A generalization of the Kaloujnine-Krasner Theorem

Tamás Dékány

Bolyai Institute, University of Szeged

June 08, 2013

Motivating theorem from group theory:

Kaloujnine–Krasner Theroem

For any groups N and H every extension of N by H is embeddable in the wreath product of N by H.

Motivating theorem from group theory:

Kaloujnine–Krasner Theroem

For any groups N and H every extension of N by H is embeddable in the wreath product of N by H.

Is there any direct generalization for semigroups?
If not, whether we can find „similar” theorem, which is still a generalization?

completely simple semigroups $\equiv$

Rees-matrix semigroups with normalized sandwich matrices

$S = \mathcal{M}(G; I, \Lambda; P)$, P normalized
$\rho \subseteq S \times S$ congruence
ρ is a group congruence of S iff $\exists\ N \lhd G$ s.t. every entry of P are from N

completely simple semigroups $\equiv$

Rees-matrix semigroups with normalized sandwich matrices

$S = \mathcal{M}(G; I, \Lambda; P)$, P normalized
$\rho \subseteq S \times S$ congruence
ρ is a group congruence of S iff $\exists\ N \lhd G$ s.t. every entry of P are from N

moreover $S/\rho \cong G/N$ and $\operatorname{Ker} \rho = \mathcal{M}(N; I, \Lambda; P)$

completely simple semigroups $\equiv$

Rees-matrix semigroups with normalized sandwich matrices

$S = \mathcal{M}(G; I, \Lambda; P)$, P normalized
$\rho \subseteq S \times S$ congruence
ρ is a group congruence of S iff $\exists\ N \lhd G$ s.t. every entry of P are from N

moreover $S/\rho \cong G/N$ and $\operatorname{Ker} \rho = \mathcal{M}(N; I, \Lambda; P)$

We say that $S = \mathcal{M}(G; I, \Lambda; P)$ is an *extension* of $K = \mathcal{M}(N; I, \Lambda; P)$ by G/N.

S semigroup, H group, H acts on S
multiplication on $S \times H$:

$$(s, A)(t, B) = (s \cdot {}^{A}t, AB)$$

this is $S \rtimes H$ — *semidirect product* of S by H, with respect to the given action of H on S

S semigroup, H group, H acts on S
multiplication on $S \times H$:

$$(s, A)(t, B) = (s \cdot {}^{A}t, AB)$$

this is $S \rtimes H$ — *semidirect product* of S by H, with respect to the given action of H on S

Special construction: semidirect product $S^H \rtimes H$ with respect to the action H on S^H defined by, for $f \in S^H,\ A \in H$:

$${}^{A}f : H \to S,\ \ B({}^{A}f) = (BA)f$$

this is the *wreath product* of S by H, denoted by $S \wr H$

S semigroup, H group, H acts on S
multiplication on $S \times H$:

$$(s, A)(t, B) = (s \cdot {}^{A}t, AB)$$

this is $S \rtimes H$ — *semidirect product* of S by H, with respect to the given action of H on S

Special construction: semidirect product $S^H \rtimes H$ with respect to the action H on S^H defined by, for $f \in S^H$, $A \in H$:

$${}^{A}f : H \to S,\ \ B({}^{A}f) = (BA)f$$

this is the *wreath product* of S by H, denoted by $S \wr H$

Important: S and H completely determines $S \wr H$.

Kaloujnine–Krasner Theroem

For any groups N and H every extension of N by H is embeddable in the wreath product of N by H.

Kaloujnine–Krasner Theroem

For any groups N and H every extension of N by H is embeddable in the wreath product of N by H.

Let G be an extension of N by H.
r_A $(A \in H)$ — transversal of the cosets modulo N in G
$f_g \in N^H$ $(g \in G)$:
An embedding:

$$\varphi\colon G \to N \wr H, \; g \mapsto (f_g, gN)$$

$$f_g\colon H \to N, \; A \mapsto r_A g r_{A \cdot gN}^{-1}$$

Let $S = \mathcal{M}(G; I, \Lambda; P)$ be an extension of the semigroup $K = \mathcal{M}(N; I, \Lambda; P)$ by the group H.

Let $S = \mathcal{M}(G; I, \Lambda; P)$ be an extension of the semigroup $K = \mathcal{M}(N; I, \Lambda; P)$ by the group H. Does there exist an embedding

$$S \to K \wr H?$$

Let $S = \mathcal{M}(G; I, \Lambda; P)$ be an extension of the semigroup $K = \mathcal{M}(N; I, \Lambda; P)$ by the group H. Does there exist an embedding

$$S \to K \wr H?$$

If G is Abelian, mimic the proof of the Kaloujnine–Krasner Theorem:

$$\varphi \colon S \to K \wr H, \ (i, g, \lambda) \mapsto (f_g^{i\lambda}, gN),$$

where

$$f_g^{i\lambda} \colon H \to K, \ A \mapsto (i, r_A g r_{A \cdot gN}^{-1}, \lambda).$$

