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Problems on Kinetic Theory and PDE’s.
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Modelling non-equilibrium phenomena in which steep gradients and rapid changes occur represents a challenging task which has been tackled by means of two complementary approaches: the *continuum approach* and the *kinetic theory*. The continuum approach consists in describing the system by means of macroscopic equations (e.g. fluid-dynamic equations) obtained on the basis of conservation laws and appropriate constitutive equations. As an example, Thermodynamics of Irreversible Processes (TIP), which relies on the assumption of local thermodynamic equilibrium (LTE), has proven to be a useful and sound theory characterized by a systematic and comprehensive theoretical structure. In this framework, the Navier-Stokes-Fourier (NSF) theory has gained much popularity mainly due to its practical usefulness in many applications. Nonetheless, TIP and NSF suffer from serious weaknesses: since the mathematical structure of this theories is characterized by a system of differential equations of parabolic type, an infinite speed is predicted for the propagation of signals – an issue which has been addressed as the *paradox of heat conduction*.
Moreover, the applicability of the classical macroscopic theory is inherently restricted to processes characterized by small Knudsen numbers (dense gas), and the transport coefficients associated to the dissipation processes are not provided by the theory except for the sign. On the other hand, the approach based on the kinetic theory, which postulates that the state of the gas can be described by a velocity distribution function whose evolution is governed by the celebrated Boltzmann equation, is applicable to processes characterized by a large Knudsen number, and transport coefficients naturally emerge from the theory itself.
In particular Extended Thermodynamics (ET) which describe a nonequilibrium phenomena beyond the assumption of local equilibrium was presented to fills the gap between the above-mentioned macroscopic and microscopic approaches and shows successful result and full agreement with Kinetic Theory.

Nevertheless the weak point of ET and KT is that the applicable range is limited to rarefied monatomic gas.

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The Extended Thermodynamics of Rarefied Monoatomic Gas

The kinetic theory describes the state of a rarefied gas through the phase density $f(x, t, \xi)$, where $f(x, t, \xi)d\xi$ is the number density of atoms at point $x$ and time $t$ that have velocities between $\xi$ and $\xi + d\xi$. The phase density obeys the Boltzmann equation

$$\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} = Q$$  \hspace{1cm} (1)

where $Q$ represents the collisional terms. Most macroscopic thermodynamic quantities are identified as moments of the phase density

$$F_{k_1 k_2 \cdots k_j} = \int_{\mathbb{R}^3} f(\xi_{k_1} \xi_{k_2} \cdots \xi_{k_j}) d\xi,$$  \hspace{1cm} (2)

and due to the Boltzmann equation (1), the moments satisfy an infinity hierarchy of balance laws in which the flux in one equation becomes the density in the next one:
\[ \partial_t F + \partial_i F_i = 0 \]
\[ \partial_t F_{k_1} + \partial_i F_{i k_1} = 0 \]

\[ \partial_t F_{k_1 k_2} + \partial_i F_{i k_1 k_2} = P_{<k_1 k_2>} \]
\[ \partial_t F_{k_1 k_2 k_3} + \partial_i F_{i k_1 k_2 k_3} = P_{k_1 k_2 k_3} \]
\[ \vdots \]
\[ \partial_t F_{k_1 k_2 \ldots k_n} + \partial_i F_{i k_1 k_2 \ldots k_n} = P_{k_1 k_2 \ldots k_n} \]
\[ \vdots \]

The hierarchy structure of the system

1. The tensorial rank of the equations increases one by one.
2. The flux in one equation becomes the density in the next equation.
Taking into account that \( P_{kk} = 0 \), the first five equations are conservation laws and coincides with the mass, momentum and energy conservation respectively, while the remaining ones are balance laws.

**Remark:** ET with this hierarchy structure is valid only for *rarefied monatomic* gases. In fact due to the previous structure we have

\[
3p = 2\rho \varepsilon \quad \Pi = 0
\]

that implies

\[
\gamma = \frac{c_p}{c_V} = \frac{5}{3},
\]

i.e. monatomic gas.

