Generalizations of the classical wave equation within the theory of fractional calculus

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Wave equation as a system of PDEs

- Classical one-dimensional wave equation

$$\frac{\partial^2}{\partial t^2} u(x,t) = c^2 \frac{\partial^2}{\partial x^2} u(x,t), \quad c = \sqrt{E/\rho},$$

  is obtained from system of PDEs:

- equation of motion

$$\frac{\partial}{\partial x} \sigma(x,t) = \rho \frac{\partial^2}{\partial t^2} u(x,t),$$

  σ - stress, ρ - mass density, u - displacement,

- constitutive equation - elastic body - Hooke law

$$\sigma(x,t) = E\varepsilon(x,t),$$

  E - Young modulus, ε - strain,

- strain

$$\varepsilon(x,t) = \frac{\partial}{\partial x} u(x,t).$$

Generalization of constitutive equation and strain

- Hereditary (history dependency, time non-locality) viscoelastic body ($\alpha \in (0,1)$)

$$\begin{aligned}
\int_0^1 \phi_\sigma(\alpha)\, {}_0\mathrm{D}_t^\alpha \sigma(x,t)\,\mathrm{d}\alpha &= E\int_0^1 \phi_\varepsilon(\alpha)\, {}_0\mathrm{D}_t^\alpha \varepsilon(x,t)\,\mathrm{d}\alpha,\\
\sigma(x,t) + a\, {}_0\mathrm{D}_t^\alpha \sigma(x,t) &= E\left(\varepsilon(x,t) + b\, {}_0\mathrm{D}_t^\alpha \varepsilon(x,t)\right),\\
\sum_{j=0}^{n} a_j\, {}_0\mathrm{D}_t^{\alpha_j} \sigma(x,t) &= \sum_{j=0}^{n} b_j\, {}_0\mathrm{D}_t^{\alpha_j} \varepsilon(x,t).
\end{aligned}$$

- Spatial non-locality ($\alpha \in (1,3)$)

$$\sigma(x,t) - l_c^\alpha\, \mathrm{D}_x^\alpha \sigma(x,t) = E\varepsilon(x,t).$$

- Spatial non-locality - introduced by generalizing strain to strain measure

$$\varepsilon(x,t) = \mathcal{E}_x^\beta u(x,t).$$

Fractional integral

- Starting form Cauchy formula ($g \in L^1_{loc}(\mathbb{R}_+)$, $t > 0$, $n \in \mathbb{N}$)

$$\mathrm{J}^n g(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} g(\tau)\, \mathrm{d}\tau = \frac{t^{n-1}}{(n-1)!} * g(t),$$

- replacing $n \in \mathbb{N}$ with $\alpha \in \mathbb{R}_+$, one obtains fractional integral

$$\mathrm{J}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau)\, \mathrm{d}\tau = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * g(t), \quad t > 0.$$

- We introduce $f_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$.
- FI satisfies the semigroup property (convolution and f_α)

$$\mathrm{J}^\alpha(\mathrm{J}^\beta u) = \mathrm{J}^\beta(\mathrm{J}^\alpha u) = \mathrm{J}^{\alpha+\beta} u.$$

- Fractional derivatives are introduced inspired by:

$$\mathrm{D}^n \mathrm{J}^n g = g \;\text{ and }\; \mathrm{J}^n \mathrm{D}^n g(t) = g(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \mathrm{D}^k g(0).$$

Riemann-Liouville fractional derivative

Definition

Let $u \in L^1_{loc}(\mathbb{R}_+)$, then the Riemann-Liouville fractional derivative of order $\alpha \in (m-1, m)$, $m \in \mathbb{N}$ is defined by

$$ {}^{RL}_{\ 0}\mathrm{D}^{\alpha}_{t} u = \mathrm{D}^m \left(f_{m-\alpha} * u \right) = \mathrm{D}^m \left(\frac{t^{m-\alpha-1}}{\Gamma(m-\alpha)} * u(t) \right). $$

- In the operator form: ${}^{RL}_{\ 0}\mathrm{D}^{\alpha}_{t} u = \mathrm{D}^m \mathrm{J}^{m-\alpha} u$.
- With the convolution explicitly written:

$$ {}^{RL}_{\ 0}\mathrm{D}^{\alpha}_{t} u(t) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^t \frac{u(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau, \quad t > 0, $$

$$ {}^{RL}_{\ 0}\mathrm{D}^{\alpha}_{t} \left(\mathrm{J}^{\alpha} u(t) \right) = u(t). $$

Caputo fractional derivative

Definition

Let $u \in AC^m(\mathbb{R}_+)$, then the Caputo fractional derivative of order $\alpha \in (m-1, m)$, $m \in \mathbb{N}$ is defined by

$$ {}_0\mathrm{D}_t^\alpha u = f_{m-\alpha} * \mathrm{D}^m u = \frac{t^{m-1-\alpha}}{\Gamma(m-\alpha)} * \mathrm{D}^m u. $$

- In the operator form: ${}_0D_t^\alpha u = \mathrm{J}^{m-\alpha} D^m u$.
- With the convolution explicitly written:

$$ {}_0\mathrm{D}_t^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\mathrm{D}^m u(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau, \quad t > 0, $$

$$ \mathrm{J}^\alpha \left({}_0\mathrm{D}_t^\alpha u(t) \right) = u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \mathrm{D}^k u(0). $$

Riesz fractional derivative - I

- Riesz fractional derivative may be of Caputo type. Then, it is defined by ($\beta \in (0,1)$, $x \in \mathbb{R}$)

$$\begin{aligned} \mathcal{E}_x^\beta u(x) &= \frac{1}{2}\left(\mathrm{D}_+^\beta u(x) - \mathrm{D}_-^\beta u(x)\right) \\ &= \frac{1}{2}\frac{1}{\Gamma(1-\beta)}|x|^{-\beta} * \frac{\mathrm{d}}{\mathrm{d}x}u(x), \end{aligned}$$

- since

$$\begin{aligned} \mathrm{D}_+^\beta u(x) &= \frac{1}{\Gamma(1-\beta)}\int_{-\infty}^{x}\frac{\frac{\mathrm{d}}{\mathrm{d}\zeta}u(\zeta)}{(x-\zeta)^\beta}\mathrm{d}\zeta, \\ \mathrm{D}_-^\beta u(x) &= -\frac{1}{\Gamma(1-\beta)}\int_{x}^{\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}\zeta}u(\zeta)}{(\zeta-x)^\beta}\mathrm{d}\zeta. \end{aligned}$$

- Riesz fractional derivative generalizes the first derivative, but not the zeroth one.

