Coordinatization of join-distributive lattices*

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Coordinatization of join-distributive lattices

All lattices will be assumed to be finite!
Semimodularity: $x \preceq y$ implies $x \lor z \preceq y \lor z$, for $\forall x, y, z \in L$.

Slimness: $J(L)$ is the union of two chains.

For example, two slim sm lattices (they are always planar):
Trajectory (slim case)

Trajectories of the diagram of a slim semimodular lattice: on the set of edges (=covering pairs), the ”opposite sides of a covering square” generates an equivalence relation, whose classes are called trajectories.

Trajectories were introduced by Czédli and E.T. Schmidt: The Jordan-Hölder theorem with uniqueness for groups and semimodular lattices; Algebra Universalis 66 (2011) 69–79.
"Traffic rules" for trajectories (slim case):

Trajectories go from left to right, from the left boundary chain to the right one, they do not split, at most one turn and only from northeast to southeast is permitted.
The **Jordan-Hölder permutation** of the **diagram** of a slim sm \( L \):
we (Cz–Schmidt, AU 2011) define \( \pi \in S_n \) by trajectories, see on the left; \( n \) denotes length(\( L \)).

The old definitions of \( \pi_L \): R.P. Stanley (1972, see also H. Abels 1991) are equivalent to ours; see Czédli and Schmidt (2013, Acta Sci. Math, to appear), where we prove that \( \pi \) determines the **diagram** and also the lattice (up to isomorphism)!

**The advantage of trajectories**: they are quite visual.
If $L$ is slim sm, then, by the "traffic rules",
a maximal chain and a trajectory always have
exactly one common edge.

Think of roads from north to south; the locomotive crosses each road exactly once.

*Definition*: **the trajectories of** $L$ **are beautiful** iff each maximal chain and each trajectory have exactly one common edge. Finite lattices with this properties are the lattices we deal with! 
(We allow the case where trajectories split.)
Join-distributivity is coming . . .

It turns out: our lattices = \{join-distributive lattices\}. \approx the most often discovered mathematical objects!

Meet-semidistributivity law: \( x \land y = x \land z \Rightarrow x \land y = x \land (y \lor z) \).

We list some equivalent definition of join-distributive lattices.

**Definition.** A finite lattice \( L \) is **join-distributive**, if one of the following twelve (equivalent) conditions hold:
• $L$ is semimodular and meet-semidistributive. (Dilworth, 1940)
• $L$ has unique meet-irreducible decompositions.
• For each $x \in L$, the interval $[x, x^*)$ is distributive.
• For each $x \in L$, the interval $[x, x^*)$ is boolean.
• The length of each maximal chain of $L$ equals $|M(L)|$.
• $L$ is semimodular and diamond-free (i.e., no $M_3$).
• $L$ is semimodular and has no cover-preserving $M_3$ sublattice.
• $L$ is a cover-preserving join-subsemilattice of a finite distributive lattice.
• $L \cong$ the lattice of open sets of a finite convex geometry.
• $L$ is dually isomorphic to the lattice of closed sets of a finite convex geometry.
• $L \cong$ the lattice of feasible sets of a finite antimatroid.
• (Adaricheva–Czédli) $L$ is semimodular with beautiful trajectories.
But we will not need: 

P.H. Edelman (1980): a pair $\langle E, \Phi \rangle$ is a **convex geometry**, if
- $E$ is a finite set, and $\Phi: P(E) \to P(E)$ is a closure operator.
- If $\Phi(A) = A \in P(E)$, $x, y \in E$, $x \notin A$, $y \notin A$, $x \neq y$, and $x \in \Phi(A \cup \{y\})$, then $y \notin \Phi(A \cup \{x\})$. (This is the so-called *anti-exchange property*.)
- $\Phi(\emptyset) = \emptyset$.

R. E. Jamison-Waldner (1980): a pair $\langle E, \mathcal{F} \rangle$ is an **antimatroid** if
- $E$ is a finite set, and $\emptyset \neq \mathcal{F} \subseteq P(E)$, $\mathcal{F}$ is union-closed, $\bigcup \mathcal{F} = E$, and for each nonempty $A \in \mathcal{F}$, $\exists x \in A$ with $A \setminus \{x\} \in \mathcal{F}$.

Complementary concepts; mutually determine each other.
Let $L$ be a join-distributive lattice of length $n$. We say that $L^* = \langle L; C_1, \ldots, C_k \rangle$ is a $k$-dimensional coordinate system (of join-width at most $k$) if the $C_i$ are maximal chains such that $J(L) \subseteq C_1 \cup \ldots \cup C_k$.

- The trajectories are beautiful $\Rightarrow$ for each (say, the $i$-th) edge (=prime interval) of $C_1$ there exists a unique edge (say, the $j$-th) of $C_t$ such that these two edges belong to the same trajectory. The rule $i \mapsto j$ defines a permutation $\pi_{1t} \in S_n$.

- The coordinate structure of $L^*$ is $\vec{\pi} = \langle \pi_{12}, \ldots, \pi_{1k} \rangle \in S_{kn}^{k-1}$. We denote $\vec{\pi}$ by $\xi(L^*)$. We say that $S_{kn}^{k-1}$ is the set of $k$-dimensional coordinate structures.