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**The Closure Problem**

When we cut the hierarchy at the density with tensor of rank \( n \), we have the problem of closure because the last flux end the production terms are not in the list of the densities.
The first idea of Rational Extended Thermodynamics Müller and Ruggeri (Springer - Verlag 1993, 1998) was to view the truncated system as a phenomenological system of continuum mechanics and then we consider the new quantities as constitutive functions:

$$F_{k_1k_2\ldots k_nk_{n+1}} \equiv F_{k_1k_2\ldots k_nk_{n+1}}(F, F_{k_1}, F_{k_1k_2}, \ldots F_{k_1k_2\ldots k_n})$$

$$P_{k_1k_2\ldots k_j} \equiv P_{k_1k_2\ldots k_j}(F, F_{k_1}, F_{k_1k_2}, \ldots F_{k_1k_2\ldots k_n}) \quad 2 \leq j \leq n.$$ 

According with the continuum theory, the restrictions on the constitutive equations come only from universal principles, i.e.: Entropy principle, Objectivity Principle and Causality and Stability (convexity of the entropy).

The most interesting physical cases was the 13 fields theory in classical framework (I.S.-Liu & I. Müller - ARMA 1983) and the 14- fields in the context of relativistic fluids (I.S.-Liu, I. Müller & T. Ruggeri -ANNALS of PHYSICS 1984).

The question is if the macroscopic system have a kinetic counterpart. We will see that in the case of rarefied gas the previous structure can be explained in clear manner. We shall, therefore, briefly describe the kinetic model for polyatomic gases and point out the important consequences related to internal energy density.

The idea is to consider an additional parameter in the distribution function $f(t, x, \xi, I)$ defined on extended domain $[0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \times [0, \infty)$. Its rate of change is determined by the Boltzmann equation which has the same form as for monatomic gas but collision integral $Q(f)$ takes into account the influence of internal degrees of freedom through collisional cross section (Bourgat, Desvillettes, Le Tallec and Perthame, see also Borgnakke and Larsen.)
Collision invariants for this model form a 5-vector:

$$\psi(\xi, I) = m \left( 1, \xi_i, \frac{1}{2} m \xi^2 + I \right)^T,$$

which lead to hydrodynamic variables in the form:

$$\begin{pmatrix} \rho \\ \rho v_i \\ \frac{1}{2} \rho v^2 + \rho \varepsilon \end{pmatrix} = \int_{\mathbb{R}^3} \int_0^\infty \psi(\xi, I) f(t, x, \xi, I) \varphi(I) \, dl \, d\xi,$$

where $\rho$, $v$ and $\varepsilon$ are mass density, hydrodynamic velocity and internal energy, respectively. A non-negative measure $\varphi(I) \, dl$ is property of the model aimed at recovering classical caloric equation of state for polyatomic gases in equilibrium. Entropy is defined by the following relation:

$$h = -k \int_{\mathbb{R}^3} \int_0^\infty f \log f \varphi(I) \, dl \, d\xi.$$

We shall introduce the peculiar velocity:

\[ C = \xi - v \] (6)

and rewrite the Eq. (4) in terms of it. Then:

\[
\begin{pmatrix}
\rho \\
\rho_0 \\
2 \rho \varepsilon
\end{pmatrix} = \int_{\mathbb{R}^3} \int_{0}^{\infty} m \begin{pmatrix}
1 \\
C_i \\
C^2 + 2l/m
\end{pmatrix} f(t, x, C, l) \varphi(l) \, dl \, dC. \tag{7}
\]

Note that the internal energy density can be divided into the translational part \( \rho \varepsilon_T \) and part related to the internal degrees of freedom \( \rho \varepsilon_I \):

\[
\rho \varepsilon_T = \int_{\mathbb{R}^3} \int_{0}^{\infty} \frac{1}{2} m C^2 f(t, x, C, l) \varphi(l) \, dl \, dC,
\]

\[
\rho \varepsilon_I = \int_{\mathbb{R}^3} \int_{0}^{\infty} l f(t, x, C, l) \varphi(l) \, dl \, dC. \tag{8}
\]

The former can be related to kinetic temperature in the following way:

\[
\varepsilon_T = \frac{3}{2} \frac{k}{m} T, \tag{9}
\]

whereas the latter should determine the contribution of internal degrees of freedom to internal energy of a polyatomic gas.
In fact if $D$ is the number of degrees of freedom of a molecule, it can be shown that $\varphi(I) = I^\alpha$ leads to appropriate caloric equation in equilibrium provided:

$$\alpha = \frac{D - 5}{2}. \tag{10}$$

In fact if the weighting function is chosen to be $\varphi(I) = I^\alpha$, internal energy of a polyatomic gas in equilibrium reads:

$$\varepsilon_{|E} = \left(\frac{5}{2} + \alpha\right) \frac{k}{m} T, \quad \alpha > -1. \tag{11}$$

