Expressibility of digraph homomorphisms in the logic LFP+Rank (joint work with C. Heggerud and F. McInerney)

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Outline

- Fixed template constraint satisfaction problem: essentially a homomorphism problem for finite relational structures.
- We are interested in membership in the class $CSP(\mathbb{A})$, a computational problem that obviously lies in the complexity class **NP**.
- Dichotomy Conjecture (Feder and Vardi): either $CSP(\mathbb{A})$ has polynomial time membership or it has **NP**-complete membership problem.

Particular cases already known to exhibit the dichotomy:

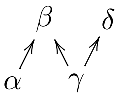
- Schaefer's dichotomy for 2-element templates;
- dichotomy for undirected graph templates due to Hell and Nešetřil
- 3-element templates (Bulatov);
- digraphs with no sources and sinks (Barto, Kozik and Niven); also some special classes of oriented trees (Barto, Bulin)
- templates in which every subset is a fundamental unary relation (list homomorphism problems; Bulatov, also Barto).

- Feder and Vardi reduced the problem of proving the dichotomy conjecture to the particular case of digraph CSPs, and even to digraph CSPs whose template is a balanced digraph (a digraph on which there is a level function).
- Specifically, for every template $\mathbb{A}$ there is a balanced digraph $\mathbb{D}$ such that $CSP(\mathbb{A})$ is polynomial time equivalent to $CSP(\mathbb{D})$.
- Some of the precise structure of $CSP(\mathbb{A})$ is necessarily altered in the transformation to $CSP(\mathbb{D})$.

- Algebraic approach to the CSP dichotomy conjecture: associate polynomial time algorithms to $Pol(\mathbb{A})$
- complexity of $CSP(\mathbb{A})$ is precisely (up to logspace reductions) determined by the polymorphisms of $\mathbb{A}$.

- Atserias (2006) revisited a construction from Feder and Vardi's original article to construct a tractable digraph CSP that is provably not solvable by the bounded width (local consistency check) algorithm.
- This construction relies heavily on finite model-theoretic machinery: quantifier preservation, cops-and-robber games (games that characterize width k), etc.

The path $\mathbb{N}$



Theorem

Let $\mathbb{A}$ be a relational structure. There exists a digraph $\mathbb{D}_{\mathbb{A}}$ such that the following holds: let Σ be any linear idempotent set of identities such that each identity in Σ is either balanced or contains at most two variables. If the digraph $\mathbb{N}$ satisfies Σ, then $\mathbb{D}_{\mathbb{A}}$ satisfies Σ if and only if $\mathbb{A}$ satisfies Σ.
The digraph $\mathbb{D}_{\mathbb{A}}$ can be constructed in logspace with respect to the size of A.

Corollary

Let $\mathbb{A}$ be a CSP template. Then each of the following hold equivalently on $\mathbb{A}$ and $\mathbb{D}_{\mathbb{A}}$.

- *Taylor polymorphism or equivalently weak near-unanimity (WNU) polymorphism or equivalently cyclic polymorphism* (*conjectured to be equivalent to being tractable if $\mathbb{A}$ is a core*);
- *Polymorphisms witnessing SD*($\wedge$) (*equivalent to bounded width*);
- (*for $k \geq 4$*) *k-ary edge polymorphism* (*equivalent to few subpowers*);
- *k-ary near-unanimity polymorphism* (*equivalent to strict width*);

Corollary

(Continued)

- *totally symmetric idempotent (TSI) polymorphisms of all arities* (*equivalent to width* 1);
- *Hobby-McKenzie polymorphisms* (*equivalent to the corresponding variety satisfying a non-trivial congruence lattice identity*);
- *Gumm polymorphisms witnessing congruence modularity;*
- *Jónsson polymorphisms witnessing congruence distributivity;*
- *polymorphisms witnessing SD*$(\vee)$*;*
- (for $n \geq 3$) *polymorphisms witnessing congruence n-permutability.*

Digraph Canonization Problem

- Consider all *finite* structures in a fixed finite relational vocabulary (may assume that the vocabulary is $\{E\}$, E-binary.)
- For a logic (i.e., a description or query language) $\mathcal{L}$, we ask for which properties P, there is a sentence φ of the language such that

$$\mathbb{A} \in P \iff A \models \varphi.$$

- Of particular interest is the case when $P \in$**P**, the class of all properties decidable in polynomial time (Canonization Problem)

- Clearly, the first-order logic cannot capture **P** on digraphs (e.g. weak/strong connectedness)

Least Fixed Point Logic (LFP)

- LFP: logic obtained from the first-order logic by closing it under formulas computing the least fixed points of monotone operators defined by positive formulas.
- On structures that come equipped with a linear order, LFP expresses precisely those properties that are in **P**.
- LFP cannot express evenness of a digraph (pebble games.)