**Main Theorem** (Czédli, 2012) The map $\xi: L^* \mapsto \vec{\pi}$ is a bijection from $\{\text{join-distributive lattices with } k\text{-dimensional coordinate systems}\}$ to the set $S_{kn}^{k-1}$ of $k$-dimensional coordinate structures.
The coordinate system is important

Main Thm. $\xi: L^* \mapsto \vec{\pi}$ is a bijection.

Remark. The coordinate structure heavily depends on the coordinate system! If $L$ is the 8-element boolean lattice with atoms $a, b, c$, then the coordinate system $C_1 = \{0, a, a \lor b, 1\}$, $C_2 = \{0, b, a \lor b, 1\}$, $C_3 = \{0, c, b \lor c, 1\}$ leads to $\pi_{12} = (12)$ and $\pi_{13} = (13)$ (two transpositions), while the choice $C'_1 = C_1$, $C'_2 = \{0, b, b \lor c, 1\}$, and $C'_3 = \{0, c, a \lor c, 1\}$ leads to $\pi'_{12} = (132)$ and $\pi'_{13} = (123)$ (two cycles of order 3).

Open problem: Give an elegant description for the pairs $\langle \vec{\pi}, \vec{\sigma} \rangle \in S_{n-1}^k \times S_{n-1}^k$ that come from the same lattice with appropriate choices of $\langle C_1, \ldots, C_k \rangle$. Solved only for $k = 2$ (the slim case).
Main Thm. $\xi: L^* \mapsto \pi$ is a bijection.

What about the coordinates of the elements of $L$?

To answer this question, let $\eta = \xi^{-1}$; we shall describe $\eta$. 
\(\bar{\pi}\)-orbits and eligible \(\bar{\pi}\)-tuples

**Main Thm.** \(\xi: L^* \mapsto \bar{\pi}\) is a bijection. \(\eta := \xi^{-1}\).

For \(\bar{\pi} \in S_{n}^{k-1}\), we define \(\eta(\bar{\pi}) = L^*(\bar{\pi}) = \langle L(\bar{\pi}); C_1(\bar{\pi}), \ldots, C_k(\bar{\pi}) \rangle\).

It is convenient to define \(\pi_{jt}(i) = \pi_{1t}(\pi_{1j}^{-1}(i))\). Note that in the model \(\langle L; C_1, \ldots, C_k\rangle\), \(\pi_{jt}\) is what the trajectories define between the chains \(C_j\) and \(C_t\).

By an **eligible \(\bar{\pi}\)-tuple** we mean a \(k\)-tuple \(\bar{x} = \langle x_1, \ldots, x_k \rangle \in \{0,1,\ldots,n\}^k\) such that \(\pi_{ij}(x_i + 1) \geq x_j + 1\) holds for all \(i,j \in \{1,\ldots,k\}\) such that \(x_i < n\). (Roughly saying: if we enlarge a component of \(\bar{x}\) by 1, then its images will be big.)
The elements are coordinatized this way

**Main Thm.** $\xi : L^* \mapsto \vec{\pi}$ is a bijection. Want: $\eta(\vec{\pi}) = L^*(\vec{\pi})$.

$\vec{x} \in \{0, \ldots, n-1\}^k$ is eligible $\iff \pi_{ij}(x_i + 1) \geq x_j + 1$ if $x_i < n$.

**Definition.** Let $L(\vec{\pi}) := \{\text{eligible $\vec{\pi}$-tuples}\}$ with the componentwise ordering. We have defined the lattice; the elements are coordinatized by eligible $\vec{\pi}$-tuples.

For $i \in \{1, \ldots, k\}$, an eligible $\vec{\pi}$-tuple $\vec{x}$ is **$i$-minimal** if for all $\vec{y} \in L(\pi)$, $x_i = y_i$ implies $\vec{x} \leq \vec{y}$. Let $C_i(\vec{\pi})$ be the set of all $i$-minimal eligible $\vec{\pi}$-tuples.

We have defined $\eta(\vec{\pi}) = L^*(\vec{\pi}) = \langle L(\vec{\pi}); C_1(\vec{\pi}), \ldots, C_k(\vec{\pi}) \rangle$.

(One has to prove that this construct works and $\eta = \xi^{-1}$.)
Main Thm. $\xi: L^* \mapsto \vec{\pi}$ is a bijection. $\vec{x}$ is eligible iff $\pi_{ij}(x_i + 1) \geq x_j + 1$. $\xi^{-1}(\vec{\pi}) = \langle \{\text{eligibles, 1-minimals, \ldots, } k\text{-minimals}\} \rangle$.


This connection is analyzed in a joint paper by Adaricheva and Czédli [ arxiv.org/1210.3376 or my web site]. In this paper, we show that my Main Theorem and the Edelman-Jamison description can mutually be derived from each other in less than a page.

Although the lattice-theoretical is somewhat longer, it makes sense by the following reasons.
Why with Lattice Theory? Czédli 2013 20′/0′

1st, it exemplifies how Lattice Theory can be applied to other fields of mathematics.

2nd, not only our methods and the motivations are different from that of Edelman and Jamison, the two results are not exactly the same even if the latter is translated to lattice theory.

3rd, trajectories led to a new characterization of join-distributive lattices.

4th, it is not yet clear which approach can be used to attack the open problem mentioned before. Thank you for your attention!

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