The relation between $\alpha$ and $D$ (10) follows directly from comparison between (11) and well-know caloric equation for polyatomic gases:

$$\varepsilon_{|E} = \frac{D}{2} \frac{k}{m} T.$$

Observe that model for a monatomic gas ($D = 3$) cannot be recovered from the one with continuous internal energy, since the value of parameter $\alpha$ in monatomic case violates the overall restriction $\alpha > -1$. 
Pavić, Ruggeri and Simić [5] firstly considered the Euler fluid with 5 moments and they considered the the maximum entropy principle expressed in terms of the following variational problem: determine the velocity distribution function $f(t, x, \xi, I)$ such that $h \rightarrow \max.$, being subjected to the constraints (4). In this way they were able to determine the following equilibrium distribution function which maximizes the entropy (5) with the constraints (4):

$$f_E = \frac{\rho}{m} \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left\{ -\frac{1}{kT} \left( \frac{1}{2} mC^2 + I \right) \right\},$$  \hspace{1cm} (12)

where $C = \xi - v$ is the peculiar velocity and

$$q(T) = \int_0^\infty \exp \left( -\frac{I}{kT} \right) \varphi(I) \, dI,$$  \hspace{1cm} (13)

that, in the case $\varphi(I) = I^\alpha$, becomes

$$q(T) = (kT)^{1+\alpha} \Gamma(1 + \alpha)$$

where $\Gamma$ is the gamma function.
Pavić, Ruggeri and Simić [5] secondly considered the case of 14 moments. This case is completely in agreement with the binary hierarchy with the moments:

\[
\begin{pmatrix}
F \\
F_{i_1} \\
F_{i_1 i_2}
\end{pmatrix}
= \int_{\mathbb{R}^3} \int_0^\infty m \begin{pmatrix}
1 \\
\xi_{i_1} \\
\xi_{i_1} \xi_{i_2}
\end{pmatrix}
\begin{array}{l}
f(t, x, \xi, I) \varphi(I) dI d\xi, \\
\end{array}
\]

\(\text{(14)}\)

\[
\begin{pmatrix}
G_{pp} \\
G_{ppk_1}
\end{pmatrix}
= \int_{\mathbb{R}^3} \int_0^\infty m \begin{pmatrix}
\xi^2 + 2 \frac{l}{m} \\
(\xi^2 + 2 \frac{l}{m}) \xi_{k_1}
\end{pmatrix}
\begin{array}{l}
f(t, x, \xi, I) \varphi(I) dI d\xi. \\
\end{array}
\]

For the entropy defined by (5), the following variational problem, expressing the maximum entropy principle, can be formulated: determine the velocity distribution function \(f(t, x, C, I)\) such that \(h \rightarrow \max\), being subjected to the constraints (14). The solution of the problem is as follows.
Near the equilibrium state the velocity distribution function, which maximizes the entropy (5) with the constraints (14) and the weighting function $\varphi(I) = I^\alpha$, has the form:

$$f = f_E \left\{ 1 - \frac{\rho}{p^2} q_i C_i + \frac{\rho}{p^2} \left[ -S_{\langle ij \rangle} + \left( \frac{5}{2} + \alpha \right) (1 + \alpha)^{-1} \prod \delta_{ij} \right] C_i C_j \right\} (15)$$

$$- \frac{3}{2(1 + \alpha)} \frac{\rho}{p^2} \prod \left( \frac{1}{2} C^2 + \frac{I}{m} \right) + \left( \frac{7}{2} + \alpha \right)^{-1} \frac{\rho^2}{p^3} q_i \left( \frac{1}{2} C^2 + \frac{I}{m} \right) C_i \right\},$$

where $f_E$ is the equilibrium distribution (12) and $q(T)$ is the auxiliary function (13). The non-equilibrium distribution (15) reduces to the velocity distribution obtained by Mallinger for gases composed of diatomic molecules ($\alpha = 0$), and, for any $\alpha > -1$, the closure gives exactly the same equations obtained before by using the macroscopic approach and the entropy principle:

