TECHNISCHE UNIVERSITÄT DRESDEN

RELATIONAL STRUCTURE THEORY—LOCALISING ALGEBRAS, AND MORE

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NSAC 2013, Novi Sad, Serbia, 8 June 2013

Outline

1. What is magic?
2. Localisation
3. Global to local correspondence
4. Local to global correspondence
5. Optimal Covers
6. Beyond algebras

Origins

- 1980s McKenzie, Hobby: Tame Congruence Theory (study polynomial structure of alg's via congruence pairs)
- ≈2000 TCT for term operations, too???
- 2001 Kearnes, Szendrei: Tame Congruence Theory is a localization theory *A Course in Tame Congruence Theory* workshop, Budapest (ideas for generalisation, answers, focussing on finite algebras)
- 2009 MB: diploma thesis
- 2012 Schneider: *A relational localisation theory for topological algebras*
- 2013 MB: *Relational Structure Theory–A Localisation Theory for Algebraic Structures*

Idea

algebra	magic	relational structure
set + op's	$\rightleftharpoons$	set + rel's
		$\downarrow$
		split into pieces
		$\downarrow$
local alg's	magic	local structures
	$\rightleftharpoons$	

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Functions and relations

Let A be a set, $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

$$\mathrm{O}_A^{(n)} := A^{A^n} \qquad \mathrm{R}_A^{(m)} := \mathcal{P}\left(A^m\right)$$

$$\mathrm{O}_A := \bigcup_{n\in\mathbb{N}} \mathrm{O}_A^{(n)} \qquad \mathrm{R}_A := \bigcup_{m\in\mathbb{N}} \mathrm{R}_A^{(m)}$$

For $f \in \mathrm{O}_A^{(n)}$ and $S \subseteq A^m$ (i.e. $S \in \mathrm{R}_A^{(m)}$):

$$f \rhd S \;:\Longleftrightarrow\; S \in \mathrm{Sub}\left(\langle A; f\rangle^m\right)$$

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$$\mathrm{O}_A := \bigcup_{n\in\mathbb{N}} \mathrm{O}_A^{(n)} \qquad \mathrm{R}_A := \bigcup_{m\in\mathbb{N}} \mathrm{R}_A^{(m)}$$

For $f \in \mathrm{O}_A^{(n)}$ and $S \subseteq A^m$ (i.e. $S \in \mathrm{R}_A^{(m)}$):

$$\begin{aligned} f \rhd S \;:\Longleftrightarrow\; & S \in \mathrm{Sub}\left(\langle A; f\rangle^m\right) \\ \Longleftrightarrow\; & f \in \mathrm{Hom}\left(\langle A; S\rangle^n ; \langle A; S\rangle\right) \end{aligned}$$

Polymorphisms and invariant relations

For $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$:

$$\begin{aligned}
\operatorname{Inv}\langle A; F\rangle &:= \operatorname{Inv}_A F := \{\, S \in \mathrm{R}_A \mid \forall f \in F : \quad f \rhd S \}\\
&= \bigcup_{m\in\mathbb{N}} \operatorname{Sub}\left(\langle A; F\rangle^m\right)\\
\operatorname{Pol}\langle A; Q\rangle &:= \operatorname{Pol}_A Q := \{\, f \in \mathrm{O}_A \mid \forall S \in Q : \quad f \rhd S \}\\
&= \bigcup_{n\in\mathbb{N}} \operatorname{Hom}\left(\langle A; Q\rangle^n ; \langle A; Q\rangle\right)
\end{aligned}$$

Galois closures

local closure of clone of term operations

$$\mathrm{Clo}\,(\mathbf{A}) := \mathrm{Pol}\,\mathrm{Inv}\,\mathbf{A} = \mathrm{Loc}_A\,\mathrm{Term}\,(\mathbf{A})$$

local closure of relational clone

$$\mathrm{Inv}\,\mathrm{Pol}\,\underset{\sim}{\mathbf{A}} = \mathrm{LOC}_A\,\left[\underset{\sim}{\mathbf{A}}\right]_{\mathrm{R}_A}$$

Clones?

Definition (Clone of operations)

= a set $F \subseteq \mathrm{O}_A$ such that

- $e_i^{(n)} \in F$ for $1 \leq i \leq n$, $n \in \mathbb{N}_+$
- $f \in F^{(n)}, g_1, \ldots, g_n \in F^{(m)} \implies f \circ (g_1, \ldots, g_n) \in F$ $(n, m \in \mathbb{N})$

Definition (Clone of relations)

= a set $Q \subseteq \mathrm{R}_A$ such that

$$\{a \circ \alpha \mid a \in A^\kappa, \forall i \in I\colon a \circ \beta_i \in \sigma_i\} =: \prod_{(\beta_i)_{i \in I}}^{\alpha} (\sigma_i)_{i \in I} \in Q$$

for all $\alpha\colon m \longrightarrow \kappa$ $\beta_i\colon m_i \longrightarrow \kappa$, $\sigma_i \in Q^{(m_i)}$, $i \in I$, $m, m_i \in \mathbb{N}$, κ an ordinal

