Shadow waves

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Cauchy Problem

Consider a system given in a general form (hyperbolic or weakly hyperbolic):

$$\partial_t f(U) + \partial_x g(U) = 0,\ x \in \mathbb{R},\ t > 0, \tag{1}$$

together with Riemann initial data

$$U|_{t=0} = \begin{cases} U_0, & x < 0, \\ U_1, & x > 0 \end{cases}$$

Examples: pressureless gas dynamics (with or without the energy conservation law), ionic gas model (Keyfitz-Kranzer system), 2×2 simplified 1D magnetohydrodynamics model, Chaplygin gas and its generalized version.

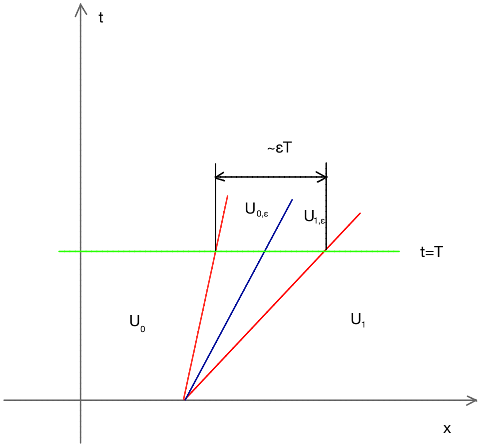
Simple SDW

(From [M.N., Shadow Waves: Entropies and Interactions for Delta and Singular Shocks, ARMA, 2010])

In general, we are seeking a solution in a form (simple shadow wave)

$$U_\varepsilon(x,t) = \begin{cases} U_0, & x < (c-\varepsilon)t \\ U_{1,\varepsilon}, & (c-\varepsilon)t < x < ct \\ U_{2,\varepsilon}, & ct < x < (c+\varepsilon)t \\ U_1, & x > (c+\varepsilon)t. \end{cases}$$

It is adapted to "catch" delta and singular shocks: $\|U_{1,\varepsilon}\|, \|U_{2,\varepsilon}\| \sim 1/\varepsilon$.


t
~εT
U0,ε
U1,ε
t=T
U0
U1
x

Basic Lemma

With the above assumptions we have the following useful relations.

$$\langle \partial_t f(U_\varepsilon), \phi \rangle \approx \int_0^\infty \Big(\lim_{\varepsilon\to 0} \left(\varepsilon f(U_{1,\varepsilon}(t)) + \varepsilon f(U_{2,\varepsilon}(t))\right)$$

$$-c\big(f(U_1) - f(U_0)\big)\Big)\, \phi(c(t), t)\, dt$$

$$+ \int_0^\infty \lim_{\varepsilon\to 0} c\Big(\varepsilon f(U_{1,\varepsilon}(t)) + \varepsilon f(U_{2,\varepsilon}(t))\Big) \partial_x \phi(c(t), t) dt$$

$$\langle \partial_x g(U_\varepsilon), \phi \rangle \approx \int_0^\infty \left(g(U_1) - g(U_0)\right) \phi(c(t), t)\, dt$$

$$- \int_0^\infty \lim_{\varepsilon\to 0} \left(\varepsilon g(U_{1,\varepsilon}(t)) + \varepsilon g(U_{2,\varepsilon}(t))\right) \partial_x \phi(c(t), t)\, dt.$$

Entropy Conditions

Let $\eta(U)$ be a (semi-) convex entropy function for (1), with an entropy-flux function $q(U)$. I.e. each smooth solution U satisfies $\partial_t \eta(U) + \partial_x q(U) = 0$ and the matrix $D^2\eta(U)$ is positive (semi-) definite in Ω. A weak solution U_ε to system (1) with initial data $U|_{t=0} = U_\varepsilon^0$ is admissible if for every $T > 0$ we have

$$\underline{\lim}_{\varepsilon\to 0} \int_{\mathbb{R}} \int_0^T \eta(U_\varepsilon)\partial_t \phi + q(U_\varepsilon)\partial_x \phi \, dt \, dx \\ + \int_{\mathbb{R}} \eta(U_\varepsilon^0(x,0))\phi(x,0)\, dx \geq 0,$$

for all non-negative test functions $\phi \in C_0^\infty(\mathbb{R} \times (-\infty, T))$.