The general hierarchy of moment equations for polyatomic gases

In non-equilibrium motivated by the idea of phenomenological ET, we shall generalize the moment equations for polyatomic gases by constructing two independent hierarchies. One will be much alike classical “momentum” hierarchy of monatomic gases ($F$–hierarchy); the other one, “energy” hierarchy, commences with the moment related to energy collision invariant and proceeds with standard increase of the order through multiplication by velocities ($G$–hierarchy). They read:

$$\partial_t F + \partial_i F_i = P, \quad \partial_t G + \partial_i G_i = Q.$$ 

[Pavić, Ruggeri & Simić Physica A,(2012)]
[Arima, Mentrelli & Ruggeri - Annals of Physics (2014)]
Moments, fluxes and productions of $F$–hierarchy are defined as:

\[
F(t, x) = \int_{\mathbb{R}^3} \int_{0}^{\infty} \Psi(\xi) f(\varphi(I)) \, dI \, d\xi,
\]

\[
F_i(t, x) = \int_{\mathbb{R}^3} \int_{0}^{\infty} \xi_i \Psi(\xi) f(\varphi(I)) \, dI \, d\xi,
\]

\[
P(t, x) = \int_{\mathbb{R}^3} \int_{0}^{\infty} \Psi(\xi) Q(f) \varphi(I) \, dI \, d\xi,
\]

with:

\[
\Psi(\xi) = m \begin{pmatrix}
1 \\
\xi_i \\
\xi_i \xi_i \\
\vdots \\
\xi_i \cdots \xi_i \\
\xi_i \cdots \xi_i
\end{pmatrix}.
\]
Moments, fluxes and productions of $G$–hierarchy are defined as:

\[
G(t, x) = \int_{\mathbb{R}^3} \int_0^\infty \Theta(\xi, I) f(\varphi(I)) dI \, d\xi,
\]

\[
G_i(t, x) = \int_{\mathbb{R}^3} \int_0^\infty \xi_i \Theta(\xi, I) f(\varphi(I)) dI \, d\xi,
\]

\[
Q(t, x) = \int_{\mathbb{R}^3} \int_0^\infty \Theta(\xi, I) Q(f) \varphi(I) dI \, d\xi,
\]

with:

\[
\Theta(\xi, I) = m \begin{pmatrix}
\xi^2 + 2\frac{I}{m} \\
(\xi^2 + 2\frac{I}{m}) \xi_{k_1} \\
(\xi^2 + 2\frac{I}{m}) \xi_{k_1} \xi_{k_2} \\
\vdots \\
(\xi^2 + 2\frac{I}{m}) \xi_{k_1} \cdots \xi_{k_m} \\
\vdots
\end{pmatrix},
\]

Note that minimal order of the moment in $F$–hierarchy is 0, while minimal order in $G$–hierarchy is 2.
Polyatomic rarefied gas with many moments

More explicit:

\[ \partial_t F + \partial_i F_i = 0, \]
\[ \partial_t F_{k_1} + \partial_i F_{i k_1} = 0, \]
\[ \partial_t F_{k_1 k_2} + \partial_i F_{i k_1 k_2} = P_{ij}, \]
\[ \partial_t F_{k_1 k_2 k_3} + \partial_i F_{i k_1 k_2 k_3} = P_{k_1 k_2 k_3}, \]
\[ \vdots \]
\[ \partial_t F_{k_1 \cdots k_N} + \partial_i F_{i k_1 \cdots k_N} = P_{k_1 \cdots k_N}, \]

For simplicity, we adopt the following notation:

\[ F_A = \begin{cases} F & \text{for } A = 0 \\ F_{k_1 k_2 \cdots k_A} & \text{for } 1 \leq A \leq N, \end{cases} \]
\[ G_{LLa} = \begin{cases} G_{LL} & \text{for } a = 0 \\ G_{LL k_1 k_2 \cdots k_a} & \text{for } 1 \leq a \leq M, \end{cases} \]

Then the system can be rewritten as simple form:

\[ \partial_t F_A + \partial_i F_{i A} = P_A, \quad \partial_t G_{LLA'} + \partial_i G_{i LLA'} = Q_{LLA'}. \]
The variational problem from which the distribution function \( f_{(N,M)} \) is obtained is connected to the functional:

\[
\mathcal{L}_{(N,M)}(f) = -k \int_{\mathbb{R}^3} \int_0^\infty f \log f \, dl \, d\xi + u'_A \left( F_A - m \int_{\mathbb{R}^3} \int_0^\infty \xi_A f l^\alpha \, dl \, d\xi \right) + \\
+ v'A' \left( G_{IIA'} - m \int_{\mathbb{R}^3} \int_0^\infty \left( \xi^2 + \frac{2l}{m} \right) \xi_A' f l^\alpha \, dl \, d\xi \right),
\]

where \( u'_A \) and \( v'_a \) are the Lagrange multipliers and

\[
\xi_A = \begin{cases} 
1 & \text{for } A = 0 \\
\xi_k \xi_k \cdots \xi_k & \text{for } 1 \leq A \leq N,
\end{cases} \quad \xi_A' = \begin{cases} 
1 & \text{for } A' = 0 \\
\xi_k \xi_k \cdots \xi_k & \text{for } 1 \leq A' \leq M.
\end{cases}
\]