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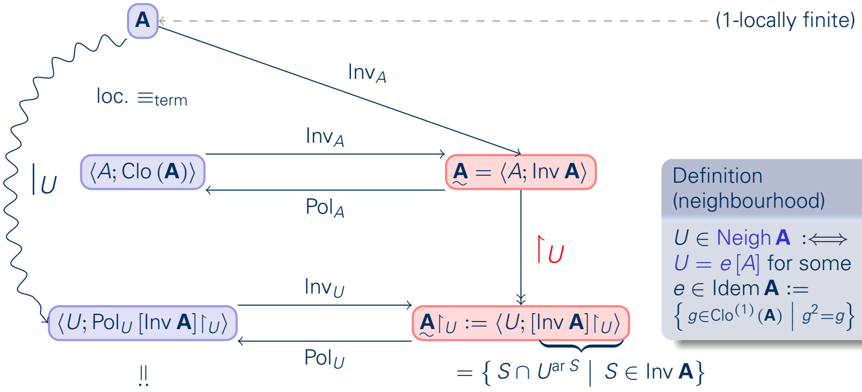
Restricting algebras to neighbourhoods



Definition (neighbourhood)

$U \in \mathrm{Neigh}\,\mathbf{A} \;:\Longleftrightarrow$ $U = e[A]$ for some $e \in \mathrm{Idem}\,\mathbf{A} := \left\{ g \in \mathrm{Clo}^{(1)}(\mathbf{A}) \;\middle|\; g^2 = g \right\}$

$$\mathbf{A}|_U = \left\langle U; \left\{ (e \circ f){\restriction}^{U}_{U^{\mathrm{ar}\, f}} \;\middle|\; f \in \mathrm{Clo}(\mathbf{A}) \right\} \right\rangle = \left\langle U; \left\{ f{\restriction}^{U}_{U^{\mathrm{ar}\, f}} \;\middle|\; f \in \mathrm{Clo}(\mathbf{A}) \;\wedge\; f \rhd U \right\} \right\rangle$$

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... explained via a toy example

$U \in \operatorname{Neigh} \mathbf{A} \implies {\restriction_U} : \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{A}}{\restriction_U}$ surj. hom betw. rel. clones

Example

. . . explained via a toy example

$U \in \operatorname{Neigh}\mathbf{A} \implies {\restriction_U} \colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{A}}\lceil_U$ surj. hom betw. rel. clones

Example

Let $S, T \in \operatorname{Inv}^{(2)}\mathbf{A}$ such that $S \circ T = T \circ S$.
${\restriction_U}$ clone hom $\implies$

. . . explained via a toy example

$U \in \operatorname{Neigh} \mathbf{A} \implies {\restriction_U}\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{A}}{\restriction_U}$ surj. hom betw. rel. clones

Example

Let $S, T \in \operatorname{Inv}^{(2)}\mathbf{A}$ such that $S \circ T = T \circ S$.
${\restriction_U}$ clone hom $\implies$

$$S{\restriction_U} \circ T{\restriction_U} = (S \circ T){\restriction_U} = (T \circ S){\restriction_U} = T{\restriction_U} \circ S{\restriction_U}$$

holds for $S{\restriction_U}, T{\restriction_U} \in \operatorname{Inv}^{(2)}\mathbf{A}|_U$

. . . explained via a toy example

$U \in \operatorname{Neigh} \mathbf{A} \implies {\restriction_U} \colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{A}}{\restriction_U}$ surj. hom betw. rel. clones

Example

Suppose $S \circ T = T \circ S$ holds for all $S, T \in \operatorname{Inv}^{(2)} \mathbf{A}$

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$U \in \operatorname{Neigh} \mathbf{A} \implies {\restriction_U} \colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{A}}{\restriction_U}$ surj. hom betw. rel. clones

Example

Suppose $S \circ T = T \circ S$ holds for all $S, T \in \operatorname{Inv}^{(2)} \mathbf{A}$

${\restriction_U}$ surjective clone hom $\implies$

$$\sigma \circ \tau = S{\restriction_U} \circ T{\restriction_U} = (S \circ T){\restriction_U} = (T \circ S){\restriction_U} = T{\restriction_U} \circ S{\restriction_U} = \tau \circ \sigma$$

holds for all $\sigma = S{\restriction_U}, \tau = T{\restriction_U} \in \operatorname{Inv}^{(2)} \mathbf{A}|_U$.

More useful properties

Lemma (Global $\longrightarrow$ local)

$\mathbf{A}$ *an algebra,* $U \in \operatorname{Neigh}\mathbf{A}$ *such that* $\langle U\rangle_{\mathbf{A}} = \mathbf{A}$

- $\restriction_U \colon \operatorname{Con}\mathbf{A} \longrightarrow \operatorname{Con}\mathbf{A}|_U$ *is a surjective complete lattice hom*
- $\mathbf{A}$ *congruence modular / distributive / k-permutable* $\implies$ $\mathbf{A}|_U$ *congruence modular / distributive k-permutable*
- *analogous statement for compatible quasiorders*

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What to expect...

Example ($U \in \operatorname{Neigh} \mathbf{A}$)

Suppose $S \circ T = T \circ S$ holds for all $S, T \in \operatorname{Inv}^{(2)}\mathbf{A}$
$\restriction_U$ surjective clone hom $\implies$

$$\sigma \circ \tau = S\restriction_U \circ T\restriction_U = (S \circ T)\restriction_U = (T \circ S)\restriction_U = T\restriction_U \circ S\restriction_U = \tau \circ \sigma$$

holds for all $\sigma = S\restriction_U, \tau = T\restriction_U \in \operatorname{Inv}^{(2)}\mathbf{A}|_U$.

What to expect...