Specially, an SDW solution is entropic if

$$
\begin{aligned}
E_1 = & \overline{\lim}_{\varepsilon\to 0} - c(\eta(U_1) - \eta(U_0)) + q(U_1) - q(U_0) \\
& + \varepsilon\eta(U_{1,\varepsilon}) + \varepsilon\eta(U_{2,\varepsilon}) \leq 0 \\
& \text{end} \\
E_2 = & \lim_{\varepsilon\to 0} -c(\varepsilon\eta(U_{1,\varepsilon}) + \varepsilon\eta(U_{2,\varepsilon})) + \varepsilon q(U_{1,\varepsilon}) + \varepsilon q(U_{2,\varepsilon}) = 0.
\end{aligned}
\tag{2}
$$

Uniqueness

An SDW solution is called weakly unique if its distributional image is unique. More precisely, a speed c of the wave has to be unique as well as the limit

$$\lim_{\varepsilon \to 0} \varepsilon U_{1,\varepsilon} + \varepsilon U_{2,\varepsilon}.$$

Let $i \in \{1, \dots, n\}$. If a limit $\lim_{\varepsilon \to 0} \varepsilon U^i_{1,\varepsilon} + \varepsilon U^i_{2,\varepsilon}$ is unique, then we say that the i-th component is unique.
We say that a solution to a system (1) is weakly unique is the solution consists of a unique combination of standard admissible elementary waves (shocks, rarefactions and contact discontinuities) and admissible SDWs.

Pressureless Gas Dynamics

Riemann problem for pressureless gas dynamics model

$$\partial_t \rho + \partial_x(\rho u) = 0$$
$$\partial_t(\rho u) + \partial_x(\rho u^2) = 0,$$

has a SDW solution

$$(\rho_\varepsilon, u_\varepsilon)(x,t) = \begin{cases} (\rho_0, u_0), & x < (c-\varepsilon)t \\ (\xi\varepsilon^{-1}, u_s), & (c-\varepsilon)t < x < (c+\varepsilon)t \\ (\rho_1, u_1), & x > (c+\varepsilon)t \end{cases}$$

where $c = u_s$, $u_s \in [u_1, u_0]$ is a solution to $u_s^2[\rho] - 2u_s[\rho u] + [\rho u^2] = 0$, and $\xi = ([\rho u] - c[\rho])/2$.
Note. Uniqueness of u_s follows from the fact that density should be nonnegative.

An enrtopy function η for the above system has to satisfy

$$D^2\eta DF = DF^T D^2\eta,\ DF = \begin{bmatrix} 0 & 1 \\ -\frac{m^2}{\rho^2} & \frac{m}{\rho} \end{bmatrix},\ m = \rho m,\ \text{i.e.}$$

$$\partial_{\rho\rho}\eta + 2\frac{m}{\rho}\partial_{\rho m} = -\frac{m^2}{\rho^2}\partial_{mm}.$$

The general solution is given by

$$\eta = \rho F\left(\frac{m}{\rho}\right) + G\left(\frac{m}{\rho}\right).$$

An apropriate entroy flux function satisfies

$$DQ = D\eta DF,\ Q = mF\left(\frac{m}{\rho}\right) + \frac{m}{\rho}G\left(\frac{m}{\rho}\right) - \left(\int G\right)\left(\frac{m}{\rho}\right).$$

Let us find when an entropy function η is positive semidefinite:

$$\det(D^2\eta) = -\frac{1}{\rho^4} G'\left(\frac{m}{\rho}\right) \geq 0$$

implies $G = 0$ (note that G equals a constant produces the same additional conservation law).

$$\eta_{\rho\rho} = \frac{m^2}{\rho^3} F''\left(\frac{m}{q}\right) \geq 0,\ \eta_{mm} = \frac{1}{\rho} F''\left(\frac{m}{q}\right) \geq 0$$

implies $F'' > 0$, i.e. F has to be a convex function.

Using the entropy conditions for SDWs (2) one has

$$E_2 = -2c\xi F(u_s) = 2\xi u_s F(u_s) = 0$$

because $c = u_s$ always.
Also, $E_1 \leq 0$ is equivalent to

$$\rho_0(u_0 - c)F(u_0) + \rho_1(c - u_2)F(u_1) - \kappa_1 F(c) \geq 0,$$

where $\kappa_1 = c[\rho] - [\rho u]$ (so called R-H deficit). Observing that $\rho_0(u_0 - c) + \rho_1(c - u_2) = \kappa_1$ and dividing the above inequality by κ_1 one gets

$$\begin{aligned} &\frac{\rho_0(u_0 - c)}{\kappa_1} F(u_0) + \frac{\rho_1(c - u_2)}{\kappa_1} F(u_1) - F(c) \\ \geq &F\left(\frac{\kappa_2}{\kappa_1}\right) - F(c) = 0. \end{aligned}$$

For the above conclusion is correct if and only if $u_0 \geq c \geq u_1$, i.e. when the SDW id overcompressive.