The distribution function \( f_{(N,M)} \) which maximizes the functional \( \mathcal{L}_{(N,M)} \) is given by:

\[
f_{(N,M)} = \exp \left( -1 - \frac{m}{k} \chi_{(N,M)} \right), \quad \chi_{(N,M)} = u'_A \xi_A + \left( \xi^2 + \frac{2l}{m} \right) v'A' \xi_A'. \tag{17}
\]
Then, the system may be rewritten as follows:

\[
\begin{pmatrix}
J^0_{AB} & J^1_{AB'} \\
J^1_{A'B} & J^2_{A'B'}
\end{pmatrix}
\frac{\partial}{\partial t}
\begin{pmatrix}
u'_B \\
v'_B'
\end{pmatrix}
\begin{pmatrix}
J^0_{iAB} & J^1_{iAB'} \\
J^1_{iA'B} & J^2_{iA'B'}
\end{pmatrix}
\frac{\partial}{\partial i}
\begin{pmatrix}
u'_B \\
v'_B'
\end{pmatrix}
= \begin{pmatrix}
P_A \\
Q_{llA'}
\end{pmatrix},
\]

where

\[
J^0_{AB} = -\frac{m}{k} \int_{R^3} \int_{0}^{\infty} f_{\xi_A\xi_B} I^\alpha dld\xi,
\]

\[
J^1_{AB'} = -\frac{m}{k} \int_{R^3} \int_{0}^{\infty} f_{\xi_A\xi_B'} \left(\xi^2 + \frac{2I}{m}\right) I^\alpha dld\xi,
\]

\[
J^2_{A'B'} = -\frac{m}{k} \int_{R^3} \int_{0}^{\infty} f_{\xi_{A'}\xi_{B'}} \left(\xi^2 + \frac{2I}{m}\right)^2 I^\alpha dld\xi.
\]

The system (18) is symmetric hyperbolic according with the general theory of systems of balance laws with a convex entropy density (Boillat & Ruggeri Continuum Mech. Thermodyn. (1997)).
**Problem** Are $N$ and $M$ independent?

The following two theorems give the answer [Arima, Mentrelli and Ruggeri- Annals of Physics (2014)]:

**Theorem**

The differential system is Galilean invariant if and only if $M \leq N - 1$.

**Theorem**

If $M < N - 1$, all characteristic velocities are independent from the internal degrees of freedom $D$ and coincides with the one of $F$-hierarchy of monatomic gases with the truncation order $N$.

The relation between $N$ and $M$ for the physically meaningful system

The requirement that the system is Galilean invariant and the characteristic velocities are function of $D$ require

$$M = N - 1$$
Examples:

**N=1, M=0 Euler system:**

\[ \partial_t F + \partial_i F_i = 0, \]
\[ \partial_t F_{k_1} + \partial_i F_{i k_1} = 0, \]
\[ \partial_t G_{LL} + \partial_i G_{iLL} = 0 \]

**N=2, M=1 14-field system:**

\[ \partial_t F + \partial_i F_i = 0, \]
\[ \partial_t F_{k_1} + \partial_i F_{i k_1} = 0, \]
\[ \partial_t F_{k_1 k_2} + \partial_i F_{i k_1 k_2} = P_{ij} \]
\[ \partial_t G_{LL} + \partial_i G_{iLL} = 0 \]
\[ \partial_t G_{LLk_1} + \partial_i G_{iLLk_1} = Q_{LLk_1} \]
Singular limit of monatomic gas as particular case of polyatomic one

We introduce

\[ I_\alpha = F_{\text{LL}\alpha} - G_{\text{LL}\alpha} \quad (0 \leq \alpha \leq M - 1(= N - 2)). \]

where

\[ F_{\text{LL}\alpha} = F_{\text{LL}k_1 k_2 \cdots k_\alpha}, \]