Example ($U \in \operatorname{Neigh} \mathbf{A}$)

Suppose $S \circ T = T \circ S$ holds for all $S, T \in \operatorname{Inv}^{(2)}\mathbf{A}$
$\restriction_U$ surjective clone hom $\Longleftarrow$

$$\sigma \circ \tau = S\restriction_U \circ T\restriction_U = (S \circ T)\restriction_U = (T \circ S)\restriction_U = T\restriction_U \circ S\restriction_U = \tau \circ \sigma$$

holds for all $\sigma = S\restriction_U, \tau = T\restriction_U \in \operatorname{Inv}^{(2)}\mathbf{A}|_U$.

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What to expect...

Example ($U \in \mathrm{Neigh}\,\mathbf{A}$)

Suppose $S \circ T = T \circ S$ holds for all $S, T \in \mathrm{Inv}^{(2)}\mathbf{A}$
$\restriction_U$ surjective clone hom $\Longleftarrow$generally WRONG

$$\sigma \circ \tau = S\restriction_U \circ T\restriction_U = (S \circ T)\restriction_U = (T \circ S)\restriction_U = T\restriction_U \circ S\restriction_U = \tau \circ \sigma$$

holds for all $\sigma = S\restriction_U, \tau = T\restriction_U \in \mathrm{Inv}^{(2)}\mathbf{A}|_U$.

Choosing more than one neighbourhood...

Idea using $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$

If for all $U \in \mathcal{U}$

$$\forall \sigma, \tau \in \operatorname{Inv}^{(2)} \mathbf{A}|_U \colon \sigma \circ \tau = \tau \circ \sigma,$$

then

$$\forall S, T \in \operatorname{Inv}^{(2)} \mathbf{A} \colon S \circ T = T \circ S.$$

Choosing more than one neighbourhood...

Idea using $\mathcal{U} \subseteq \text{Neigh}\,\mathbf{A}$

If for all $U \in \mathcal{U}$

$$\forall\, S, T \in \text{Inv}^{(2)}\mathbf{A}\colon\ S{\restriction_U} \circ T{\restriction_U} = T{\restriction_U} \circ S{\restriction_U},$$

then

$$\forall\, S, T \in \text{Inv}^{(2)}\mathbf{A}\colon\ S \circ T = T \circ S.$$

Choosing more than one neighbourhood...

Idea using $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$

If for all $U \in \mathcal{U}$

$$\forall\, S, T \in \operatorname{Inv}^{(2)}\mathbf{A}\colon\; (S \circ T){\restriction_U} = (T \circ S){\restriction_U},$$

then

$$\forall\, S, T \in \operatorname{Inv}^{(2)}\mathbf{A}\colon S \circ T = T \circ S.$$

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If $\forall\, S, T \in \operatorname{Inv}^{(2)} \mathbf{A}$

$$\forall\, U \in \mathcal{U}\colon\ (S \circ T){\restriction_U} = (T \circ S){\restriction_U},$$

then $\forall\, S, T \in \operatorname{Inv}^{(2)} \mathbf{A}$

$$S \circ T = T \circ S.$$

Choosing more than one neighbourhood...

Idea using $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$

Sufficient condition: for all $S, T \in \operatorname{Inv}^{(2)} \mathbf{A}$, if

$$\forall\, U \in \mathcal{U} \colon S{\restriction_U} = T{\restriction_U},$$

then

$$S = T.$$

Choosing more than one neighbourhood...

Idea using $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$

Contrapositive: for all $S, T \in \operatorname{Inv}^{(2)}\mathbf{A}$, if

$$S \neq T,$$

then

$$\exists\, U \in \mathcal{U}\colon S{\restriction_U} \neq T{\restriction_U}.$$

Choosing more than one neighbourhood...

Idea using $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$

Contrapositive: for all $S, T \in \operatorname{Inv}^{(2)} \mathbf{A}$, if

$$S{\restriction_A} \neq T{\restriction_A},$$

then

$$\exists\, U \in \mathcal{U} \colon S{\restriction_U} \neq T{\restriction_U}.$$

Choosing more than one neighbourhood. . .

Idea using $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$

Contrapositive: for all $S, T \in \operatorname{Inv}^{(2)} \mathbf{A}$, if

$$(S, T) \in \operatorname{Sep}_{\mathbf{A}}(A),$$

then

$$\exists\, U \in \mathcal{U} \colon (S, T) \in \operatorname{Sep}_{\mathbf{A}}(U)$$

where

$\operatorname{Sep}_{\mathbf{A}}(U) := \left\{ (S, T) \in \operatorname{Inv} \mathbf{A}^2 \;\middle|\; \operatorname{ar} S = \operatorname{ar} T, S{\restriction_U} \neq T{\restriction_U} \right\}$.

Choosing more than one neighbourhood. . .

Idea using $\mathcal{U} \subseteq \mathrm{Neigh}\,\mathbf{A}$

Contrapositive: for all $S, T \in \mathrm{Inv}^{(2)}\mathbf{A}$, if

$$(S, T) \in \mathrm{Sep}_{\mathbf{A}}(A),$$

then

$$(S, T) \in \mathrm{Sep}_{\mathbf{A}}(\mathcal{U})$$

where $\mathrm{Sep}_{\mathbf{A}}(\mathcal{U}) := \bigcup_{U \in \mathcal{U}} \mathrm{Sep}_{\mathbf{A}}(U)$.

Separation, covering

Definition (Separation, covering relation)

Let $m \in \mathbb{N}$, $S, T \in \operatorname{Inv}^{(m)} \mathbf{A} := \operatorname{Sub} \mathbf{A}^m$, $V \in \operatorname{Neigh} \mathbf{A}$ and $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$.