Simplified 2×2 MHD model

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = 0$$
$$\partial_t v + \partial_x \left((u-1)v \right) = 0.$$

An SDW solution to the Riemann problem for it is given by

$$(u_\varepsilon, v_\varepsilon)(x,t) = \begin{cases} (u_0, v_0), & x < ct - \varepsilon t \\ (u_{s,0}, \xi_0 \varepsilon^{-1}), & ct - \varepsilon t < x < ct \\ (u_{s,1}, \xi_1 \varepsilon^{-1}), & ct < x < ct + \varepsilon t \\ (u_1, v_1), & x > ct + \varepsilon t, \end{cases}$$

where $\xi_0 + \xi_1 = \kappa_2 = c[v] - [(u-1)v]$, $c = \frac{u_0 + u_1}{2}$. Either $u_{s,0} = u_{s,1} = 1 + c$ without further restrictions on ξ_0 and ξ_1, or:

$$\xi_0 = \frac{\kappa_2 (u_{s,1} - c)}{u_{s,1} - u_{s,0}}, \ \xi_1 = \frac{\kappa_2 (c - u_{s,0})}{u_{s,1} - u_{s,0}}, \ u_{s,0} \neq u_{s,1}.$$

Entropy pairs are given by

$$\eta = e^u g(v^{-u}) + f(u),$$
$$Q = e^u(u-1)\left(g(ve^{-u}) + uf(u) - \left(\int f\right)(u)\right).$$

An entropy function is (semi-)convex if and only if f is (semi-)convex and g is also (semi-)convex satisfying $g'(x) \leq g(x)x^{-1}$. The last inequality implies that g is either sublinear or linear function. So let us assume that $\eta = \alpha v + f(u) + e^u g_1(ve^{-u})$ and check entropy inequality for each of these three addends.
In this case, E_2 is true only if $u_{s,0} = u_{s,1} = c + 1$.
E_1 is true in all these three cases.

Keyfitz–Kranzer System

For some Riemann data, the system ([Keyfitz-Kranzer, 1995])

$$\partial_t u + \partial_x(u^2 - v) = 0$$
$$\partial_t v + \partial_x(u^3/3 - u) = 0$$

has a solution

$$(u_\varepsilon, v_\varepsilon)(x,t) = \begin{cases} (u_0, v_0), & x < ct - \varepsilon t \\ (\bar{u}_{1,\varepsilon} + z_{1,\varepsilon}, v_{1,\varepsilon}), & ct - \varepsilon t < x < ct \\ (\bar{u}_{2,\varepsilon} + z_{2,\varepsilon}, v_{2,\varepsilon}), & ct < x < ct + \varepsilon t \\ (u_1, v_1), & x > ct + \varepsilon t, \end{cases}$$

where $c = \frac{[u^2 - v]}{[u]}$, $\varepsilon(v_{1,\varepsilon} + v_{2,\varepsilon}) = c[v] - \left[\frac{u^3}{3} - u\right]$, $\bar{u}_{i,\varepsilon}^2 = v_{i,\varepsilon}$, $z_{i,\varepsilon} = c$, $i = 1, 2$ and $\bar{u}_{1,\varepsilon} = -\bar{u}_{2,\varepsilon}$.

Convex entropy functions and their corresponding fluxes are given by

$$\eta(u,v) = ce^{\gamma(v-u^2/2-u)}e^{-\gamma(v-u^2/2+u)/(1+4\gamma)},$$
$$q(u,v) = (u+1+\frac{1}{2\gamma})\eta(u,v),\ c>0,\ \gamma<-1/4.$$

Entropy conditions for SDWs are satisfied (for each γ) if and only if

$$u_0 - 1 \geq c \geq u_1 + 1,$$

i.e. the wave is overcompressive.
The SDW solution is weakly unique.
An interaction involving SDW can be always solved using only constant SDWs.