Then the basic equations can be transformed to the equivalent one:

\[ \partial_t F_A + \partial_i F_{iA} = P_A \quad 0 \leq A \leq N \]
\[ \partial_t G_{\text{LL}\alpha} + \partial_i G_{i\text{LL}\alpha} = Q_{\text{LL}\alpha} \quad 0 \leq \alpha \leq M - 1 \]
\[ \partial_t G_{\text{LL}M} + \partial_i G_{i\text{LL}M} = Q_{\text{LL}M} \]
\[ \partial_t I_\alpha + \partial_i I_{i\alpha} = R_\alpha \quad 0 \leq \alpha \leq M - 1 \]

where \( I_{i\alpha} = F_{i\text{LL}\alpha} - G_{i\text{LL}\alpha} \) and \( R_\alpha = P_\alpha - Q_{\text{LL}\alpha} \).
We define

$$\Pi_\alpha = \lim_{D \to 3} I_\alpha,$$

and we can prove that

$$\lim_{D \to 3} R_\alpha = -\Psi_{\alpha\beta} \Pi_\beta.$$

where $\Psi_{\alpha\beta}$ are the coefficient depending on the collisional integral. In the case of BGK model,

$$\Psi_{\alpha\beta} = \frac{1}{T} \delta_{\alpha\beta}.$$

In the limit $D \to 3$, we can obtain the balance equations of $\Pi_\alpha$:

$$\partial_t \Pi_\alpha + J_{ia\alpha}^M J_{a\beta}^M J_{a\beta}^{-1} \partial_i \Pi_\beta = - \left( \Psi_{\alpha\beta} + \partial_i J_{ia\alpha}^M J_{a\beta}^M J_{a\beta}^{-1} \right) \Pi_\beta.$$

with

$$J_{a\alpha}^M = \int f_M \xi_a \xi_\alpha \, dc,$$

where $f_M$ is the Maxwellian distribution function.
When we impose the initial data compatible with rarefied monatomic gases:

\[ \Pi_\alpha(0, x) = 0, \]

we obtain

\[ \Pi_\alpha(t, x) = 0, \quad \forall t \]

Furthermore we obtain

\[ \lim_{D \to 3} G_{LLM} = \lim_{D \to 3} F_{LLM} \]
Therefore we proved in the limit $D \to 3$ the system of polyatomic gases formed by $\frac{1}{6}(N + 1)(N + 2)(2N + 3)$ equations have the same solution of the system of rarefied monatomic gases formed by $\frac{1}{6}(N + 1)(N^2 + 8N + 6)$ equations.

Example $N = 2$: The solutions of 14-field theory for polyatomic gases, when $D \to 3$, have the same solutions of 13-field theory for monatomic gases. [Arima, Taniguchi, Ruggeri & Sugiyama, Phys. Lett. A. 2013]:
Maximum characteristic velocity

It has been proved that, in the case that \( D \rightarrow 3 \), the solutions of the ET theory for rarefied polyatomic gases converge to those of corresponding rarefied monatomic gases. Therefore, when \( D \rightarrow 3 \), all the characteristic velocities converge to the one of the monatomic one as in figure and in particular the \( \lambda_{\text{max}} \):

\[
\lim_{D \to 3} \lambda_{\text{max}}
\]

Moreover we have the following property with respect to \( \lambda_{\text{max}} \).

**Theorem**

*When \( D \rightarrow \infty \), \( \lambda_{\text{max}} \) coincides with the one of the system of rarefied monatomic gases with truncation order \( N \).*
The dependence of the maximum characteristic velocity on $D$ and $N$:
Using the property of the principal subsystem [Boillat & Ruggeri (1997)], we can prove that the lower bound of monatomic gases remains also for polyatomic gases.

\[
\frac{\lambda_{\text{max}}^{(N)}}{c_s} \geq \frac{\lambda_{\text{max}}^{(N)}|_{D \to \infty}}{c_s} = \frac{\lambda_{\text{max}}^{(N)}|_{\text{mon}}}{c_s} \geq \sqrt{\frac{6}{5} \left( N - \frac{1}{2} \right)}
\]

Therefore, also for the case of polyatomic gases, \( \lambda_{\text{max}} \) is unbounded when \( N \to \infty \).