- $(S, T) \in \operatorname{Sep}_{\mathbf{A}}(V)$ iff $S{\restriction_V} \neq T{\restriction_V}$. $\quad (S{\restriction_V} := S \cap V^m)$
- $\mathcal{U}$ separates S and T iff $\exists U \in \mathcal{U}\colon U$ separates S and T (i.e. $(S, T) \in \operatorname{Sep}_{\mathbf{A}}(\mathcal{U}) = \bigcup_{U \in \mathcal{U}} \operatorname{Sep}_{\mathbf{A}}(U)$)
- $\mathcal{V} \leq_{\text{cov}} \mathcal{U} \;:\Longleftrightarrow\; \operatorname{Sep}_{\mathbf{A}}(\mathcal{V}) \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{U})$
  $\mathcal{V}$ is covered by $\mathcal{U}$

Interesting for local to global correspondence

Definition (Cover)

$\mathrm{Cov}\,(\mathbf{A}) := \{\mathcal{U} \subseteq \mathrm{Neigh}\,\mathbf{A} \mid \{A\} \leq_{\mathrm{cov}} \mathcal{U}\}$

Characterisation of covers of finite algebras

Theorem (Kearnes, Á. Szendrei, 2001)

Let $\mathbf{A}$ *be a finite algebra and* $E \subseteq \operatorname{Idem}\mathbf{A}$.
Put $\mathcal{U} := \{\operatorname{im} e \mid e \in E\}$. *T.f.a.e.:*

1. $\mathcal{U}$ *covers* $\mathbf{A}$.
2. $\exists q \in \mathbb{N}\, \exists e_1, \ldots, e_q \in E\, \exists g_1, \ldots, g_q \in \operatorname{Clo}^{(1)}(\mathbf{A})$
   $\exists \lambda \in \operatorname{Clo}^{(q)}(\mathbf{A}) : \ \lambda \circ (e_1 \circ g_1, \ldots, e_q \circ g_q) = \operatorname{id}_A$.
3. $\exists q \in \mathbb{N}\, \exists (U_1, \ldots, U_q) \in \mathcal{U}^q :$
   $\underset{\sim}{\mathbf{A}}$ *retract of* $\underset{\sim}{\mathbf{A}}\rceil_{U_1} \times \cdots \times \underset{\sim}{\mathbf{A}}\rceil_{U_q}$

Covers for general algebras

Theorem

Let $\mathbf{A}$ *be an algebra,* $e \in \operatorname{Idem}\mathbf{A}$ *and* $E \subseteq \operatorname{Idem}\mathbf{A}$, $U := \operatorname{im} e \in \operatorname{Neigh}\mathbf{A}$ *and* $\mathcal{V} := \left\{ \operatorname{im} e' \mid e' \in E \right\} \subseteq \operatorname{Neigh}\mathbf{A}$. *Furthermore, let* $T_0 := \operatorname{Clo}^{(1)}(\mathbf{A})$ *and* $S_0 := \langle F \rangle_{\mathbf{A}^A}$, *where* $F := \left\{ f \in \operatorname{Clo}^{(1)}(\mathbf{A}) \;\middle|\; \exists V \in \mathcal{V} \colon \operatorname{im} f \subseteq V \right\}$. *Moreover, we fix a choice function* $V \colon F \longrightarrow \mathcal{V}$ *satisfying* $\operatorname{im} f \subseteq V(f)$ *for every* $f \in F$. *For a fixed finite cardinal* $m \in \mathbb{N}$ *we let* $\mu := \min\{m, |U|\}$. *Then t.f.a.e:*

(a) $\mathcal{V}$ *separates all at most m-ary invariant relations of* $\mathbf{A}$ *that are separated in* U.

(b) $\mathcal{V}$ *separates all pairs* $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$, $S \subseteq T$ *that are separated in* U, *and if* $\emptyset \in \operatorname{Inv}\mathbf{A}$, *then* $\mathcal{V} \neq \emptyset$.

(c) $\mathcal{V}$ *separates all pairs* $S, T \in \operatorname{Inv}^{(\mu)}\mathbf{A}$, $S \subseteq T$ *of* μ*-ary invariant relations that are separated in* U.

(d) *For every* $X \subseteq U$ *of cardinality* $|X| = \mu$ *the collection* $\mathcal{V}$ *separates the invariant relations* $\operatorname{pr}^A_X S_0$ *and* $\operatorname{pr}^A_X T_0$ *belonging to* $\operatorname{Inv}^{(\mu)}\mathbf{A}$ *if their restrictions to* U *are distinct.*

(e) *For every* $X \subseteq U$ *of cardinality* $|X| = \mu$, *we have* $\left(\operatorname{pr}^A_X S_0\right)\restriction_U = \left(\operatorname{pr}^A_X T_0\right)\restriction_U$.

(f) *For every* $X \subseteq U$ *of cardinality* $|X| = \mu$, *we have* $e|^A_X \in \operatorname{pr}^A_X (e \circ [S_0])$.

(g) *For every* $X \subseteq A$ *of cardinality* $|X| \leq m$ *there exists* $n \in \mathbb{N}$, $\lambda \in \operatorname{Term}^{(n)}(\mathbf{A})$ *and* $(f_1, \ldots, f_n) \in F^n$ *such that* $(e \circ \lambda \circ (f_1, \ldots, f_n))\,|^A_X = e|^A_X$.