Riesz fractional derivative - II

- Riesz fractional derivative of order $\alpha \in (1,2)$ of Caputo type

$$\mathrm{D}_x^\alpha u = \frac{1}{2}\left(\mathrm{D}_+^\alpha + \mathrm{D}_-^\alpha\right) u = \frac{1}{2\Gamma(2-\alpha)} \frac{1}{|x|^{\alpha-1}} * \frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x).$$

- Riesz fractional derivative of order $\alpha \in (2,3)$ of Caputo type

$$\mathrm{D}_x^\alpha u = \frac{1}{2}\left(\mathrm{D}_+^\alpha + \mathrm{D}_-^\alpha\right) u = \frac{1}{2\Gamma(3-\alpha)} \frac{\operatorname{sgn} x}{|x|^{\alpha-2}} * \frac{\mathrm{d}^3}{\mathrm{d}x^3} u(x).$$

- Both definitions generalize the second derivative.
- Definitions ($\alpha \in (n-1, n)$):

$$\mathrm{D}_+^\alpha u(x) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{x} \frac{\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} u(\zeta)}{(x-\zeta)^{\alpha-n+1}} \mathrm{d}\zeta,$$

$$\mathrm{D}_-^\alpha u(x) = (-1)^n \frac{1}{\Gamma(n-\beta)} \int_{x}^{\infty} \frac{\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} u(\zeta)}{(\zeta-x)^{\alpha-n+1}} \mathrm{d}\zeta.$$

Distributional fractional derivative - I

- Distributional fractional derivative - defined as inverse operator of FI J^α.
- Family of tempered distributions with support in $[0, \infty)$

$$f_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)} H(t) & t \in \mathbb{R},\ \alpha > 0, \\ \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} H(t)\right] & t \in \mathbb{R},\ n \in \mathbb{N},\ \alpha + n > 0. \end{cases}$$

- Semigroup property

$$f_\alpha * f_\beta = f_{\alpha+\beta},\quad \alpha, \beta \in \mathbb{R}.$$

Distributional fractional derivative - II

Definition

Let $h \in \mathcal{S}'_+(\mathbb{R})$, then the distributional fractional derivative of order $\alpha \in (m-1, m)$, $m \in \mathbb{N}$, is defined by

$$\mathrm{D}_t^\alpha h = f_{m-\alpha} * \mathrm{D}^m h = \mathrm{D}^m \left[f_{m-\alpha} * h \right].$$

- Left (right) inverse operator of FI J^α

$$\begin{aligned}
\mathrm{D}_t^\alpha \mathrm{J}^\alpha h &= \mathrm{D}^m \left[f_{m-\alpha} * (f_\alpha * h) \right] = \mathrm{D}^m \left[f_m * h \right] = \mathrm{D}^m f_m * h = h, \\
\mathrm{J}^\alpha \mathrm{D}_t^\alpha h &= f_\alpha * \left[f_{m-\alpha} * \mathrm{D}^m h \right] = f_m * \mathrm{D}^m h = \mathrm{D}^m f_m * h = h.
\end{aligned}$$

Complex order fractional derivative

Definition

Complex order fractional derivative fractional derivative of order α, β, with $\alpha \in (0,1)$, $\beta \in \mathbb{R}_+$, is defined as

$$ {}_0\overline{\mathrm{D}}_t^{\alpha,\beta} = \frac{1}{2}\left({}_0\mathrm{D}_t^{\alpha+\mathrm{i}\beta} + {}_0\mathrm{D}_t^{\alpha-\mathrm{i}\beta}\right), $$

where ${}_0\mathrm{D}_t^{\alpha\pm\mathrm{i}\beta}$ is the Riemann-Liouville fractional derivative.

- Property: applied to real valued function returns the real valued function for real argument.

Constitutive equation

- Complex fractional order Kelvin-Voigt type CE

$$\sigma(t) = E\left(1 + a\,{}_0\mathrm{D}_t^{\alpha} + 2b\,{}_0\overline{\mathrm{D}}_t^{\alpha,\beta}\right)\varepsilon(t), \quad t > 0.$$

- Thermodynamical restrictions

$$a \geq 2b\cosh\frac{\beta\pi}{2}\sqrt{1 + \left(\cot\frac{\alpha\pi}{2}\tanh\frac{\beta\pi}{2}\right)^2},$$

$$a \geq 2b\cosh\frac{\beta\pi}{2}\sqrt{1 + \left(\tan\frac{\alpha\pi}{2}\tanh\frac{\beta\pi}{2}\right)^2},$$

- follow from the complex modulus $E = E' + \mathrm{i}E''$

$$E(\omega) = 1 + a(\mathrm{i}\omega)^{\alpha} + b\left((\mathrm{i}\omega)^{\alpha+\mathrm{i}\beta} + (\mathrm{i}\omega)^{\alpha-\mathrm{i}\beta}\right), \quad \omega \in \mathbb{R}_+,$$

by requiring $E' \geq 0$ and $E'' \geq 0$ for all $\omega \in \mathbb{R}_+$.

Creep experiment - I

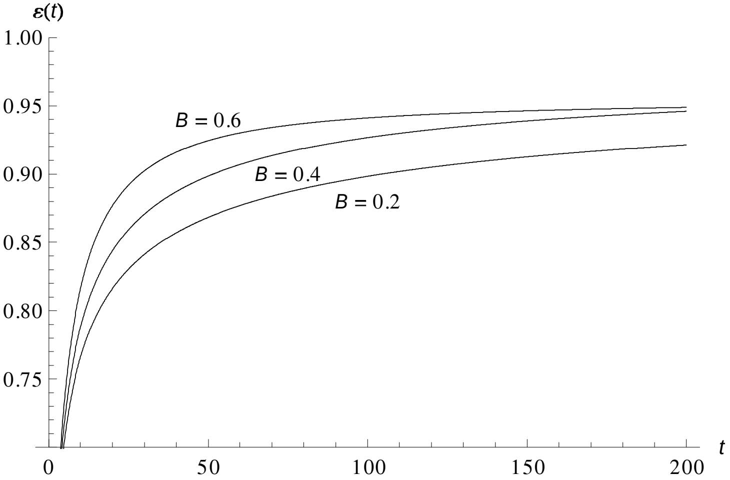

Figure: Creep curves for $\alpha = 0.4$ and $B \in \{0.2, 0.4, 0.6\}$, $a = 0.8$, $b = 0.1$, $t \in [0, 200]$.

