The incompressible Navier-Stokes limit of the Boltzmann equation for mixtures

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Framework

We wish to obtain a coherent formal passage from rarefied gases (over 60 Km of altitude) to dense gases (under 60 Km of altitude), in situations as realistic as possible.

- Mixtures (O_2 and N_2)

- Monoatomic and diatomic (O_2 and O)

- Basic chemistry ($O_2 \leftrightarrow O + O$)

- Diffusion terms (viscosity, thermal conductivity, etc.)

- Boundary conditions

Asymptotics

- Hilbert expansion (Boltzmann $\Rightarrow$ compressible Euler)

- Chapman-Enskog asymptotics (Boltzmann $\Rightarrow$ compressible Navier-Stokes)

- Incompressible limits (Boltzmann $\Rightarrow$ incompressible Euler and Navier-Stokes), Bardos, Golse, Levermore, 90s

Boltzmann operator for monoatomic rarefied gases with power law potential

$$Q(f,f)(v) = \int_{\mathbb{R}^3}\int_{S^2}\left[f\left(\frac{v+w}{2}+\frac{|v-w|}{2}\,\sigma\right)f\left(\frac{v+w}{2}-\frac{|v-w|}{2}\,\sigma\right)\right.$$
$$\left. - f(v)\,f(w)\right]|v-w|^{\gamma}\,b\left(\left(\widehat{\frac{v-w}{|v-w|},\sigma}\right)\right)\,d\sigma\,dw,$$

where $\gamma = \frac{s-5}{s-1}$ if the intermolecular forces scales like $F \sim r^{-s}$.

Conserved quantities (mass, momentum, energy):

$$\int_{\mathbb{R}^3} Q(f,f)(v)\begin{pmatrix}1\\ v\\ |v|^2/2\end{pmatrix}dv = 0.$$

Second part of Boltzmann's H-theorem:

$$Q(f,f)(v) = 0 \qquad \Rightarrow \qquad f(v) = \frac{\rho}{(2\pi\,T)^{3/2}}\,\exp\left(-\frac{|v-u|^2}{2\,T}\right).$$

Boltzmann equation for one gas

Unknown: $f := f(t,x,v) \geq 0$ density of molecules which at time t and point x move with velocity v.

Boltzmann equation:

$$\frac{\partial f}{\partial t}(t,x,v) + v \cdot \nabla_x f(t,x,v) = Q(f(t,x,\cdot), f(t,x,\cdot))(v).$$

Rescaled Boltzmann equation ($\varepsilon := \frac{\lambda}{L}$ is the Knudsen number):

$$\frac{\partial f^\varepsilon}{\partial t}(t,x,v) + v \cdot \nabla_x f^\varepsilon(t,x,v) = \frac{1}{\varepsilon}\, Q(f^\varepsilon(t,x,\cdot), f^\varepsilon(t,x,\cdot))(v).$$

Related non-closed conservations laws:

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f(t,x,v) \begin{pmatrix} 1 \\ v_i \\ |v|^2/2 \end{pmatrix} dv + \nabla_x \cdot \int_{\mathbb{R}^3} f^\varepsilon(t,x,v) \begin{pmatrix} 1 \\ v_i \\ |v|^2/2 \end{pmatrix} v\, dv = 0.$$

Hilbert expansion

$$\frac{\partial f^\varepsilon}{\partial t}(t,x,v) + v\cdot\nabla_x f^\varepsilon(t,x,v) = \frac{1}{\varepsilon}\,Q(f^\varepsilon(t,x,\cdot),f^\varepsilon(t,x,\cdot))(v).$$

so that

$$f^\varepsilon(t,x,v) = \frac{\rho^\varepsilon(t,x)}{(2\pi\,T^\varepsilon(t,x))^{3/2}}\,\exp\left(-\frac{|v-u^\varepsilon(t,x)|^2}{2\,T^\varepsilon(t,x)}\right) + O(\varepsilon).$$

and one gets the Euler equations (up to order 1) of compressible perfect monoatomic gases.

