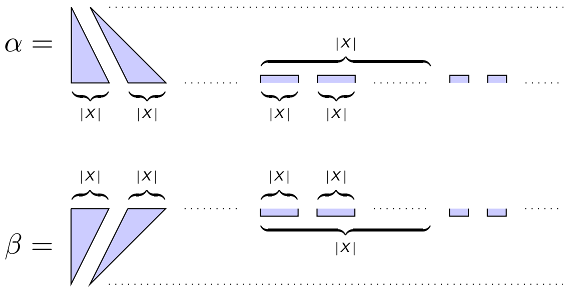

Here:

- $c_u(\alpha, 2) = 0 = d_u(\alpha, 1)$,
- $c_l(\alpha, |X|) = |X| = d_l(\alpha, |X|)$,
- α is "injective",
- $c_l(\beta, 2) = 0 = d_l(\beta, 1)$,
- $c_u(\beta, |X|) = |X| = d_u(\beta, |X|)$,

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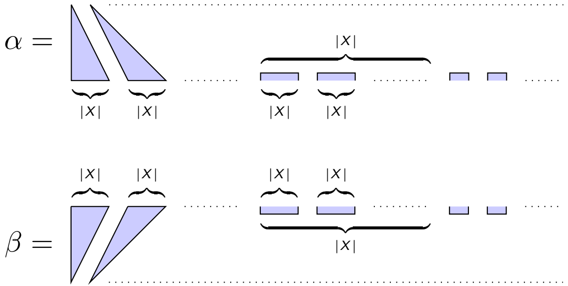

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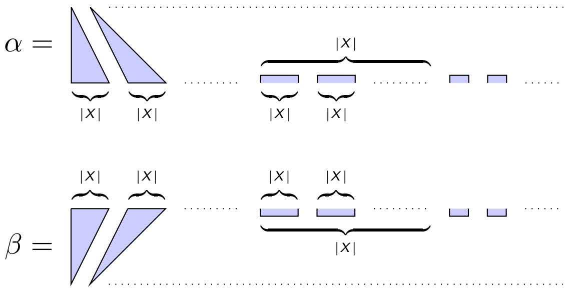
- $c_u(\alpha, 2) = 0 = d_u(\alpha, 1)$,
- $c_l(\alpha, |X|) = |X| = d_l(\alpha, |X|)$,
- α is "injective",
- $c_l(\beta, 2) = 0 = d_l(\beta, 1)$,
- $c_u(\beta, |X|) = |X| = d_u(\beta, |X|)$,
- β is "co-injective",

2. Partition monoids.

From theorem:



Here:

- $c_u(\alpha, 2) = 0 = d_u(\alpha, 1)$,
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- α is "injective",
- α is NOT "co-injective",
- $c_l(\beta, 2) = 0 = d_l(\beta, 1)$,
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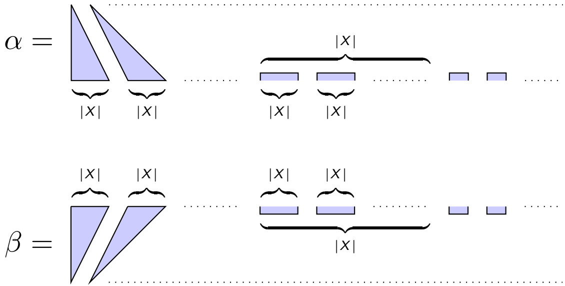

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- $c_u(\beta, |X|) = |X| = d_u(\beta, |X|)$,
- β is "co-injective",
- β is NOT "injective".

Theorem

$\mathcal{P}_X = \langle \mathcal{S}_X, \alpha, \beta \rangle$ if

- $c_u(\alpha, 2) = d_u(\alpha, 1) = 0$,
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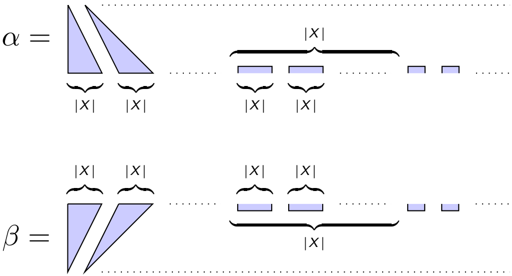
- $c_u(\beta, |X|) = d_u(\beta, |X|) = |X|$.