Creep experiment - II

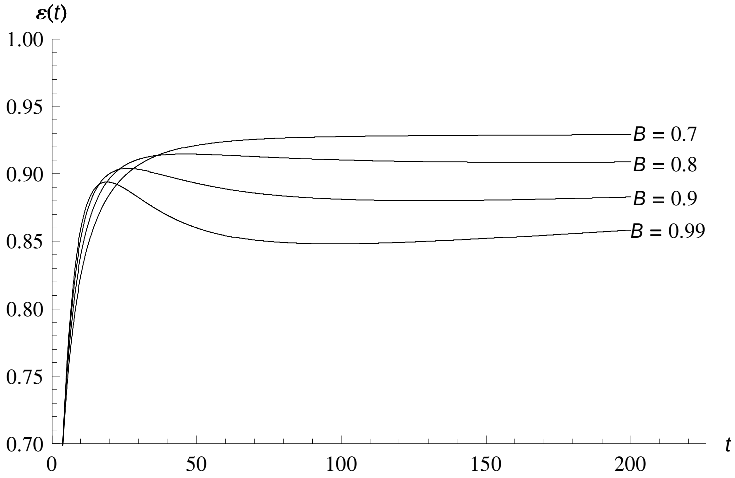

Figure: Creep curves for $\alpha = 0.4$ and $B \in \{0.7, 0.8, 0.9, 0.99\}$, $a = 0.8$, $b = 0.1$, $t \in [0, 200]$.

Stress relaxation experiment

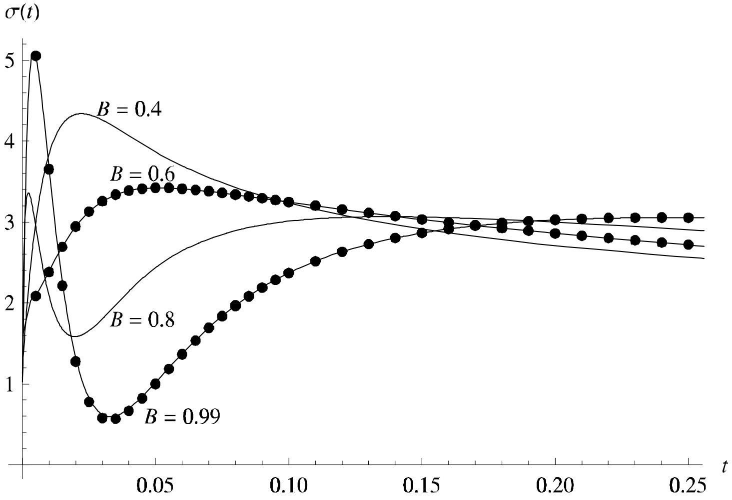

Figure: Stress relaxation curves for $\alpha = 0.4$ and $B \in \{0.4, 0.6, 0.8, 0.99\}$, $a = 0.8$, $b = 0.1$, $t \in [0, 0.25]$.

Space-time FWE of Zener type

Space-time FWE of Zener type is represented by the system of equations with initial and boundary data ($x \in \mathbb{R}$, $t > 0$)

$$\partial_x \sigma(x,t) = \partial_{tt} u(x,t), \;\; \varepsilon(x,t) = \mathcal{E}_x^{\beta} u(x,t),$$
$$\sigma(x,t) + \tau\, {}_0\mathrm{D}_t^{\alpha} \sigma(x,t) = \varepsilon(x,t) + {}_0\mathrm{D}_t^{\alpha} \varepsilon(x,t), \;\; \tau < 1,$$
$$u(x,0) = u_0(x), \;\; \partial_t u(x,0) = v_0(x), \;\; \sigma(x,0) = 0, \;\; \varepsilon(x,0) = 0,$$
$$\lim_{x \to \pm\infty} u(x,t) = 0, \quad \lim_{x \to \pm\infty} \sigma(x,t) = 0.$$

In distributional setting:

$$\partial_{tt} u(x,t) = L_t^{\alpha} \partial_x \mathcal{E}_x^{\beta} u(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t), \;\; in \;\; \mathcal{K}'(\mathbb{R}^2),$$
$$L_t^{\alpha} = \mathcal{L}^{-1}\left[\frac{1+s^{\alpha}}{1+\tau s^{\alpha}}\right] *_t = \left(\frac{1}{\tau}\delta(t) + \left(\frac{1}{\tau} - 1\right) e'_{\alpha}(t)\right) *_t, \;\; t > 0.$$

Space-time FWE of Zener type

Space-time FWE of Zener type is represented by the system of equations ($x \in \mathbb{R}$, $t > 0$)

$$
\begin{aligned}
\partial_x \sigma(x,t) &= \rho\, \partial_{tt} u(x,t), \\
\sigma(x,t) &= \frac{E}{2\ell^{1-\beta}\Gamma(1-\beta)} \left(\frac{\tau_\varepsilon}{\tau_\sigma} |x|^{-\beta} \delta(t) \right. \\
&\quad \left. + \left(\frac{\tau_\varepsilon}{\tau_\sigma} - 1 \right) |x|^{-\beta} e'_\alpha(t) \right) *_{x,t} \varepsilon_{cl}(x,t), \\
\varepsilon_{cl}(x,t) &= \partial_x u(x,t),
\end{aligned}
$$

subject to initial and boundary data

$$
u(x,0) = u_0(x), \;\; \partial_t u(x,0) = v_0(x), \;\; \sigma(x,0) = 0, \;\; \varepsilon(x,0) = 0,
$$

$$
\lim_{x \to \pm\infty} u(x,t) = 0, \quad \lim_{x \to \pm\infty} \sigma(x,t) = 0.
$$

Theorem

Let $\alpha \in [0,1)$, $\beta \in [0,1)$, $\tau \in (0,1)$ *and* $u_0, v_0 \in L^1(\mathbb{R})$. *Then there exists a unique generalized solution* $u \in \mathcal{K}'(\mathbb{R}^2)$, $\operatorname{supp} u \subset \mathbb{R} \times [0,\infty)$, *to the distributional space-time FWE of Zener type* ($x \in \mathbb{R}$, $t > 0$)

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi^2}\left(\delta'(t)\, u_0(x) + \delta(t)\, v_0(x)\right) *_{x,t} P(x,t), \\ P(x,t) &= I(x,t) - \left(\frac{\partial}{\partial t} J_1(x,t) + \frac{\partial^2}{\partial t^2} J_2(x,t)\right) \mathrm{e}^{s_0 t}, \end{aligned}$$

where

$$J_1 = \mathrm{i}\left(J_1^+ - J_1^-\right), \quad J_2 = J_2^+ + J_2^-.$$

Functions I, J_1^+, J_1^-, J_2^+ *i* J_2^- *bounded and continuous on* $\mathbb{R}$; *for fixed* $t \geq 0$ *and continuous, exponentially bounded on* $[0,\infty)$, *for fixed* $x \in R$.