(h) *For every* $X \subseteq U$ *of cardinality* $|X| = \mu$ *there exists* $n \in \mathbb{N}$, $\lambda \in \operatorname{Clo}^{(n)}(\mathbf{A})$ *and* $(f_1, \ldots, f_n) \in F^n$ *such that* $(\lambda \circ (f_1, \ldots, f_n))\,|^A_X = e|^A_X$.

(i) $\underset{\sim}{\mathbf{A}}\restriction_U$ *is a jointly finite m-local retract of* $\left(\underset{\sim}{\mathbf{A}}\restriction_{V(f)}\right)_{f \in F}$.

(j) $\underset{\sim}{\mathbf{A}}\restriction_U$ *is a* μ*-local retract of* $\prod_{f \in F} \underset{\sim}{\mathbf{A}}\restriction_{V(f)}$.

(k) *There is an index set* Φ *and a mapping* $\tilde{V} \colon \Phi \longrightarrow \mathcal{V}$ *such that* $\underset{\sim}{\mathbf{A}}\restriction_U$ *is a* μ*-local retract of* $\prod_{\varphi \in \Phi} \underset{\sim}{\mathbf{A}}\restriction_{\tilde{V}(\varphi)}$.

Covers for general algebras

Theorem

Let $\mathbf{A}$ be an algebra, $e \in \operatorname{Idem}\mathbf{A}$ and $E \subseteq \operatorname{Idem}\mathbf{A}$, $U := \operatorname{im} e \in \operatorname{Neigh}\mathbf{A}$ and $\mathcal{V} := \{\operatorname{im} e' \mid e' \in E\} \subseteq \operatorname{Neigh}\mathbf{A}$. Furthermore, let $T_0 := \operatorname{Clo}^{(1)}(\mathbf{A})$ and $S_0 := \langle F\rangle_{\mathbf{A}^A}$, where $F := \left\{ f \in \operatorname{Clo}^{(1)}(\mathbf{A}) \,\middle|\, \exists V \in \mathcal{V}\colon \operatorname{im} f \subseteq V\right\}$. Moreover, we fix a choice function $V\colon F \longrightarrow \mathcal{V}$ satisfying $\operatorname{im} f \subseteq V(f)$ for every $f \in F$. For a fixed finite cardinal $m \in \mathbb{N}$ we let $\mu := \min\{m, |U|\}$. Then t.f.a.e:

- (a) *$\mathcal{V}$ separates all at most m-ary invariant relations of $\mathbf{A}$ that are separated in U.*
- (b) *$\mathcal{V}$ separates all pairs $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$, $S \subseteq T$ that are separated in U, and if $\emptyset \in \operatorname{Inv}\mathbf{A}$, then $\mathcal{V} \neq \emptyset$.*
- (c) *$\mathcal{V}$ separates all pairs $S, T \in \operatorname{Inv}^{(\mu)}\mathbf{A}$, $S \subseteq T$ of μ-ary invariant relations that are separated in U.*
- (d) *For every $X \subseteq U$ of cardinality $|X| = \mu$ the collection $\mathcal{V}$ separates the invariant relations $\operatorname{pr}^A_X S_0$ and $\operatorname{pr}^A_X T_0$ belonging to $\operatorname{Inv}^{(\mu)}\mathbf{A}$ if their restrictions to U are distinct.*
- (e) *For every $X \subseteq U$ of cardinality $|X| = \mu$, we have $\left(\operatorname{pr}^A_X S_0\right)\restriction_U = \left(\operatorname{pr}^A_X T_0\right)\restriction_U$.*
- (f) *For every $X \subseteq U$ of cardinality $|X| = \mu$, we have $e|^A_X \in \operatorname{pr}^A_X(e \circ [S_0])$.*
- (g) *For every $X \subseteq A$ of cardinality $|X| \leq m$ there exists $n \in \mathbb{N}$, $\lambda \in \operatorname{Term}^{(n)}(\mathbf{A})$ and $(f_1, \dots, f_n) \in F^n$ such that $(e \circ \lambda \circ (f_1, \dots, f_n))|^A_X = e|^A_X$.*
- (h) *For every $X \subseteq U$ of cardinality $|X| = \mu$ there exists $n \in \mathbb{N}$, $\lambda \in \operatorname{Clo}^{(n)}(\mathbf{A})$ and $(f_1, \dots, f_n) \in F^n$ such that $(\lambda \circ (f_1, \dots, f_n))|^A_X = e|^A_X$.*
- (i) *$\underset{\sim}{\mathbf{A}}\restriction_U$ is a jointly finite m-local retract of $\left(\underset{\sim}{\mathbf{A}}\restriction_{V(f)}\right)_{f\in F}$.*
- (j) *$\underset{\sim}{\mathbf{A}}\restriction_U$ is a μ-local retract of $\prod_{f\in F}\underset{\sim}{\mathbf{A}}\restriction_{V(f)}$.*
- (k) *There is an index set Φ and a mapping $\tilde{V}\colon \Phi \longrightarrow \mathcal{V}$ such that $\underset{\sim}{\mathbf{A}}\restriction_U$ is a μ-local retract of $\prod_{\varphi\in\Phi}\underset{\sim}{\mathbf{A}}\restriction_{\tilde{V}(\varphi)}$.*

