On multipalindromic sequences

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Definition

We call a number a *palindrome in base b* if for its expansion in base b, say $\langle c_{d-1}, c_{d-2}, \ldots, c_0\rangle_b$, $c_{d-1} \neq 0$, the equality $c_j = c_{d-1-j}$ holds for every $0 \leqslant j \leqslant d-1$.

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- We are interested in numbers that are (roughly said) palindromes simultaneously in more different bases.

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Theorem (Goins, 2009)

There are exactly 203 positive integers that are d-digit palindrome in base 10 and d-digit palindrome in another base, ranging from 22 to 9986831781362631871386899 ($d = 2$ to $d = 25$).

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  $\langle 9,8,9\rangle_{10} = \langle 3,7,3\rangle_{17} = \langle 2,5,2\rangle_{21} = \langle 1,12,1\rangle_{26}$.

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Is it possible to find more than four different bases such that there is a number that is a d-digit palindrome simultaneously in all those bases?

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Is it possible to find more than four different bases such that there is a number that is a d-digit palindrome simultaneously in all those bases? If possible, then what is the largest such list?

The number of bases is unbounded

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Theorem

Given any $K \in \mathbb{N}$ *and* $d \geqslant 2$*, there exists* $n \in \mathbb{N}$ *and a list of bases* $\{b_1, b_2, \ldots, b_K\}$ *such that, for each* $1 \leqslant i \leqslant K$*,* n *is a* d*-digit palindrome in base* b_i*.*

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- $n = \left\langle \binom{d-1}{d-1} a_i, \binom{d-1}{d-2} a_i, \dots, \binom{d-1}{1} a_i, \binom{d-1}{0} a_i \right\rangle_{b_i}$

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- For $d = 2$, these are precisely all the palindromic sequences of length 2.

Easy comes first: palindromes of variable length

Theorem

Let $d \geqslant 2$ and a palindromic sequence $\langle c_{d-1}, c_{d-2}, \ldots, c_0 \rangle$, $c_{d-1} \neq 0$, be given. Then for any $K \in \mathbb{N}$ there exists $n \in \mathbb{N}$ and a list of bases $\{b_1, b_2, \ldots, b_K\}$ such that, for each i such that $1 \leqslant i \leqslant K$, n is a palindrome with at least d digits in base b_i, and that, for some i_0 such that $1 \leqslant i_0 \leqslant K$, we have $\langle c_{d-1}, c_{d-2}, \ldots, c_0 \rangle_{b_{i_0}} = n$.

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- $$m \equiv -\frac{c_1}{c_0^2(K-2)!} \pmod{s - c_0(K-2)!}$$
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Three digits — the main result

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  - The core of the second construction seems to provide some space for optimization in order to get a smaller number a (and thus a smaller number n).
  - There are some arguments that suggest that for $d > 3$ the numbers we are looking for become much rarer; thus, it is not at all impossible that a construction that produces large values in the case $d = 3$ can be adapted to be of some use also for $d > 3$, while the one that produces small values in the case $d = 3$ actually only picks some exceptions whose existence essentially relies on the assumption $d = 3$.

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- Could the number K from the previous question be equal to 1 for some sequence?