Theorem

Let all conditions of previous theorem be satisfied. Let $u \in K'\left(\mathbb{R}^2\right)$, with support in $\mathbb{R} \times [0, \infty)$, be generalized solution to space-time FWE of Zener type. Then u takes the form($x \in \mathbb{R}$, $t > 0$)

$$u(x,t) = \left(u_0(x)\delta(t) + v_0(x)H(t)\right) *_{x,t} K(x,t),$$

where K is the distributional limit in $K'\left(\mathbb{R}^2\right)$:

$$\begin{aligned} K(x,t) &= \lim_{\varepsilon \to 0} K_\varepsilon(x,t), \\ K_\varepsilon(x,t) &= \frac{1}{\pi}\int_0^\infty S(\rho,t)\cos(\rho x)\,\mathrm{e}^{-\frac{(\varepsilon\rho)^2}{4}}\mathrm{d}\rho, \end{aligned}$$

with

$$S(\rho,t) = \frac{1}{2\pi \mathrm{i}} \int_0^\infty \left(\frac{1}{q^2 + \frac{1+q^\alpha e^{\mathrm{i}\alpha\pi}}{1+\tau q^\alpha e^{\mathrm{i}\alpha\pi}} \rho^{1+\beta} \sin\frac{\beta\pi}{2}} - \frac{1}{q^2 + \frac{1+q^\alpha e^{-\mathrm{i}\alpha\pi}}{1+\tau q^\alpha e^{-\mathrm{i}\alpha\pi}} \rho^{1+\beta} \sin\frac{\beta\pi}{2}} \right) q e^{-qt} dq$$
$$+ \left. \frac{s e^{st}}{2s + \frac{\alpha(1-\tau)s^{\alpha-1}}{(1+\tau s^\alpha)^2} \rho^{1+\beta} \sin\frac{\beta\pi}{2}} \right|_{s=s_z(\rho)} + \left. \frac{s e^{st}}{2s + \frac{\alpha(1-\tau)s^{\alpha-1}}{(1+\tau s^\alpha)^2} \rho^{1+\beta} \sin\frac{\beta\pi}{2}} \right|_{s=\bar{s}_z(\rho)}.$$

The zeros of $\Psi_\alpha(s) = s^2 + \theta\frac{1+s^\alpha}{1+\tau s^\alpha}$, $\theta = \rho^{1+\beta}\sin\frac{\beta\pi}{2}$, *are denoted by* s_z. *For suitably chosen* $s_0 > 0$, $K_\varepsilon(x,t)\,e^{-s_0 t}$ *is continuous and bounded function for* $x \in \mathbb{R}$, $t > 0$, *for every* $\varepsilon \in (0,1]$.

Special cases - I

- Starting form

$$u(x,t) = (u_0(x)\,\delta(t) + v_0(x)\,H(t)) *_{x,t} K_{\alpha,\beta}(x,t),$$

where

$$\tilde{K}_{\alpha,\beta}(x,s) = \frac{1}{\pi}\int_0^\infty \frac{s}{s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha}\rho^{1+\beta}\sin\frac{\beta\pi}{2}}\cos(\rho x)\mathrm{d}\rho.$$

- If $\alpha \to 0$, then

$$\tilde{K}_{0,\beta}(x,s) = \frac{1}{\pi}\int_0^\infty \frac{s}{s^2 + \frac{2}{1+\tau}\rho^{1+\beta}\sin\frac{\beta\pi}{2}}\cos(\rho x)\,\mathrm{d}\rho,$$

or equivalently

$$K_{0,\beta}(x,t) = \frac{1}{\pi}\int_0^\infty \cos\left(t\sqrt{\frac{2}{1+\tau}\rho^{1+\beta}\sin\frac{\beta\pi}{2}}\right)\cos(\rho x)\,\mathrm{d}\rho.$$

Special cases - II

- In the sense of distributions, with $c = \sqrt{2/\left(1+\tau\right)}$,

$$K_{0,\beta}\left(x,t\right) = \frac{1}{2\pi}\int_0^\infty \left(\cos\left(\left(x+ct\sqrt{\frac{1}{\rho^{1-\beta}}\sin\frac{\beta\pi}{2}}\right)\rho\right) + \cos\left(\left(x-ct\sqrt{\frac{1}{\rho^{1-\beta}}\sin\frac{\beta\pi}{2}}\right)\rho\right)\right)\mathrm{d}\rho.$$

- If $\beta = 0$ then

$$K_{0,0}\left(x,t\right) = \frac{1}{\pi}\int_0^\infty \cos\left(x\rho\right)\mathrm{d}\rho = \delta\left(x\right).$$

- If $\beta = 1$ then

$$\begin{aligned} K_{0,1}\left(x,t\right) &= \frac{1}{2\pi}\int_0^\infty \left(\cos\left(\left(x+ct\right)\rho\right) + \cos\left(\left(x-ct\right)\rho\right)\right)\mathrm{d}\rho, \\ &= \frac{1}{2}\left(\delta\left(x+ct\right) + \delta\left(x-ct\right)\right). \end{aligned}$$

Plots of solution u for $u_0 = \delta$ and $v_0 = 0$ - I

Let: $\alpha = 0.25$, $\beta = 0.45$, $\tau = 0.1$, $\varepsilon = 0.01$.