- Immermann: proposed LFP+C, a two sorted extension of LFP with a mechanism that allows counting.
- There are existential quantifiers that count the number of elements of the structure which satisfy a formula φ. Also, we have a linear order built into one of the sort (essentially, positive integers.)
- FO quantifiers are bounded over the integer sort.
- There are polynomial time properties of digraphs not definable in LFP+C (Cai-Fürer-Immermann graphs; Bijection games)
- Atserias, Bulatov, Dawar (2007): LFP+C cannot express solvability of linear equations over $\mathbb{F}_2$.

Expressibility of HOM($\mathbb{D}$) in extensions of LFP

- **Problem:** Is there an extension of first-order logic $\mathcal{L}$ which is poly-time testable on finite structures such that $\neg$ HOM($\mathbb{D}$) can be expressed in $\mathcal{L}$ if and only if HOM($\mathbb{D}$) is in **P** ($\mathbb{D}$ - a finite digraph)?
- LFP+C is not such a logic, by the Atserias-Bulatov-Dawar result.
- What is lacking?

- What can be expressed in LFP+C?
- Over a finite field $\mathbb{F}_p$, we can express matrix multiplication, non-singularity of matrices, the inverse of a matrix, determinants, the characteristic polynomial... (Dawar, Grohe, Holm, Laubner, 2010)
- What cannot be expressed?

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- What cannot be expressed? The rank of the matrix.

Logic LFP+Rank

- LFP+Rank is the logic obtained from LFP by adding the ability to compute the rank of a matrix over a finite field $\mathbb{F}_q$. It is a proper extension of LFP+C.
- Integer sort is equipped with the usual operations and relations $(+, \times, <)$; Quantifiers $\forall, \exists$ are still bounded over this sort.
- LFP+Rank is poly-time testable on finite structures.
- All known examples of non-expressible properties in LFP+C can be handled in this logic. (Dawar, Grohe, Holm, Laubner)
- There is a back-and-forth game that captures this logic.

Matrix Of A Logical Formula

- $x_1, x_2, \ldots, x_n$ - vertices of a finite digraph $\mathbb{D}$;
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$M(\phi; x_1, \ldots, x_n)$ - the $n \times n$-matrix over $\mathbb{F}_p$ defined by:

$$M(\phi; x_1, \ldots, x_n)[i,j] = 1 \quad \Leftrightarrow \quad \phi(x_i, x_j) \text{ holds in } \mathbb{D};$$

otherwise, the entry is 0.

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This can be generalized in several ways: we can use tuples of any fixed lenth instead of individual variables x_i's (consequently, we may end up with non-square matrices) or, we can work with any finite number of formulas instead of a single formula ϕ (consequently, we no longer get $\{0, 1\}$-valued matrices only.)

An addition to Bulin-D.-Jackson-Niven result

Theorem

(D., Heggerud, McInerney, 2013) Let $\mathbb{A}$ *be a finite relational structure and* $\mathbb{D}_A$ *the balanced digraph obtained by Bulin-D.-Jackson-Niven construction. Then,* $\neg\,\mathrm{HOM}(\mathbb{A})$ *is expressible in LFP+Rank if and only if* $\neg\,\mathrm{HOM}(\mathbb{D}_A)$ *is expressible in LFP+Rank.*

Open Problem

If a finite digraph admits a weak near unanimity polymorphism, is $\neg\,\mathrm{HOM}(\mathbb{D})$ expressible in LFP+Rank?

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If a finite digraph admits a weak near unanimity polymorphism, is $\neg\,\mathrm{HOM}(\mathbb{D})$ expressible in LFP+Rank?

If the answer to this question is affirmative, the Dichotomy Conjecture for digraphs is true.

Problem: If a digraph $\mathbb{D}$ admits a k-ary edge polymorphism (for some k), is $\neg\,\mathrm{HOM}(\mathbb{D})$ expressible in LFP+Rank?