Chaplyigin Gas

From [Y. Brenier, Solutions with concentration to the Riemann problem for the one-dimensional Chaplygin gas equations, J Math Fluid Mech, 2005] and [M.N., Higher order shadow waves and delta shock blow up in the Chaplygin gas, JDE, 2014]

$$\partial_t \rho + \partial_x q = 0$$
$$\partial_t q + \partial_x \Big(\frac{q^2 - 1}{\rho} \Big) = 0,$$

with $q = \rho u$. It posses an infinite number of convex entropy – entropy flux pairs,

$$\eta = \frac{\rho}{2} \bigg(F\Big(\frac{q-1}{\rho}\Big) + G\Big(\frac{q+1}{\rho}\Big) \bigg)$$
$$Q = \frac{1}{2} \bigg((q+1) F\Big(\frac{q-1}{\rho}\Big) + (q-1) G\Big(\frac{q+1}{\rho}\Big) \bigg).$$

The entropy function η is convex if and only if both F and G are convex.

The main additional conservation law is the energy conservation

$$\partial_t\Big(\frac{q^2+1}{\rho}\Big)+\partial_x\Big(\frac{q}{\rho}\frac{q^2-1}{\rho}\Big)=0.$$

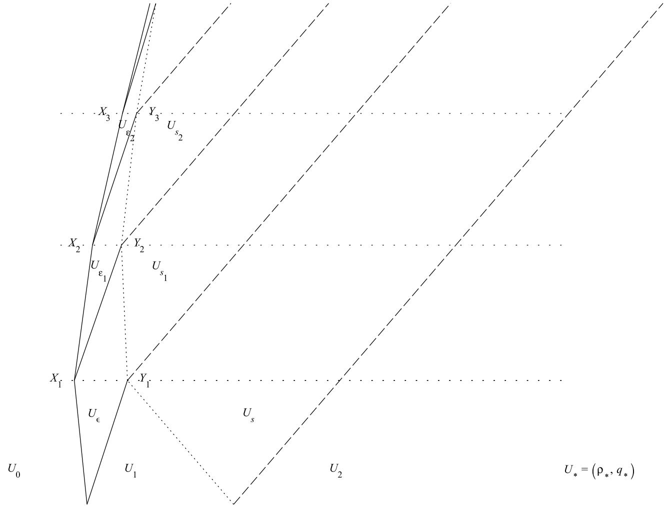
The following assertions are equivalent:

An SDW satisfies the entropy inequalities for energy.

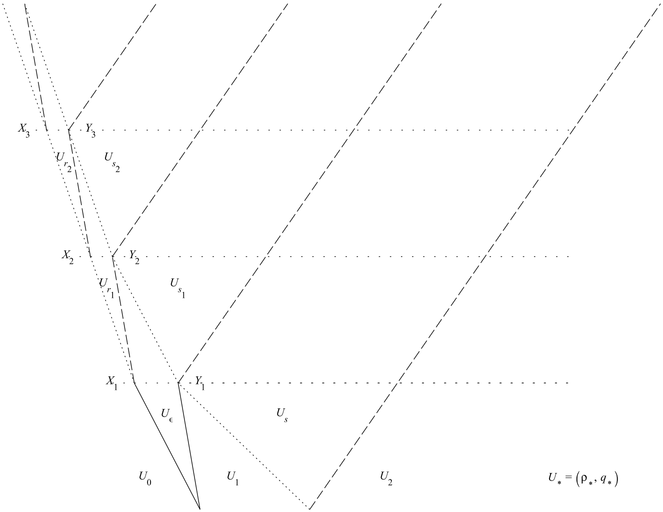
An SDW satisfies the entropy inequalities for every convex F and G.

An SDW is overcompressive.

Blow up scenario



Marginal case



Generalized Chaplygin Gas

Let $0 < a < 1$.

$$\partial_t \rho + \partial_x q = 0$$
$$\partial_t q + \partial_x \Big(\frac{q^2}{\rho} - \frac{1}{\rho^\alpha} \Big) = 0.$$

The energy entropy pair is now

$$\eta = \frac{1}{2} \frac{q^2}{\rho} - \frac{1}{1+\alpha} \rho^{-\alpha}, \; Q = \frac{1}{2} \frac{q^3}{\rho^2} - \frac{\alpha}{1+\alpha} q \rho^{-(1+\alpha)}.$$

Both the energy entropy and overcompressibility are not enough to fill the gap in classical Riemann solutions until now.