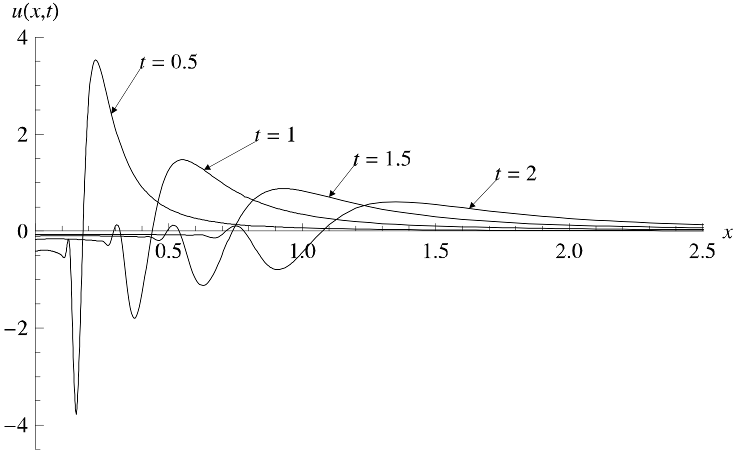

Figure: Displacement $u(x,t)$ for $t \in \{0.5, 1, 1.5, 2\}$ as a function of x.

Plots of solution u for $u_0 = \delta$ and $v_0 = 0$ - II

Let: $\alpha = 0.25$, $\tau = 0.1$, $\varepsilon = 0.01$.

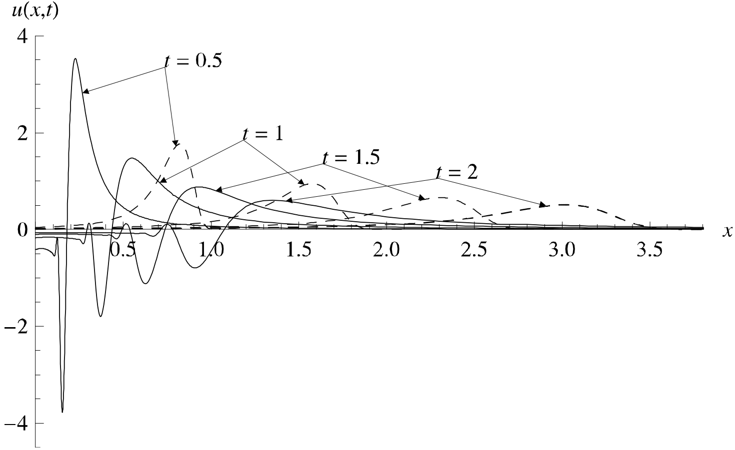

Figure: Displacement $u(x,t)$ for $t \in \{0.5, 1, 1.5, 2\}$ as a function of x: solid line - $\beta = 0.45$, dashed line - $\beta = 1$.

Plots of solution u for $u_0 = \delta$ and $v_0 = 0$ - III

Let: $\alpha = 0.25$, $\tau = 0.1$, $\varepsilon = 0.01$.

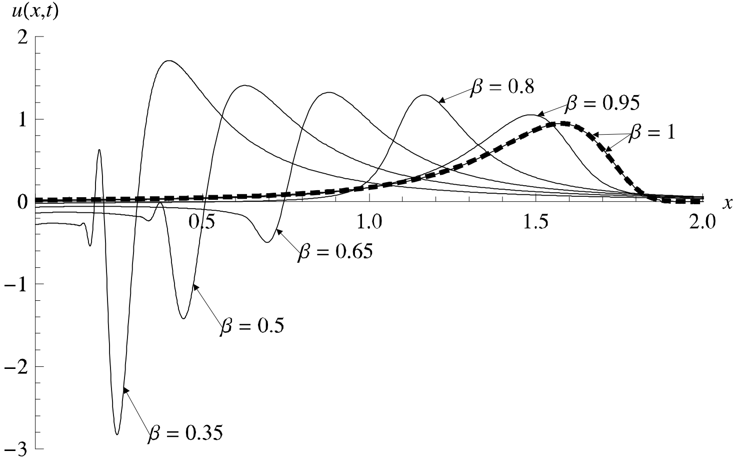

Figure: Displacement $u(x,t)$ for $t = 1$ as a function of x.

Time FWE of Zener type

Time FWE of Zener type is represented by system of equations with initial and boundary data ($x \in \mathbb{R}$, $t > 0$)

$$
\begin{gathered}
\partial_x \sigma(x,t) = \partial_{tt} u(x,t), \;\; \varepsilon(x,t) = \partial_x u(x,t), \\
\sigma(x,t) + \tau\, {}_0\mathrm{D}_t^\alpha \sigma(x,t) = \varepsilon(x,t) + {}_0\mathrm{D}_t^\alpha \varepsilon(x,t), \;\; \tau < 1, \\
u(x,0) = u_0(x), \;\; \partial_t u(x,0) = v_0(x), \;\; \sigma(x,0) = 0, \;\; \varepsilon(x,0) = 0, \\
\lim_{x \to \pm\infty} u(x,t) = 0, \quad \lim_{x \to \pm\infty} \sigma(x,t) = 0.
\end{gathered}
$$

In distributional setting

$$
\begin{gathered}
\partial_{tt} u(x,t) = \mathcal{L}^{-1}\left[\frac{1+s^\alpha}{1+\tau s^\alpha}\right] *_t \partial_{xx} u(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t), \\
L_t^\alpha = \mathcal{L}^{-1}\left[\frac{1+s^\alpha}{1+\tau s^\alpha}\right] *_t = \left(\frac{1}{\tau}\delta(t) + \left(\frac{1}{\tau} - 1\right) e_\alpha'(t)\right) *_t, \quad t > 0.
\end{gathered}
$$

Theorem

Let $u_0, v_0 \in S'(R)$. Then there exists unique solution $u \in S'(R \times R_+)$ to time FWE of Zener type

$$u(x,t) = S(x,t) *_{x,t} (u_0(x)\delta'(t) + v_0(x)\delta(t)),$$

where

$$S(x,t) = \frac{1}{2} + \frac{1}{4\pi \mathrm{i}} \int_0^\infty \left(\sqrt{\frac{1+\tau q^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi}}{1+q^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi}}} \mathrm{e}^{|x| q \sqrt{\frac{1+\tau q^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi}}{1+q^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi}}}} - \sqrt{\frac{1+\tau q^\alpha \mathrm{e}^{-\mathrm{i}\alpha\pi}}{1+q^\alpha \mathrm{e}^{-\mathrm{i}\alpha\pi}}} \mathrm{e}^{|x| q \sqrt{\frac{1+\tau q^\alpha \mathrm{e}^{-\mathrm{i}\alpha\pi}}{1+q^\alpha \mathrm{e}^{-\mathrm{i}\alpha\pi}}}} \right) \frac{\mathrm{e}^{-qt}}{q} \mathrm{d}q,$$

is the fundamental solution $S \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$ with support in the cone $|x| < \frac{t}{\sqrt{\tau}}$.