Theorem

$\mathcal{P}_X = \langle \mathcal{S}_X, \alpha, \beta \rangle$ if

- $c_u(\alpha, 2) = d_u(\alpha, 1) = 0$,
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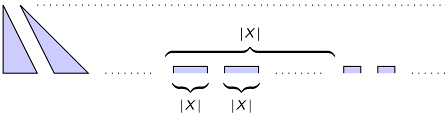
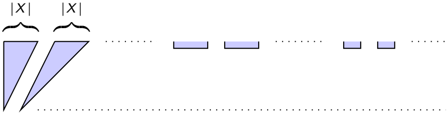
- $d_l(\alpha, 1) = |X|$,
- $c_l(\beta, 2) = d_l(\beta, 1) = 0$,
- $c_u(\beta, |X|) + d_u(\beta, |X|) = |X|$,
- $d_u(\beta, 1) = |X|$.

$\alpha =$ 

$\beta =$ 

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- $c_u(\alpha, 2) = d_u(\alpha, 1) = 0$,
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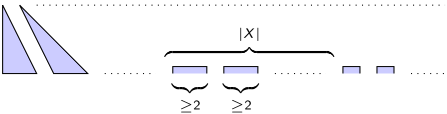
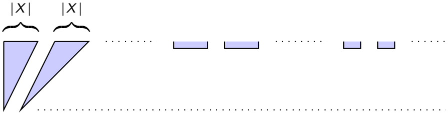
- $d_l(\alpha, 1) = |X|$,
- $c_l(\beta, 2) = d_l(\beta, 1) = 0$,
- $c_u(\beta, |X|) + d_u(\beta, |X|) = |X|$,
- $d_u(\beta, 1) = |X|$.

$\alpha =$ 

$\beta =$ 

2. Partition monoids.

Theorem

$\mathcal{P}_X = \langle \mathcal{S}_X, \alpha, \beta \rangle$ if

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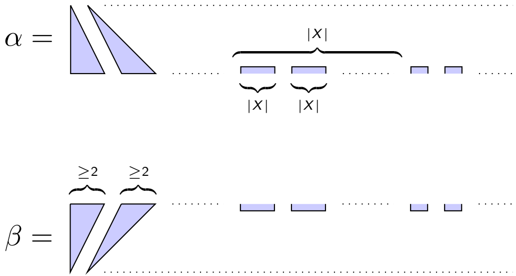
- $d_l(\alpha, 1) = |X|$,
- $c_l(\beta, 2) = d_l(\beta, 1) = 0$,
- $c_u(\beta, 2) + d_u(\beta, 2) = |X|$,
- $d_u(\beta, 1) = |X|$.



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Theorem

$\mathcal{P}_X = \langle \mathcal{S}_X, \alpha, \beta \rangle$ if

- $c_u(\alpha, 2) = d_u(\alpha, 1) = 0$,
- $d_l(\alpha, 1) = |X|$,
- $c_l(\beta, 2) = d_l(\beta, 1) = 0$,
- $d_u(\beta, 1) = |X|$,

and either

- $c_l(\alpha, 2) + d_l(\alpha, 2) = |X|$,
- $c_u(\beta, |X|) + d_u(\beta, |X|) = |X|$,

or

- $c_l(\alpha, |X|) + d_l(\alpha, |X|) = |X|$,
- $c_u(\beta, 2) + d_u(\beta, 2) = |X|$.

2. Partition monoids.

Theorem

If X is uncountable and regular, then $\mathcal{P}_X = \langle \mathcal{S}_X, \alpha, \beta \rangle$ iff

- $c_u(\alpha, 2) = d_u(\alpha, 1) = 0$,
- $d_l(\alpha, 1) = |X|$,
- $c_l(\beta, 2) = d_l(\beta, 1) = 0$,
- $d_u(\beta, 1) = |X|$,

and either

- $c_l(\alpha, 2) + d_l(\alpha, 2) = |X|$,
- $c_u(\beta, |X|) + d_u(\beta, |X|) = |X|$,

or

- $c_l(\alpha, |X|) + d_l(\alpha, |X|) = |X|$,
- $c_u(\beta, 2) + d_u(\beta, 2) = |X|$.