Separating m-ary relations

Theorem

Let $\mathbf{A}$ *be an algebra,* $e \in \operatorname{Idem}\mathbf{A}$ *and* $E \subseteq \operatorname{Idem}\mathbf{A}$*,* $U := \operatorname{im} e \in \operatorname{Neigh}\mathbf{A}$ *and* $\mathcal{V} := \{\operatorname{im} e' \mid e' \in E\} \subseteq \operatorname{Neigh}\mathbf{A}$*. Furthermore, let* $T_0 := \operatorname{Clo}^{(1)}(\mathbf{A})$ *and* $S_0 := \langle F \rangle_{\mathbf{A}^A}$*, where* $F := \left\{ f \in \operatorname{Clo}^{(1)}(\mathbf{A}) \;\middle|\; \exists V \in \mathcal{V}\colon \operatorname{im} f \subseteq V \right\}$*. Fix a choice function* $V\colon F \longrightarrow \mathcal{V}$ *satisfying* $\operatorname{im} f \subseteq V(f)$ *for every* $f \in F$*. For* $m \in \mathbb{N}$ *let* $\mu := \min\{m, |U|\}$*. Then t.f.a.e:*

1. $\operatorname{Sep}_{\mathbf{A}}(U) \cap \left(\operatorname{Inv}^{(\leq m)}\mathbf{A}\right)^2 \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$
2. $\forall X \subseteq U\colon |X| = \mu \implies$
   $\exists\, n \in \mathbb{N}\, \exists\, \lambda \in \operatorname{Clo}^{(n)}(\mathbf{A})\, \exists\, (f_1, \ldots, f_n) \in F^n\colon\ (\lambda \circ (f_1, \ldots, f_n))|_X^A = e|_X^A.$
3. $\underset{\sim}{\mathbf{A}}\restriction_U$ *is a jointly finite m-local retract of* $\left(\underset{\sim}{\mathbf{A}}\restriction_{V(f)}\right)_{f \in F}$.
4. $\exists\, \Phi\, \exists\, \tilde{V}\colon \Phi \longrightarrow \mathcal{V}\colon\ \underset{\sim}{\mathbf{A}}\restriction_U$ *is a* μ*-local retract of* $\prod_{\varphi \in \Phi} \underset{\sim}{\mathbf{A}}\restriction_{\tilde{V}(\varphi)}$.

Covers

Corollary

Let $\mathbf{A}$ *be an algebra,* $e \in \operatorname{Idem} \mathbf{A}$ *and* $E \subseteq \operatorname{Idem} \mathbf{A}$,
$U := \operatorname{im} e \in \operatorname{Neigh} \mathbf{A}$ *and* $\mathcal{V} := \{\operatorname{im} e' \mid e' \in E\} \subseteq \operatorname{Neigh} \mathbf{A}$.
Furthermore, let $T_0 := \operatorname{Clo}^{(1)}(\mathbf{A})$ *and* $S_0 := \langle F \rangle_{\mathbf{A}^A}$, *where*
$F := \left\{ f \in \operatorname{Clo}^{(1)}(\mathbf{A}) \;\middle|\; \exists V \in \mathcal{V} \colon \operatorname{im} f \subseteq V \right\}$. *Choose*
$V \colon F \longrightarrow \mathcal{V}$ *such that* $\operatorname{im} f \subseteq V(f)$ *for every* $f \in F$. *T.f.a.e:*

1. $U \leq_{\mathrm{cov}} \mathcal{V}$
2. $\forall X \subseteq_{\mathrm{fin}} U \exists n \in \mathbb{N} \exists \lambda \in \operatorname{Clo}^{(n)}(\mathbf{A}) \exists (f_1, \ldots, f_n) \in F^n \colon$ $(\lambda \circ (f_1, \ldots, f_n))|_X^A = e|_X^A$.
3. $\underset{\sim}{\mathbf{A}}\restriction_U$ *is a jointly finite local retract of* $\left(\underset{\sim}{\mathbf{A}}\restriction_{V(f)}\right)_{f \in F}$.
4. $\forall m \in \mathbb{N}, m \leq |U| \exists \Phi_m \exists \tilde{V}_m \colon \Phi_m \longrightarrow \mathcal{V} \colon$ $\underset{\sim}{\mathbf{A}}\restriction_U$ *is an m-local retract of* $\prod_{\varphi \in \Phi_m} \underset{\sim}{\mathbf{A}}\restriction_{\tilde{V}_m(\varphi)}$.

Outline

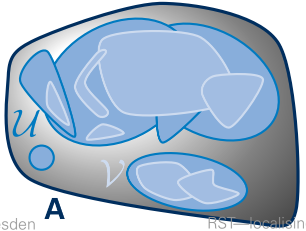
1 What is magic?
2 Localisation
3 Global to local correspondence
4 Local to global correspondence
5 Optimal Covers
6 Beyond algebras

Quord's on sets of neighbourhoods

For $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ we define:

Definition (Downset quasiorder)

$$\mathcal{V} \sqsubseteq (\subseteq)\, \mathcal{U} \;:\Longleftrightarrow\; \forall\, V \in \mathcal{V}\, \exists\, U \in \mathcal{U} \colon V \subseteq U$$
$$\Longleftrightarrow\; {\downarrow}_{(\operatorname{Neigh}\mathbf{A},\subseteq)}\, \mathcal{V} \subseteq {\downarrow}_{(\operatorname{Neigh}\mathbf{A},\subseteq)}\, \mathcal{U}$$

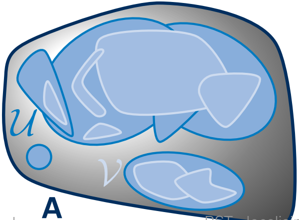


Quord's on sets of neighbourhoods

For $\mathcal{U}, \mathcal{V} \subseteq \text{Neigh}\,\mathbf{A}$ we define:

Definition (Downset quasiorder, for $q \subseteq \leq_{\text{cov}}$ quasiorder)

$$\mathcal{V} \sqsubseteq (q)\, \mathcal{U} \;:\Longleftrightarrow\; \forall\, V \in \mathcal{V}\, \exists\, U \in \mathcal{U} \colon V \; q \; U$$
$$\Longleftrightarrow\; {\downarrow}_{(\text{Neigh}\,\mathbf{A}, q)}\, \mathcal{V} \subseteq {\downarrow}_{(\text{Neigh}\,\mathbf{A}, q)}\, \mathcal{U}$$



Quord's on sets of neighbourhoods

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$$\begin{aligned}\mathcal{V} \sqsubseteq (q)\, \mathcal{U} \;&:\Longleftrightarrow\; \forall V \in \mathcal{V}\, \exists U \in \mathcal{U} \colon V \; q \; U \\ &\Longleftrightarrow\; {\downarrow_{(\text{Neigh}\,\mathbf{A}, q)}} \mathcal{V} \subseteq {\downarrow_{(\text{Neigh}\,\mathbf{A}, q)}} \mathcal{U}\end{aligned}$$

Definition (q-Refinement quasiorder for a quord $q \subseteq \leq_{\mathrm{cov}}$)

$$\mathcal{V} \leq_{\mathrm{ref}} (q)\, \mathcal{U} \;:\Longleftrightarrow\; \mathcal{V} \sqsubseteq (q)\, \mathcal{U} \wedge \mathcal{U} \leq_{\mathrm{cov}} \mathcal{V}$$

Quord's on sets of neighbourhoods

For $\mathcal{U}, \mathcal{V} \subseteq \text{Neigh}\,\mathbf{A}$ we define:

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Lemma (q-Refinement for covers, $q \subseteq \leq_{\mathrm{cov}}$)

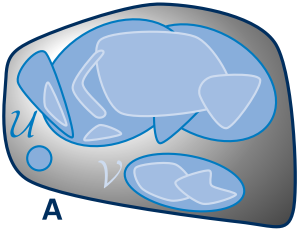
For $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}\,(\mathbf{A})$ we have $\mathcal{V} \leq_{\mathrm{ref}} (q)\, \mathcal{U}$ iff $\mathcal{V} \sqsubseteq (q)\, \mathcal{U}$.

q-Non-refinable covers

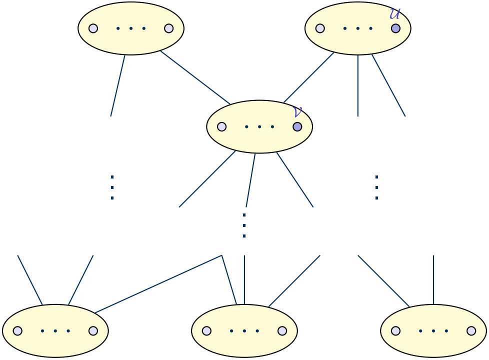
For $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}(\mathbf{A})$:

$\mathcal{V} \leq_{\mathrm{ref}} (q)\, \mathcal{U}$ quasiorder

$:\Longleftrightarrow \forall V \in \mathcal{V} \exists U \in \mathcal{U}\colon\ V\, q\, U$



q-Non-refinable covers



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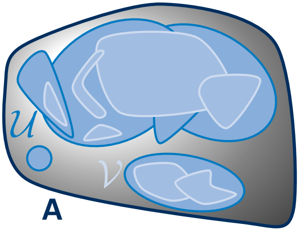
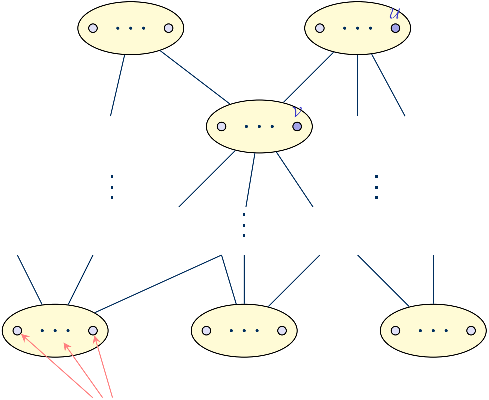
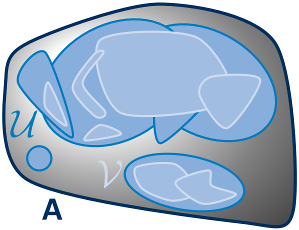


q-Non-refinable covers

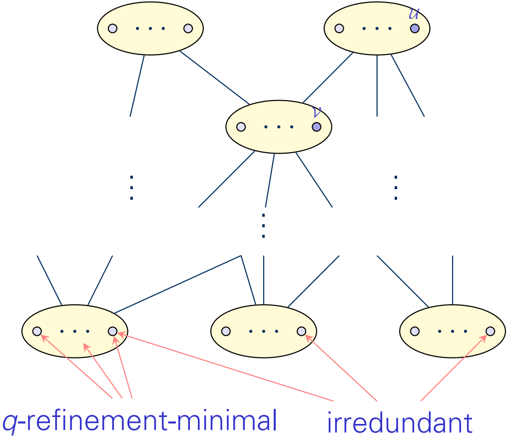


q-refinement-minimal

For $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}\,(\mathbf{A})$:
$\mathcal{V} \leq_{\mathrm{ref}} (q)\; \mathcal{U}$ quasiorder
$:\Longleftrightarrow \forall V \in \mathcal{V} \exists U \in \mathcal{U} \colon V\; q\; U$