Using the standard procedure one can find that entropy function satisfies tho following PDE

$$\partial_{\rho\rho}\eta + \frac{2q}{\rho}\partial_{\rho q} + \Big(\frac{q^2}{\rho^2} - \frac{\alpha}{\rho^{1+\alpha}}\Big)\partial_{qq}\eta = 0,$$

or, after a change of variables,

$$v = \frac{q}{\rho} + \frac{2\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}} \text{ and } w = \frac{q}{\rho} - \frac{2\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}},$$

$$(v-w)\partial_{vw} = \frac{3+\alpha}{2(1+\alpha)}(\partial_v\eta - \partial_w\eta).$$

Separating the variables we have founded the following families of convex entropy pairs ($\lambda \geq 0$):

$$
\begin{aligned}
\eta_1(\rho, q) :=& e^{\frac{2q}{\rho}\lambda} \rho^{\frac{1}{2}} K_{\frac{1}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big),\\
Q_1(\rho, q) :=& \frac{1}{2\lambda}\rho^{\frac{1}{2}} e^{\frac{2q}{\rho}\lambda}\Bigg((2\lambda q-\rho)K_{\frac{1}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big)\\
&+2\lambda\sqrt{\alpha}\rho^{\frac{1-\alpha}{2}} K_{\frac{2+\alpha}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big)\Bigg)\\
\eta_2(\rho, q) :=& e^{-\frac{2q}{\rho}\lambda} \rho^{\frac{1}{2}} K_{\frac{1}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big),\\
Q_2(\rho, q) :=& \frac{1}{2\lambda}\rho^{\frac{1}{2}} e^{\frac{2q}{\rho}\lambda}\Bigg((2\lambda q+\rho)K_{\frac{1}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big)\\
&-2\lambda\sqrt{\alpha}\rho^{\frac{1-\alpha}{2}} K_{\frac{2+\alpha}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big)\Bigg).
\end{aligned}
$$

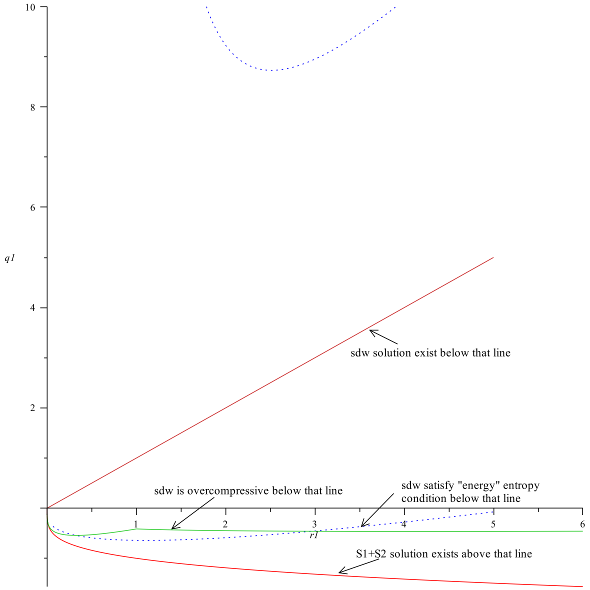


Figure : Energy inequality and overcompressibility condition

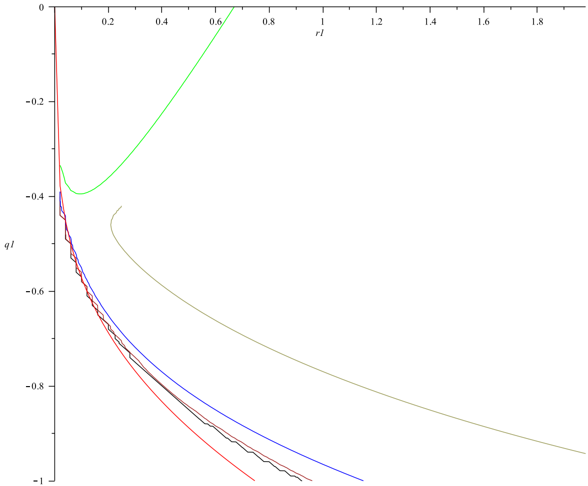


Figure : Entropy condition of the first family

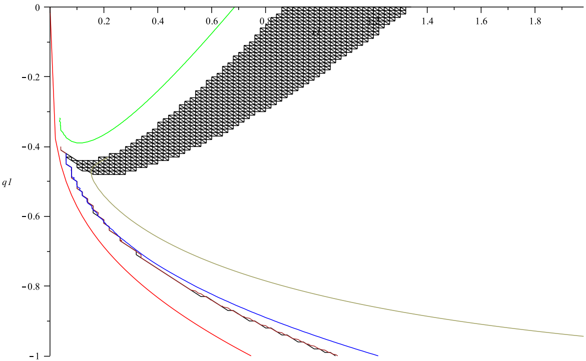


Figure : Entropy condition of the second family