$$\frac{\partial\rho^\varepsilon}{\partial t} + \nabla_x\cdot(\rho^\varepsilon\,u^\varepsilon) = O(\varepsilon),$$

$$\frac{\partial(\rho^\varepsilon\,u^\varepsilon)}{\partial t} + \nabla_x\cdot(\rho^\varepsilon\,u^\varepsilon\otimes u^\varepsilon + \rho^\varepsilon\,T^\varepsilon) = O(\varepsilon),$$

$$\frac{\partial}{\partial t}\left(\rho^\varepsilon\,\frac{|u^\varepsilon|^2}{2} + \frac{3}{2}\,\rho^\varepsilon\,T^\varepsilon\right) + \nabla_x\cdot\left(\rho^\varepsilon\,\frac{|u^\varepsilon|^2}{2}\,u^\varepsilon + \frac{5}{2}\,\rho^\varepsilon\,T^\varepsilon\,u^\varepsilon\right) = O(\varepsilon).$$

Chapman-Enskog expansion

Navier-Stokes-(Fourier) equations (up to order 2) of compressible perfect monoatomic gases.

$$\frac{\partial \rho^\varepsilon}{\partial t} + \nabla_x \cdot (\rho^\varepsilon\, u^\varepsilon) = O(\varepsilon^2),$$

$$\frac{\partial (\rho^\varepsilon\, u^\varepsilon)}{\partial t} + \nabla_x \cdot (\rho^\varepsilon\, u^\varepsilon \otimes u^\varepsilon + \rho^\varepsilon\, T^\varepsilon) = \varepsilon\, \nabla_x \cdot \Pi^\varepsilon + O(\varepsilon^2),$$

$$\Pi^\varepsilon = \left(\kappa(T^\varepsilon) - \frac{2}{3}\eta(T^\varepsilon)\right) (\nabla_x \cdot u^\varepsilon)\, \mathbf{I} + \eta(T^\varepsilon)\, \left(\nabla_x u^\varepsilon + (\nabla_x u^\varepsilon)^T\right),$$

$$\frac{\partial}{\partial t}\left[\rho^\varepsilon\, \frac{|u^\varepsilon|^2}{2} + \frac{3}{2}\,\rho^\varepsilon\, T^\varepsilon\right] + \nabla_x \cdot \left[\rho^\varepsilon\, \frac{|u^\varepsilon|^2}{2}\, u^\varepsilon + \frac{5}{2}\,\rho^\varepsilon\, T^\varepsilon\, u^\varepsilon\right] = \varepsilon\, \nabla_x \cdot (\Pi^\varepsilon \cdot u^\varepsilon)$$

$$+\varepsilon\, \nabla_x \cdot q^\varepsilon + O(\varepsilon^2), \qquad q^\varepsilon = \mu(T^\varepsilon)\, \nabla_x T^\varepsilon\,.$$

Low Mach number scaling

Rescaled Navier-Stokes-(Fourier) equations of compressible perfect monoatomic gases.

$$\frac{\partial \rho^\delta}{\partial t} + \nabla_x \cdot (\rho^\delta \, u^\delta) = 0,$$

$$\frac{\partial (\rho^\delta \, u^\delta)}{\partial t} + \nabla_x \cdot (\rho^\delta \, u^\delta \otimes u^\delta + \delta^{-2} \rho^\delta \, T^\delta) = \nabla_x \cdot \Pi^\delta,$$

$$\Pi^\delta = \left(\kappa(T^\delta) - \frac{2}{3}\eta(T^\delta)\right) (\nabla_x \cdot u^\delta) \, \mathbf{I} + \eta(T^\delta) \, \left(\nabla_x u^\delta + (\nabla_x u^\delta)^T\right),$$