q-Non-refinable covers



For $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}(\mathbf{A})$:
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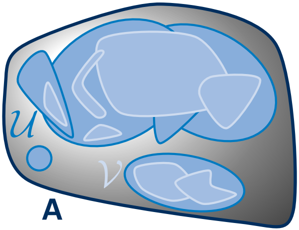
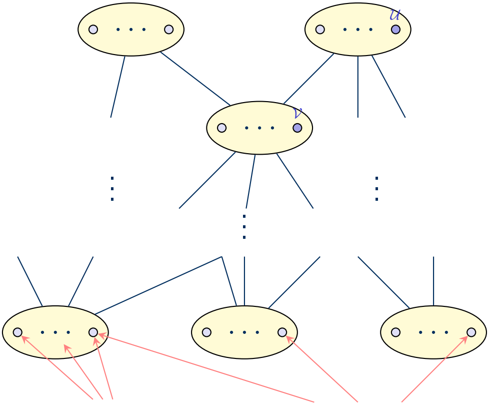
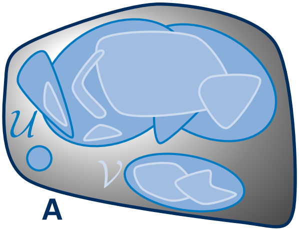


q-Non-refinable covers



For $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}\,(\mathbf{A})$:
$\mathcal{V} \leq_{\mathrm{ref}} (q)\ \mathcal{U}$ quasiorder
$:\Longleftrightarrow \forall V \in \mathcal{V} \exists U \in \mathcal{U} \colon V\ q\ U$



q-refinement-minimal $\wedge$ irredundant $\Longleftrightarrow: \mathcal{U} \in \mathrm{Cov}\,(\mathbf{A})$ q-non-refinab

q-Cover (pre)bases

Definition (q-Cover prebase for a quord $q \subseteq \leq_{\mathrm{cov}}$)

$\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ q-cover prebase $:\Longleftrightarrow \forall \mathcal{U} \in \operatorname{Cov}(\mathbf{A}) : \mathcal{V} \sqsubseteq (q)\, \mathcal{U}$.

Definition (q-Cover base)

$\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ q-cover base $:\Longleftrightarrow$ $\mathcal{V} \in \operatorname{Cov}(\mathbf{A}) \wedge$
$\mathcal{V}$ q-cover prebase

Definition (Irredundant q-cover base)

$\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ irredundant q-cover base
$\Longleftrightarrow$ q-cover base and irredundant cover

Constructing q-non-refinable covers

Lemma

$\mathcal{V} \in \mathrm{Cov}(\mathbf{A})$ *q-cover base* + $(\mathcal{V}, q\restriction_{\mathcal{V}})$ *has ACC* $\rightsquigarrow \mathcal{W}$ *irredundant q-cover base.*

Theorem

$\mathcal{V} \in \mathrm{Cov}(\mathbf{A})$ *an irredundant q-cover base* $\implies$

- $\mathcal{V}$ *q-non-refinable cover*
- *Every q-non-refinable cover* $\mathcal{U} \in \mathrm{Cov}(\mathbf{A})$ *satisfies* $\mathcal{U} \cong_q \mathcal{V}$

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Hands on *q*-non-refinable covers

$\mathbf{A}$ poly-Artinian + Var $\mathbf{A}$ 1-locally finite

$\Longrightarrow$ strictly irreducible neigh's = embedding-cover base.

$\Longrightarrow$ embedding-non-refinable covers exist and are unique

$\Longrightarrow$ = non-refinable covers exist and are unique

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A general localisation theory for structures seems possible

- structures $\mathbf{A}$ with carrier sets A
- local subsets $A \in \text{Neigh}\,\mathbf{A} \subseteq \mathcal{P}(A)$
- interesting properties $\text{Int}\,\mathbf{A}$
  RST: $\text{Int}\,\mathbf{A} = \text{Sep}_{\mathbf{A}}(A)$.
- for $U \in \text{Neigh}\,\mathbf{A}$ manageable properties
  $\text{Man}_{\mathbf{A}}(U) \subseteq \text{Int}\,\mathbf{A}$; assume $\text{Man}_{\mathbf{A}}(A) = \text{Int}\,\mathbf{A}$
  RST: $\text{Man}_{\mathbf{A}}(U) = \text{Sep}_{\mathbf{A}}(U)$
- define $\text{Man}_{\mathbf{A}}(\mathcal{U}) := \bigcup_{U \in \mathcal{U}} \text{Man}_{\mathbf{A}}(U)$ for $\mathcal{U} \subseteq \text{Neigh}\,\mathbf{A}$
- let $\mathcal{U} \leq_{\text{cov}} \mathcal{V} :\Longleftrightarrow \text{Man}_{\mathbf{A}}(\mathcal{U}) \subseteq \text{Man}_{\mathbf{A}}(\mathcal{V})$
- **goal:** $\text{Cov}(\mathbf{A}) := \{\mathcal{V} \subseteq \text{Neigh}\,\mathbf{A} \mid \{A\} \leq_{\text{cov}} \mathcal{V}\}$
- q-refinement, q-non-refinability, q-cover (pre)bases and their related assertions **still make sense**

Thank you for your attention