Plots of solution u for $u_0 = \delta$ and $v_0 = 0$ - I

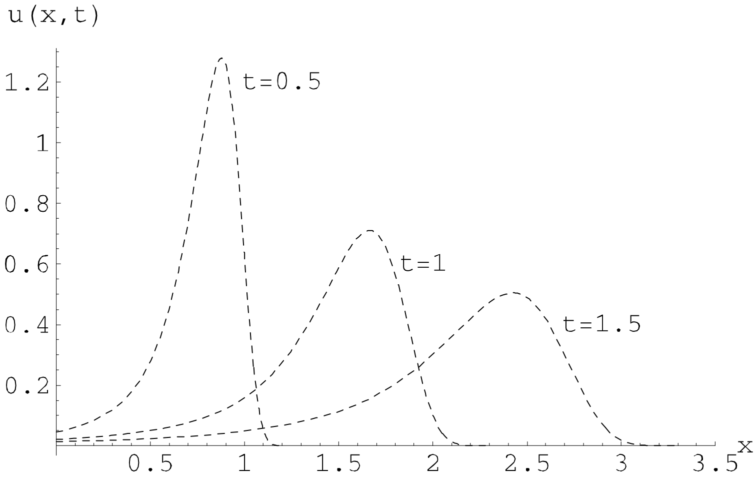

Figure: Displacement $u(x,t)$, $x \in (0,3)$, $t \in \{0.5, 1, 1.5\}$ for $\alpha = 0.25$.

Plots of solution u for $u_0 = \delta$ and $v_0 = 0$ - II

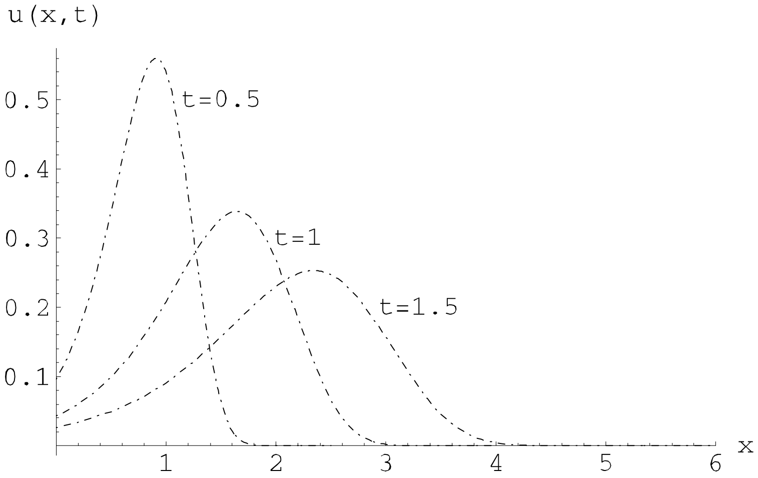

Figure: Displacement $u(x,t)$, $x \in (0,6)$, $t \in \{0.5, 1, 1.5\}$ for $\alpha = 0.5$.

Plots of solution u for $u_0 = \delta$ and $v_0 = 0$ - III

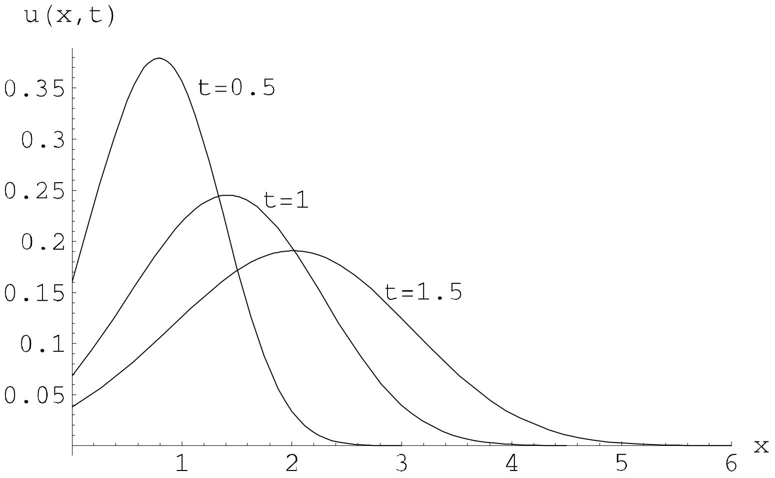

Figure: Displacement $u(x,t)$, $x \in (0,6)$, $t \in \{0.5, 1, 1.5\}$ for $\alpha = 0.75$.

Plots of solution u for $u_0 = \delta$ and $v_0 = 0$ - IV

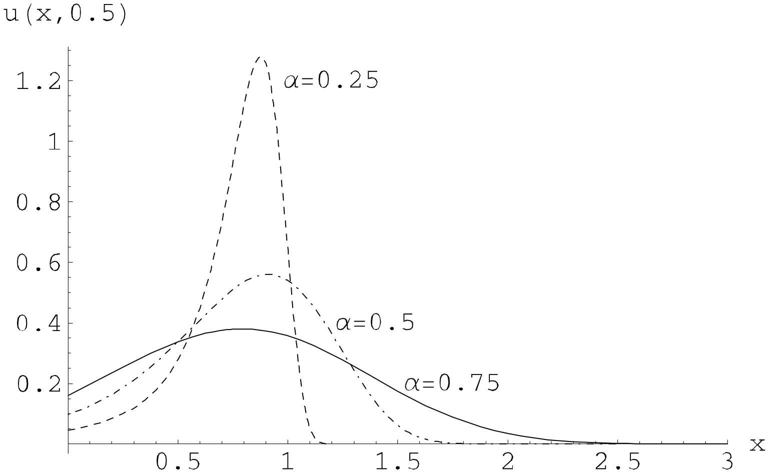

Figure: Displacement $u(x,t)$, $x \in (0,3)$, $t = 0.5$ for $\alpha \in \{0.25, 0.5, 0.75\}$.