$$\frac{\partial}{\partial t} \left[\rho^\delta \, \frac{|u^\delta|^2}{2} + \frac{3}{2} \, \rho^\delta \, T^\delta\right] + \nabla_x \cdot \left[\rho^\delta \, \frac{|u^\delta|^2}{2} \, u^\delta + \frac{5}{2} \, \rho^\delta \, T^\delta \, u^\delta\right] = \delta^2 \, \nabla_x \cdot (\Pi^\delta \cdot u^\delta)$$

$$+ \nabla_x \cdot q^\delta, \qquad q^\delta = \mu(T^\delta) \, \nabla_x T^\delta \, .$$

Low Mach number scaling

$$\rho^\delta(t,x) = \rho_0 + \delta\,\rho_1(t,x) + O(\delta^2), \qquad T^\delta(t,x) = T_0 + \delta\,T_1(t,x) + O(\delta^2),$$

$$u^\delta(t,x) = u_0(t,x) + O(\delta).$$

Incompressible limit:

$$\nabla_x \cdot u_0 = 0\,,$$

Boussinesq identity:

$$\nabla_x \left(\frac{T_0}{\rho_0}\,\rho_1 + T_1 \right) = 0\,,$$

Low Mach number scaling

Convection-diffusion equation for the momentum:

$$\partial_t u_0 + u_0 \cdot \nabla_x u_0 + \nabla_x p = d_1 \ \Delta_x u_0,$$

Convection-diffusion equation for the temperature:

$$\partial_t T_1 + u_0 \cdot \nabla_x T_1 = d_2 \, \Delta_x T_1.$$

Diffusion coefficient: $d_1 = \frac{\eta(T_0)}{\rho_0}$, Thermal conduction coefficient: $d_2 = \frac{2}{5} \, \frac{\mu(T_0)}{\rho_0}$.

Bardos-Golse-Levermore

Scaling:

$$\varepsilon\,\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon = \frac{1}{\varepsilon}\,Q(f^\varepsilon, f^\varepsilon),$$

Solutions are searched under the form

$$f^\varepsilon = M\,(1+\varepsilon\,g^\varepsilon), \qquad M(v) = \left(\frac{1}{2\pi}\right)^{3/2} e^{-\frac{1}{2}|v|^2}.$$

Then

$$g^\varepsilon(v) = \rho + v\cdot u + \left(\frac{1}{2}|v|^2 - \frac{3}{2}\right) T + O(\varepsilon)\,,$$

Direct asymptotics from Boltzmann to incompressible NS

Incompressible limit:

$$\nabla_x \cdot u = 0\,,$$

Boussinesq identity:

$$\nabla_x\,(\rho + T) = 0\,,$$

Convection-diffusion equation for the momentum:

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = d_1\ \Delta_x u,$$

Convection-diffusion equation for the temperature:

$$\partial_t T + u \cdot \nabla_x T = d_2\,\Delta_x T.$$

Diffusion coefficient: $d_1 > 0$ and thermal conduction coefficient: $d_2 > 0$ are given by a linear problem involving the linearized Boltzmann operator (around M).

Boltzmann equation for a mixture of gases

Unknowns: $f^s := f^s(t,x,v) \geq 0$ density of molecules of species s which at time t and point x move with velocity v.

Boltzmann equation:

$$\frac{\partial f^s}{\partial t}(t,x,v) + v \cdot \nabla_x f^s(t,x,v) = \sum_{r=1}^{N} Q^{sr}(f^s(t,x,\cdot), f^r(t,x,\cdot))(v),$$

$$Q^{sr}(f^s,f^r)(v) = \iint \Big(f^s(v')\, f^r(w') - f^s(v)\, f^r(w) \Big)$$

$$\times\, |v-w|^{\gamma^{sr}}\, b^{sr}\left(\left(\widehat{\frac{v-w}{|v-w|}, \sigma} \right) \right)\, dw\, d\sigma,$$

$$v' = \frac{m^s}{m^s+m^r}\, v + \frac{m^r}{m^s+m^r}\, w + \frac{m^r}{m^s+m^r}\, |v-w|\, \sigma\,,$$

$$w' = \frac{m^s}{m^s+m^r}\, v + \frac{m^r}{m^s+m^r}\, w - \frac{m^s}{m^s+m^r}\, |v-w|\, \sigma\,.$$