The 14 moments system for polyatomic gases

In the case of 14 moments and polyatomic gas $p = \frac{k}{m} \rho T$, $\varepsilon = \frac{D}{2} \frac{k}{m} T$, the system become:

\[
\dot{\rho} + \rho \frac{\partial v_k}{\partial x_k} = 0,
\]

\[
\rho \dot{\nu}_i + \frac{\partial p}{\partial x_i} + \frac{\partial \Pi}{\partial x_i} - \frac{\partial S_{\langle ij \rangle}}{\partial x_j} = 0,
\]

\[
\dot{T} + \frac{2}{D \frac{k_B}{m} \rho} (p + \Pi) \frac{\partial v_k}{\partial x_k} - \frac{2}{D \frac{k_B}{m} \rho} \frac{\partial v_i}{\partial x_i} S_{\langle ik \rangle} + \frac{2}{D \frac{k_B}{m} \rho} \frac{\partial q_k}{\partial x_k} = 0,
\]

\[
\dot{S}_{\langle ij \rangle} + S_{\langle ij \rangle} \frac{\partial v_k}{\partial x_k} - 2\Pi \frac{\partial v_i}{\partial x_j} + 2 \frac{\partial v_i}{\partial x_k} S_{\langle jk \rangle} - \frac{4}{D + 2} \frac{\partial q_i}{\partial x_j} - 2p \frac{\partial v_i}{\partial x_j} = -\frac{1}{\tau S} S_{\langle ij \rangle},
\]

\[
\dot{\Pi} + \frac{5D - 6}{3D} \Pi \frac{\partial v_k}{\partial x_k} - \frac{2(D - 3)}{3D} \frac{\partial v_i}{\partial x_k} S_{\langle ik \rangle} + \frac{4(D - 3)}{3D(D + 2)} \frac{\partial q_k}{\partial x_k} + \frac{2(D - 3)}{3D} \rho \frac{\partial v_k}{\partial x_k} = -\frac{1}{\tau \Pi} \Pi,
\]

\[
\dot{q}_i + \frac{D + 4}{D + 2} q_i \frac{\partial v_k}{\partial x_k} + \frac{2}{D + 2} q_k \frac{\partial v_k}{\partial x_i} + \frac{D + 4}{D + 2} q_k \frac{\partial v_i}{\partial x_k}
\]

\[
+ \frac{k_B}{m} T \frac{\partial \Pi}{\partial x_i} - \frac{k_B}{m} T \frac{\partial S_{\langle ik \rangle}}{\partial x_k} + \Pi \left[ -\frac{k_B}{m} T \frac{\partial \rho}{\partial x_i} + \frac{D + 2}{2} \frac{k_B}{m} \frac{\partial T}{\partial x_i} - \frac{1}{\rho} \frac{\partial \Pi}{\partial x_i} + \frac{1}{\rho} \frac{\partial S_{\langle pk \rangle}}{\partial x_k} \right]
\]

\[
- S_{\langle ik \rangle} \left[ -\frac{k_B}{m} T \frac{\partial \rho}{\partial x_k} + \frac{D + 2}{2} \frac{k_B}{m} \frac{\partial T}{\partial x_k} - \frac{1}{\rho} \frac{\partial \Pi}{\partial x_k} + \frac{1}{\rho} \frac{\partial S_{\langle pk \rangle}}{\partial x_k} \right] + \frac{D + 2}{2} \left( \frac{k_B}{m} \right)^2 \rho T \frac{\partial T}{\partial x_i} = -\frac{1}{\tau q} q_i.
\]
The 6 Moments case

The most simple case of dissipative polyatomic gas is the one in which we neglect heat conductivity and shear viscosity and we suppose that only the bulk viscosity is not negligible. In this case we have the most simple model after Euler in the presence of dissipation due to the role of the dynamical pressure (Arima, Taniguchi, Ruggeri and Sugiyama Physics Lett. A -2012). The field equations are

\[
\begin{align*}
\dot{\rho} + \rho \text{div} \mathbf{v} &= 0, \\
\rho \dot{v}_i + \frac{\partial}{\partial x_i} (p + \Pi) &= 0, \\
\rho \dot{\varepsilon} + (p + \Pi) \text{div} \mathbf{v} &= 0, \\
\tau \dot{\Pi} + \left( \nu + \tau \frac{5D - 6}{3D} \Pi \right) \text{div} \mathbf{v} &= -\Pi,
\end{align*}
\]