2. Partition monoids.

Theorem

If X is countable, then $\mathcal{P}_X = \langle \mathcal{S}_X, \alpha, \beta \rangle$ iff

- $c_u(\alpha, 2) = d_u(\alpha, 1) = 0$,
- $d_l(\alpha, 1) = |X|$,
- $c_l(\beta, 2) = d_l(\beta, 1) = 0$,
- $d_u(\beta, 1) = |X|$,

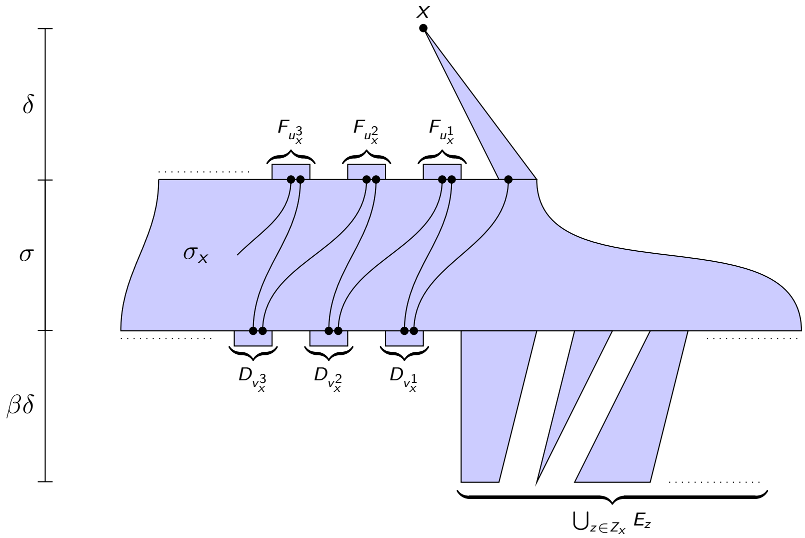
and either

- $c_l(\alpha, 2) + d_l(\alpha, 2) = |X|$,
- $c_u(\beta, 2) + d_u(\beta, 3) = |X|$,

or

- $c_l(\alpha, 2) + d_l(\alpha, 3) = |X|$,
- $c_u(\beta, 2) + d_u(\beta, 2) = |X|$.

2. Partition monoids.



2. Partition monoids.

Theorem

If X is singular, then $\mathcal{P}_X = \langle \mathcal{S}_X, \alpha, \beta \rangle$ iff

- $c_u(\alpha, 2) = d_u(\alpha, 1) = 0$,
- $d_l(\alpha, 1) = |X|$,
- $c_l(\beta, 2) = d_l(\beta, 1) = 0$,
- $d_u(\beta, 1) = |X|$,

and either

- $c_l(\alpha, 2) + d_l(\alpha, 2) = |X|$,
- $c_u(\beta, \mu) + d_u(\beta, \mu) = |X|$ for all cardinals $\mu < |X|$,

or

- $c_l(\alpha, \mu) + d_l(\alpha, \mu) = |X|$ for all cardinals $\mu < |X|$,
- $c_u(\beta, 2) + d_u(\beta, 2) = |X|$.

2. Partition monoids.

Corollary 1

Any countable subset of $\mathcal{P}_X$ is contained in a 4-generated subsemigroup of $\mathcal{P}_X$.

2. Partition monoids.

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Any countable subset of $\mathcal{P}_X$ is contained in a 4-generated subsemigroup of $\mathcal{P}_X$.

Proof: Follows from general results of Mitchell and Péresse, and:

- any countable subset of $\mathcal{S}_X$ is contained in a 2-generated subsemigroup of $\mathcal{S}_X$ (Galvin), and
- $\mathcal{P}_X = \langle \mathcal{S}_X, \alpha, \beta \rangle$.

Corollary 1

Any countable subset of $\mathcal{P}_X$ is contained in a 4-generated subsemigroup of $\mathcal{P}_X$.

Proof: Let $\gamma_1, \gamma_2, \ldots \in \mathcal{P}_X$.

Corollary 1

Any countable subset of $\mathcal{P}_X$ is contained in a 4-generated subsemigroup of $\mathcal{P}_X$.

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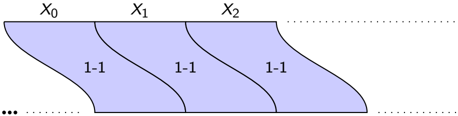
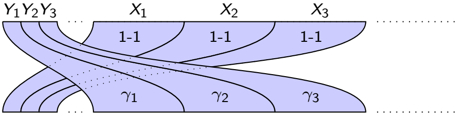
Then $\gamma_n = \alpha\beta\alpha^n\beta^2\alpha^{-n}\beta^{-1}\alpha^{-1}$ where:

$\alpha =$ 

$\beta =$ 

Corollary 2

If $\mathcal{P}_X = \langle U \rangle$, then $\mathcal{P}_X = U \cup U^2 \cup \cdots \cup U^n$ for some n.

2. Partition monoids.

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Proof: Follows from general results of Maltcev, Mitchell and Ruškuc, and:

- $\mathcal{P}_X$ is "strongly distorted" (Corollary 1).

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