Conserved quantities

Mass of each species, momentum, energy:

$$\int_{\mathbb{R}^3} Q^{sr}(f^s, f^r)(v)\, dv = 0.$$

$$m^s \int_{\mathbb{R}^3} Q^{sr}(f^s, f^r)(v)\, v\, dv + m^r \int_{\mathbb{R}^3} Q^{rs}(f^r, f^s)(v)\, v\, dv = 0,$$

$$m^s \int_{\mathbb{R}^3} Q^{sr}(f^s, f^r)(v)\, \frac{|v|^2}{2}\, dv + m^r \int_{\mathbb{R}^3} Q^{rs}(f^r, f^s)(v)\, \frac{|v|^2}{2}\, dv = 0.$$

Second part of Boltzmann's H-theorem:

$$\forall s, r, \quad Q^{sr}(f^s, f^r)(v) = 0$$

$$\iff \qquad m^r\, f^r(v) = \frac{\rho^r}{(2\pi\, T/m^r)^{3/2}} \exp\left(-\frac{m_r\, |v-u|^2}{2\, T}\right).$$

Hilbert expansion

Rescaled Boltzmann equation ($\varepsilon := \frac{\lambda}{L}$ is the Knudsen number):

$$\frac{\partial f^s_\varepsilon}{\partial t}(t,x,v) + v \cdot \nabla_x f^s_\varepsilon(t,x,v) = \frac{1}{\varepsilon} \sum_{r=1}^{N} Q^{sr}(f^s_\varepsilon(t,x,\cdot), f^r_\varepsilon(t,x,\cdot))(v).$$

Related non-closed conservations laws:

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} f^s_\varepsilon(t,x,v)\, dv + \nabla_x \cdot \int_{\mathbb{R}^3} f^s_\varepsilon(t,x,v)\, v\, dv = 0,$$

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} \sum_s m^s\, f^s_\varepsilon(t,x,v)\, v\, dv + \nabla_x \cdot \int_{\mathbb{R}^3} \sum_s m^s\, f^s_\varepsilon(t,x,v)\, v \otimes v\, dv = 0,$$

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} \sum_s m^s\, f^s_\varepsilon(t,x,v)\, \frac{|v|^2}{2}\, dv + \nabla_x \cdot \int_{\mathbb{R}^3} \sum_s m^s\, f^s_\varepsilon(t,x,v)\, \frac{|v|^2}{2}\, v\, dv = 0.$$

Hilbert expansion

$$m^s\, f^s_\varepsilon(t,x,v) = \frac{\rho^s_\varepsilon(t,x)}{(2\pi\, T^\varepsilon(t,x)/m^s)^{3/2}}\, \exp\left(-\frac{m^s\, |v-u^\varepsilon(t,x)|^2}{2\, T^\varepsilon(t,x)}\right) + O(\varepsilon).$$

and one gets the Euler equations (up to order 1) of mixtures of compressible perfect monoatomic gases.