(20)

where the bulk viscosity \( \nu \propto D - 3 \). When \( D \rightarrow 3 \) (monatomic gas) the previous system have the same solution of the Euler fluid provided \( \Pi(x, 0) = 0 \).
The parabolic case $\tau \to 0$:

\[ \rho \dot{\mathbf{v}} + \rho \text{div} \mathbf{v} = 0, \]
\[ \rho \dot{v}_i + \frac{\partial}{\partial x_i} (p + \Pi) = 0, \]
\[ \rho \dot{\varepsilon} + (p + \Pi) \text{div} \mathbf{v} = 0, \]
\[ \nu \text{div} \mathbf{v} = -\Pi, \]

was studied in same papers, e.g.:
Applications of the $\text{ET}_{14}$ theory to specific problems have been made. In this section we review briefly some of them:

- Dispersion relation for sound in rarefied diatomic gases
- Light scattering
- Shock structure
- Riemann problem
The dispersion relation for sound in rarefied diatomic gases; hydrogen, deuterium and hydrogen deuteride gases basing on the ET_{14} theory was recently studied in detail in Arima, T., Taniguchi, S., Ruggeri, T. and Sugiyama, M. Cont. Mech. Thermodyn., (2013). The relation was compared with those obtained in experiments and by the NSF theory. As is expected, the applicable frequency-range of the ET theory was shown to be much wider than that of the NSF theory. The values of the bulk viscosity and the relaxation times involved in nonequilibrium processes were evaluated. It was found that the relaxation time related to the dynamic pressure has a possibility to become much larger than the other relaxation times related to the shear stress and the heat flux. The isotope effects on sound propagation were also clarified.
4. An application of the theory to ultrasonic sounds
A test of the ET theory for rarefied diatomic gases: Dispersion relation for sound

Normal hydrogen (n-H$_2$) gases:

Figure: Dependence of the dimensionless phase velocity $v_{ph}/c_0$ (left) and the attenuation factor $c_0(\tau_S)_0\alpha$ (right) on the dimensionless frequency $\Omega(=\tau_S\omega)$ for n-H$_2$ gases. The squares and triangles in the left figure are the experimental data at $T_0 = 273.5$ and 296.8K, respectively, and the circles in the right figure are those at $T_0 = 293$K. The solid and dashed lines are predictions at 293K by the ET and NSF theories, respectively.
The shock wave structure in a rarefied polyatomic gas is, under some conditions, quite different from the shock wave structure in a rarefied monatomic gas due to the presence of the microscopic internal modes in a polyatomic molecule such as the rotational and vibrational modes. For examples: 1) The shock wave thickness in a rarefied monatomic gas is of the order of the mean free path. On the other hand, owing to the slow relaxation process involving the internal modes, the thickness of a shock wave in a rarefied polyatomic gas is several orders larger than the mean free path. 2) As the Mach number increases from unity, the profile of the shock wave structure in a polyatomic rarefied gas changes from the nearly symmetric profile (Type A) to the asymmetric profile (Type B), and then changes further to the profile composed of thin and thick layers (Type C)
Schematic profiles of the mass density are shown in Figure 3. Such change of the shock wave profile with the Mach number cannot be observed in a monatomic gas.

**Figure:** Schematic representation of three types of the shock wave structure in a rarefied polyatomic gas, where $\rho$ and $x$ are the mass density and the position, respectively. As the Mach number increases from unity, the profile of the shock wave structure changes from Type A to Type B, and then to Type C that consists of the thin layer $\Delta$ and the thick layer $\Psi$.
In order to explain the shock wave structure in a rarefied polyatomic gas, there have been two well-known approaches. One was proposed by Bethe and Teller and the other is proposed by Gilbarg and Paolucci. Although the Bethe-Teller theory can describe qualitatively the shock wave structure of Type C, its theoretical basis is not clear enough. The Gilbarg-Paolucci theory, on the other hand, cannot explain asymmetric shock wave structure (Type B) nor thin layer (Type C). Recently it was shown that the ET$_{14}$ theory can describe the shock wave structure of all Types A to C in a rarefied polyatomic gas. In other words the ET$_{14}$ theory has overcome the difficulties encountered in the previous two approaches. This new approach indicates clearly the usefulness of the ET theory for the analysis of shock wave phenomena.
Shock structure in ET$_{14}$

S. Taniguchi, T. Arima, T. Ruggeri, and M. Sugiyama
Riemann Problem in ET$_6$

S. Taniguchi (in preparation (2014)).