Let $S = \mathcal{M}(G; I, \Lambda; P)$ be an extension of the semigroup $K = \mathcal{M}(N; I, \Lambda; P)$ by the group H. Does there exist an embedding

$$S \to K \wr H?$$

If G is Abelian, mimic the proof of the Kaloujnine–Krasner Theorem:

$$\varphi\colon S \to K \wr H,\ (i, g, \lambda) \mapsto (f_g^{i\lambda}, gN),$$

where

$$f_g^{i\lambda}\colon H \to K,\ A \mapsto (i, r_A g r_{A \cdot gN}^{-1}, \lambda).$$

If $G = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ then an embedding exists, but it is not „natural.”

conjecture: embedding does not exist in general
$\Rightarrow$ look for a counterexample

conjecture: embedding does not exist in general
$\Rightarrow$ look for a counterexample
first we would like to express the wreath product in a semidirect product form:

$$K \wr H = K^H \rtimes H \cong \mathcal{M}(N^H; I^H, \Lambda^H; P^H) \rtimes H,$$

where $P^H = (p^H_{\xi\eta})$ and for any $\xi \in \Lambda^H$, $\eta \in I^H$:

$$p^H_{\xi\eta} \colon H \to N, Ap^H_{\xi\eta} = p_{A\xi, A\eta} \ (A \in H)$$

$\mathbb{Z}_n \rtimes \mathbb{Z}_2$ is not good because of $\mathbb{Z}_2$ is too „small"
the source of the problem is in the sandwich matrix of $K \wr H$, where the entries are strongly related to each other

it suffices to work a 2×2 sandwich matrix if $\mathbb{Z}_2$ is replaced by $\mathbb{Z}_3$

the proof uses that one entry of G has order 3, and the image of this element can be expressed by means of the entries of P^H
so we do not have enough freedom to choose it appropriately

it suffices to work a 2×2 sandwich matrix if $\mathbb{Z}_2$ is replaced by $\mathbb{Z}_3$

the proof uses that one entry of G has order 3, and the image of this element can be expressed by means of the entries of P^H
so we do not have enough freedom to choose it appropriately

$$h = p^H_{\xi_1 \eta_1} (p^H_{\xi_2 \eta_1})^{-1} p^H_{\xi_2 \eta_2} (p^H_{\xi_1 \eta_2})^{-1}$$

Theorem

Let $G = \mathbb{Z}_7 \rtimes \mathbb{Z}_3$, $I = \Lambda = \{1, 2\}$, P be the sandwich matrix for which $p_{11} = p_{12} = p_{21} = (\overline{0}, \overline{0})$ and $p_{22} = (\overline{1}, \overline{0})$, and $N = \{(a, \overline{0}) : a \in \mathbb{Z}_7\}$. Let $S = \mathcal{M}(G; I, \Lambda; P)$ and $K = \mathcal{M}(N; I, \Lambda; P)$. Then there exists no embedding

$$S \to K \wr H.$$

Theorem

Let $G = \mathbb{Z}_7 \rtimes \mathbb{Z}_3$, $I = \Lambda = \{1, 2\}$, P be the sandwich matrix for which $p_{11} = p_{12} = p_{21} = (\overline{0}, \overline{0})$ and $p_{22} = (\overline{1}, \overline{0})$, and $N = \{(a, \overline{0}) : a \in \mathbb{Z}_7\}$. Let $S = \mathcal{M}(G; I, \Lambda; P)$ and $K = \mathcal{M}(N; I, \Lambda; P)$. Then there exists no embedding

$$S \to K \wr H.$$

Important: there is no embedding at all, not just a „nice" embeddings like in the Kaloujnine–Krasner Theorem

How can we obtain a positive result with a similar construction?

How can we obtain a positive result with a similar construction?
Idea: wreath product $\longrightarrow$ semidirect product

How can we obtain a positive result with a similar construction?
Idea: wreath product $\longrightarrow$ semidirect product

We are looking for an embedding

$$S = \mathcal{M}(G; I, \Lambda; P) \to \mathcal{M}(N'; I', \Lambda'; P') \rtimes H,$$

and we don't want to go far from the Kaloujnine–Krasner Theorem let $N' = N^H$, $I' = I$, $\Lambda' = H \times \Lambda$, and the entries of P' are „nice" maps let $N' = N^H$, $I' = I$, $\Lambda' = H \times \Lambda$, and the entries of P' are „nice" maps

Theorem

For any extension $S = \mathcal{M}(G; I, \Lambda; P)$ of $K = \mathcal{M}(N; I, \Lambda; P)$ by a group H, there exists an embedding

$$S \to \mathcal{M}(N^H; I, H \times \Lambda; Q) \rtimes H,$$

where the restriction of this embedding to maximal subgroups of S coincides with that in the proof of the Kaloujnine–Krasner Theorem, and the entries of Q can be expressed by means of the ingredients there, too.