$$s = 1,..,N \qquad \frac{\partial \rho^s_\varepsilon}{\partial t} + \nabla_x \cdot (\rho^s_\varepsilon\, u_\varepsilon) = O(\varepsilon),$$

$$\frac{\partial}{\partial t}\left(\sum_s \rho^s_\varepsilon\, u_\varepsilon\right) + \nabla_x \cdot \left(\sum_s \left[\rho^s_\varepsilon\, u_\varepsilon \otimes u_\varepsilon + \frac{\rho^s_\varepsilon}{m^s}\, T^\varepsilon\right]\right) = O(\varepsilon),$$

$$\frac{\partial}{\partial t}\left(\sum_s \left[\rho^s_\varepsilon\, \frac{|u_\varepsilon|^2}{2} + \frac{3}{2}\, \frac{\rho^s_\varepsilon}{m^s}\, T_\varepsilon\right]\right)$$

$$+ \nabla_x \cdot \left(\sum_s \left[\rho^s_\varepsilon\, \frac{|u_\varepsilon|^2}{2}\, u_\varepsilon + \frac{5}{2}\, \frac{\rho^s_\varepsilon}{m^s}\, T_\varepsilon\, u_\varepsilon\right]\right) = O(\varepsilon).$$

Chapman-Enskog expansion

Navier-Stokes-(Fourier) equations (up to order 2) of mixtures of compressible perfect monoatomic gases.

$$s = 1, .., N \qquad \frac{\partial \rho^s_\varepsilon}{\partial t} + \nabla_x \cdot (\rho^s_\varepsilon \, u_\varepsilon) = \varepsilon \, \nabla_x \cdot F^s_\varepsilon + O(\varepsilon^2),$$

$$\frac{\partial}{\partial t} \left(\sum_s \rho^s_\varepsilon \, u_\varepsilon \right) + \nabla_x \cdot \left(\sum_s \left[\rho^s_\varepsilon \, u_\varepsilon \otimes u_\varepsilon + \frac{\rho^s_\varepsilon}{m^s} \, T_\varepsilon \right] \right) = \varepsilon \, \nabla_x \cdot \Pi_\varepsilon + O(\varepsilon^2),$$

$$\frac{\partial}{\partial t} \left(\sum_s \left[\rho^s_\varepsilon \frac{|u_\varepsilon|^2}{2} + \frac{3}{2} \frac{\rho^s_\varepsilon}{m^s} \, T_\varepsilon \right] \right) + \nabla_x \cdot \left(\sum_s \left[\rho^s_\varepsilon \frac{|u_\varepsilon|^2}{2} \, u_\varepsilon + \frac{5}{2} \frac{\rho^s_\varepsilon}{m^s} \, T_\varepsilon \, u_\varepsilon \right] \right)$$

$$= \varepsilon \, \nabla_x \cdot (\Pi_\varepsilon \cdot u_\varepsilon) + \varepsilon \nabla_x \cdot q_\varepsilon + O(\varepsilon^2).$$

Chapman-Enskog expansion, diffusion terms

$$F_\varepsilon^s = \sum_{j=1}^N L_{sj}(T_\varepsilon)\,\nabla_x \left(\frac{\rho_\varepsilon^j}{(2\pi\, T_\varepsilon/m^j)^{3/2}} \right) + L_{sq}(T_\varepsilon)\,\nabla_x(1/T_\varepsilon),$$

$$\Pi_\varepsilon = \left(\kappa(T_\varepsilon) - \frac{2}{3}\,\eta(T_\varepsilon) \right)\, (\nabla_x \cdot \mathbf{u}_\varepsilon)\, \mathbf{I} + \eta(T_\varepsilon)\, \left(\nabla_x u_\varepsilon + (\nabla_x u_\varepsilon)^T\right)$$

$$q_\varepsilon = \sum_{j=1}^N L_{qj}(T_\varepsilon)\,\nabla_x \left(\frac{\rho_\varepsilon^j}{(2\pi\, T_\varepsilon/m^j)^{3/2}} \right) - L_{qq}(T_\varepsilon)\,\nabla_x(1/T_\varepsilon)\,.$$

Rescaled Navier-Stokes-(Fourier) equations of mixtures of compressible perfect monoatomic gases.