Plots of solution u for $u_0 = \delta$ and $v_0 = 0$ - V

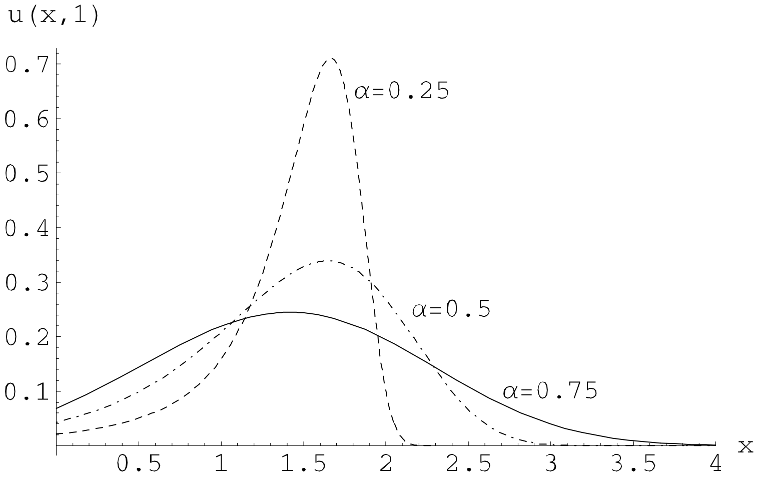

Figure: Displacement $u(x,t)$, $x \in (0,4)$, $t = 1$ for $\alpha \in \{0.25, 0.5, 0.75\}$.

Plots of solution u for $u_0 = \delta$ and $v_0 = 0$ - VI

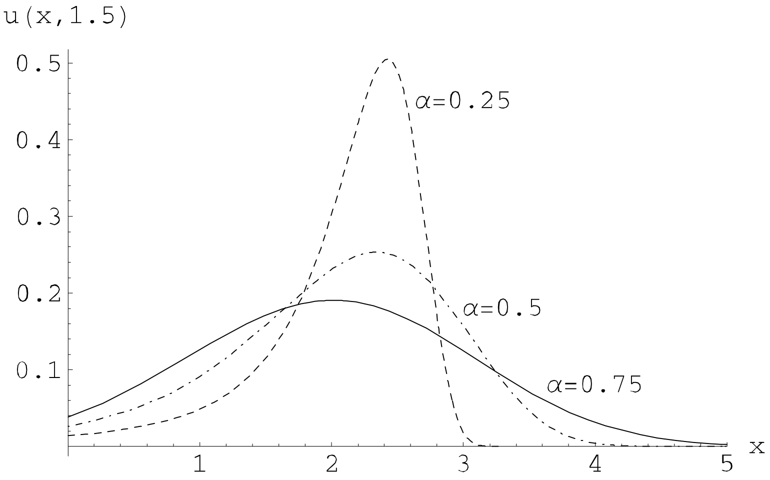

Figure: Displacement $u(x,t)$, $x \in (0,5)$, $t = 1.5$ for $\alpha \in \{0.25, 0.5, 0.75\}$.

Time FWE of linear type

Time FWE of linear type is represented by the system of equations with initial and boundary data ($x \in \mathbb{R}$, $t > 0$)

$$\partial_x \sigma(x,t) = \partial_{tt} u(x,t), \;\; \varepsilon(x,t) = \partial_x u(x,t),$$
$$\sum_{k=0}^{n} a_k \, {}_0\mathrm{D}_t^{\alpha_k} \sigma(x,t) = \sum_{k=0}^{n} b_k \, {}_0\mathrm{D}_t^{\alpha_k} \varepsilon(x,t), \;\; \frac{a_0}{b_0} \geq \frac{a_1}{b_1} \geq \ldots \geq \frac{a_n}{b_n},$$
$$u(x,0) = u_0(x), \;\; \partial_t u(x,0) = v_0(x), \;\; \sigma(x,0) = 0, \;\; \varepsilon(x,0) = 0,$$
$$\lim_{x \to \pm\infty} u(x,t) = 0, \;\; \lim_{x \to \pm\infty} \sigma(x,t) = 0.$$

In distributional setting

$$\partial_{tt} u(x,t) = \mathcal{L}^{-1}\left[\frac{\sum_{k=0}^{n} b_k s^{\alpha_k}}{\sum_{k=0}^{n} a_k s^{\alpha_k}}\right] *_t \partial_{xx} u(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t).$$

Theorem

Let $u_0, v_0 \in \mathcal{S}'(\mathbb{R})$. Then there exists unique solution $u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$ to time FWE of linear type

$$u(x,t) = S(x,t) *_{x,t} \left(u_0(x)\delta'(t) + v_0(x)\delta(t)\right),$$

where

$$\begin{aligned} S(x,t) \;=\; & \frac{1}{2}\sqrt{\frac{a_0}{b_0}} + \frac{1}{4\pi\mathrm{i}} \int_0^\infty \left(\sqrt{\frac{\sum_{k=0}^n a_k q^{\alpha_k} \mathrm{e}^{\mathrm{i}\pi\alpha_k}}{\sum_{k=0}^n b_k q^{\alpha_k} \mathrm{e}^{\mathrm{i}\pi\alpha_k}}} \mathrm{e}^{|x| q \sqrt{\frac{\sum_{k=0}^n a_k q^{\alpha_k} \mathrm{e}^{\mathrm{i}\pi\alpha_k}}{\sum_{k=0}^n b_k q^{\alpha_k} \mathrm{e}^{\mathrm{i}\pi\alpha_k}}}} \right. \\ & \left. - \sqrt{\frac{\sum_{k=0}^n a_k q^{\alpha_k} \mathrm{e}^{-\mathrm{i}\pi\alpha_k}}{\sum_{k=0}^n b_k q^{\alpha_k} \mathrm{e}^{-\mathrm{i}\pi\alpha_k}}} \mathrm{e}^{|x| q \sqrt{\frac{\sum_{k=0}^n a_k q^{\alpha_k} \mathrm{e}^{-\mathrm{i}\pi\alpha_k}}{\sum_{k=0}^n b_k q^{\alpha_k} \mathrm{e}^{-\mathrm{i}\pi\alpha_k}}}} \right) \frac{\mathrm{e}^{-qt}}{q} \, \mathrm{d}q, \end{aligned}$$

is the fundamental solution $S \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$ with support in the cone $|x| < ct$, $c = \sqrt{\frac{a_n}{b_n}}$.

Plots of solution u for $u_0 = \delta$ and $v_0 = 0$ - I

Let: $\alpha_0 = 0.25$, $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $a_0 = 1.25$, $a_1 = 1.1$, $a_2 = 1.2$, $b_0 = 1.4$, $b_1 = 1.3$, $b_2 = 1.5$.