$$s = 1, .., N \qquad \frac{\partial \rho_\delta^s}{\partial t} + \nabla_x \cdot (\rho_\delta^s \, u_\delta) = \nabla_x \cdot F_\delta^s,$$

$$\frac{\partial}{\partial t} \left(\sum_s \rho_\delta^s \, u_\delta \right) + \nabla_x \cdot \left(\sum_s \left[\rho_\delta^s \, u_\delta \otimes u_\delta + \delta^{-2} \, \frac{\rho_\delta^s}{m^s} \, T_\delta \right] \right) = \nabla_x \cdot \Pi_\delta,$$

$$\frac{\partial}{\partial t} \left(\sum_s \left[\rho_\delta^s \, \frac{|u_\delta|^2}{2} + \frac{3}{2} \, \frac{\rho_\delta^s}{m^s} \, T_\delta \right] \right) + \nabla_x \cdot \left(\sum_s \left[\rho_\delta^s \, \frac{|u_\delta|^2}{2} \, u_\delta + \frac{5}{2} \, \frac{\rho_\delta^s}{m^s} \, T_\delta \, u_\delta \right] \right)$$

$$= \delta^2 \, \nabla_x \cdot (\Pi_\delta \cdot u_\delta) + \nabla_x \cdot q_\delta.$$

Low Mach number scaling

$$\rho_\delta^s(t,x) = \rho_0^s + \delta\, \rho_1^s(t,x) + O(\delta^2), \qquad T^\delta(t,x) = T_0 + \delta\, T_1(t,x) + O(\delta^2),$$

$$u^\delta(t,x) = u_0(t,x) + O(\delta).$$

Incompressible limit:

$$\nabla_x \cdot u_0 = 0\,,$$

Boussinesq identity:

$$\nabla_x \left(\frac{T_0}{\sum_s \frac{\rho_0^s}{m_s}} \sum_s \frac{\rho_1^s}{m_s} + T_1 \right) = 0\,,$$

Low Mach number scaling, case without Dufour and Soret effect

Convection-diffusion equation for the momentum:

$$\partial_t u_0 + u_0 \cdot \nabla_x u_0 + \nabla_x p = d_1 \, \Delta_x u_0,$$

Convection-diffusion equation for the temperature:

$$\partial_t T_1 + u_0 \cdot \nabla_x T_1 = d_2 \, \Delta_x T_1.$$

Diffusion coefficient: $d_1 = \frac{\eta(T_0)}{\sum_s \frac{\rho_0^s}{m_s}}$,

Thermal conduction coefficient: $d_2 = \frac{2}{5} \frac{L_{qq}(T_0)}{\sum_s \frac{\rho_0^s}{m_s} T_0^2}$.

Boltzmann equations for a mixture of rarefied monoatomic gases with Maxwell molecules cross sections:

$$\varepsilon \, \frac{\partial f^s}{\partial t}(t,x,v) + v \cdot \nabla_x f^s(t,x,v) = \frac{1}{\varepsilon} \sum_{r=1}^{N} Q^{sr}(f^s(t,x,\cdot), f^r(t,x,\cdot))(v),$$

$$Q^{sr}(f^s, f^r)(v) = \iint \Big(f^s(v')\, f^r(w') - f^s(v)\, f^r(w) \Big)$$

$$\times \; \sigma^{sr} \left(\arccos \left(\widehat{\frac{v-w}{|v-w|}, \sigma} \right) \right) \; dw \, d\sigma,$$

$$v' = \frac{m^s}{m^s + m^r}\, v + \frac{m^r}{m^s + m^r}\, w + \frac{m^r}{m^s + m^r}\, |v-w|\, \sigma\,,$$

$$w' = \frac{m^s}{m^s + m^r}\, v + \frac{m^r}{m^s + m^r}\, w - \frac{m^s}{m^s + m^r}\, |v-w|\, \sigma\,.$$

Expansion for the distribution functions:

$$f_\varepsilon^s = \rho_s \, M_s \, (1 + \varepsilon \, g_\varepsilon^s), \qquad M_s(v) = \left(\frac{m_s}{2\pi}\right)^{3/2} \mathrm{e}^{-\frac{m_s}{2} |v|^2} .$$

Then

$$g^\varepsilon(v) = \alpha^s + m^s \, v \cdot u + \left(\frac{m_s}{2} |v|^2 - \frac{3}{2}\right) T + O(\varepsilon) \, ,$$

$$\nabla_x \cdot u = 0\,,$$

Boussinesq identity:

$$\nabla_x \left(\left(\rho^s \, \alpha^s \right) + T \right) = 0\,,$$

Convection-diffusion equations for the densities of the species:

$$\partial_t \left[\sum_{r \neq s} \rho^r \mu^{sr} \, \kappa^{sr} (\alpha^s - \alpha^r) \right] + u \cdot \nabla_x \left[\sum_{r \neq s} \rho^r \mu^{sr} \kappa^{sr} (\alpha^s - \alpha^r) \right]$$
$$= \Delta_x \left[\sum_{r \neq s} \rho^r (\alpha^s - \alpha^r) \right], \qquad s = 1, \dots, N-1,$$

where $\mu^{sr} = \frac{m^s \, m^r}{m^s + m^r}$ is the reduced mass.

Angular integrals of the (Maxwell molecules) cross sections:

$$\kappa^{sr} = 2\pi \int_0^\pi \sigma^{sr}(\chi)(1 - \cos\chi)\sin\chi\, d\chi\,,$$

$$\nu^{sr} = 2\pi \int_0^\pi \sigma^{sr}(\chi)(1 - \cos^2\chi)\sin\chi\, d\chi\,.$$

Convection-diffusion equation for the momentum:

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = d_1 \, \Delta_x u,$$

Convection-diffusion equation for the temperature:

$$\partial_t T + u \cdot \nabla_x T = d_2 \, \Delta_x T.$$

Diffusion rate:

$$d_1 = \sum_{s=1}^{N} \rho^s \, \theta^s,$$

where the parameters θ^s are the unique solution of the linear system

$$\left[\tfrac{3}{4}\,\rho^s \nu^{ss} + \textstyle\sum_{r\neq s} \rho^r \,\frac{\mu^{sr}}{m^s+m^r}\, \left(2\kappa^{sr} + \tfrac{3}{2}\, \frac{m^r}{m^s} \nu^{sr}\right)\right] \theta^s$$

$$+ \textstyle\sum_{r\neq s} \rho^r \,\frac{\mu^{sr}}{m^s+m^r}\, \left(-\,2\kappa^{sr} + \tfrac{3}{2}\, \nu^{sr}\right) \theta^r = \left(\textstyle\sum_{s=1}^{N} \rho^s \, m^s \right)^{-1}, \qquad s = 1, \dots, N\,;$$

Thermal conduction rate:

$$d_2 = \sum_{s=1}^{N} \frac{\rho^s}{\sqrt{m^s}}\, \eta^s,$$

where the parameters η^s are the unique solution of the linear system

$$\tfrac{1}{2}\, \rho^s (m^s)^{1/2} \nu^{ss} + \textstyle\sum_{r\neq s} \rho^r \frac{\mu^{sr}}{(m^s+m^r)^2} \Big[(m^s)^{-1/2} \big(3(m^s)^2 + (m^r)^2\big) \kappa^{sr}$$
$$+2(m^s)^{1/2} m^r \nu^{sr} \Big] \eta^s$$
$$+ \textstyle\sum_{r\neq s} \rho^r \frac{\mu^{sr}}{(m^s+m^r)^2}\, m^s (m^r)^{1/2} \big(-\ 4\kappa^{sr} + 2\, \nu^{sr} \big) \eta^r = 1\,, \qquad s = 1, \ldots, N\,.$$

Extensions to get closer to real rarefied gases:

- General cross sections
- Polyatomic gases (mixtures)
- Chemistry
- Boundary conditions