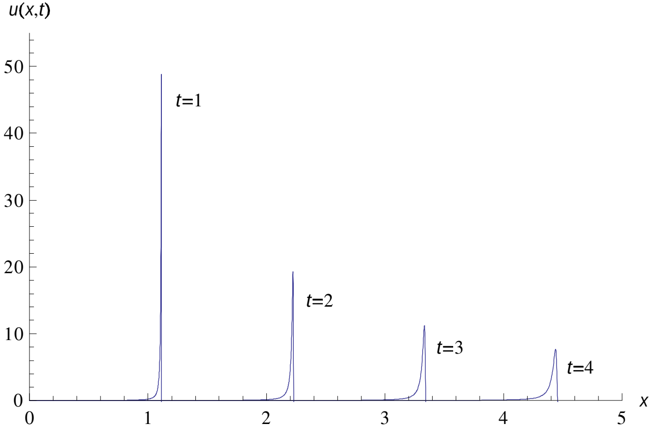

Figure: Displacement $u(x,t)$, $x \in (0,5)$, $t \in \{1,2,3,4\}$.

Plots of solution u for $u_0 = \delta$ and $v_0 = 0$ - II

Let: $\alpha_0 = 0.25$, $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $a_0 = 0.008$, $a_1 = 0.006$, $a_2 = 0.004$, $b_0 = 1.6$, $b_1 = 1.4$, $b_2 = 1.2$.

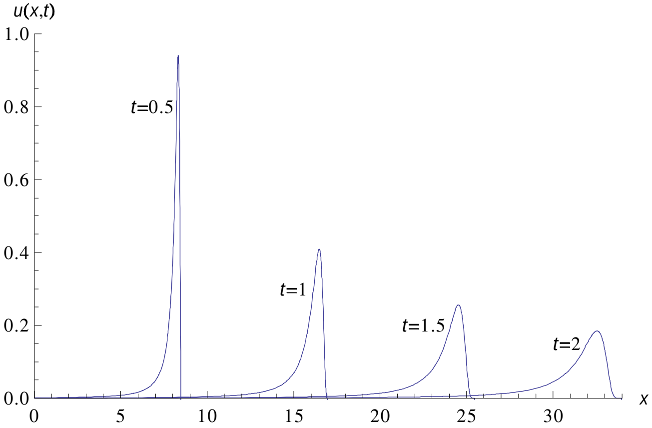

Figure: Displacement $u(x,t)$, $x \in (0,35)$, $t \in \{0.5, 1, 1.5, 2\}$.

Space FWE of Eringen type

Space FWE of Eringen type is represented by the system of equations ($x \in \mathbb{R},\ t > 0$)

$$\frac{\partial}{\partial x}\sigma(x,t) = \frac{\partial^2}{\partial t^2}u(x,t), \quad \varepsilon(x,t) = \frac{\partial}{\partial x}u(x,t),$$
$$\sigma(x,t) - l_c^{\alpha}\,\mathrm{D}_x^{\alpha}\sigma(x,t) = E\varepsilon(x,t)\,.$$

Assuming the solution to space FWE of Eringen type as a harmonic function

$$u(x,t) = u_0 \mathrm{e}^{\mathrm{i}(\omega t - kx)}, \quad \omega > 0,\ k \in \mathbb{R}.$$

Dispersion equation

- One obtains the dispersion equation

$$\omega(k) = \pm \frac{c_0 k}{\sqrt{1 - \cos\frac{\alpha\pi}{2}\left(l_c\,|k|\right)^{\alpha}}}, \quad c_0 = \sqrt{E/\rho}.$$

- For $k > 0$ the dispersion equation

$$\frac{a}{c_0}\omega(k) = ka \frac{1}{\sqrt{1 - \left(\frac{l_c}{a}\right)^{\alpha}(ka)^{\alpha}\cos\frac{\alpha\pi}{2}}},$$

is compared to dispersion equation for Born-Kármán model

$$\frac{a}{c_0}\omega_{bk}(k) = 2\sin\frac{ka}{2}.$$

- Optimal $\alpha_0 = 2.833$ and $l_c \cong 0.587a$ (for Eringen model: $l_c \cong 0.386a$).

Plots of dispersion curves - I

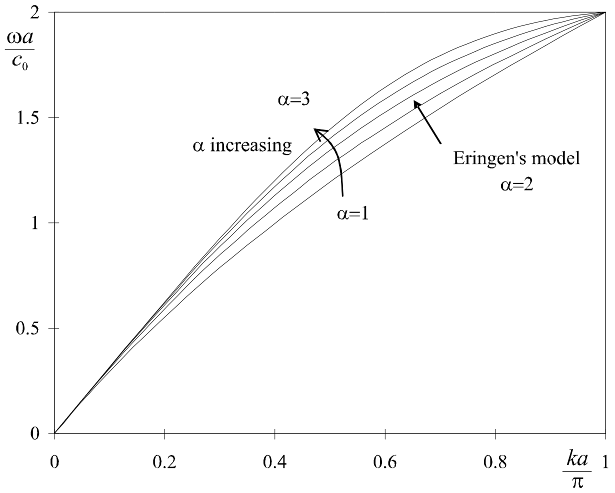

Figure: Dispersion curves for $\alpha \in \{1, 1.5, 2, 2.5, 3\}$.

Plots of dispersion curves - II

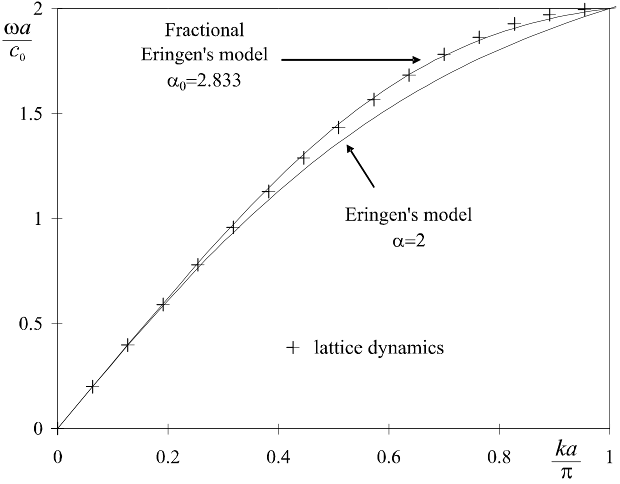

Figure: Dispersion curves.